NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Modeling for Interest; Proof for Power

Daniel Teague
NC School of Science and Mathematics

NCTM Annual Meeting
April 28, 2018

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Proof for Interest; Modeling for Power

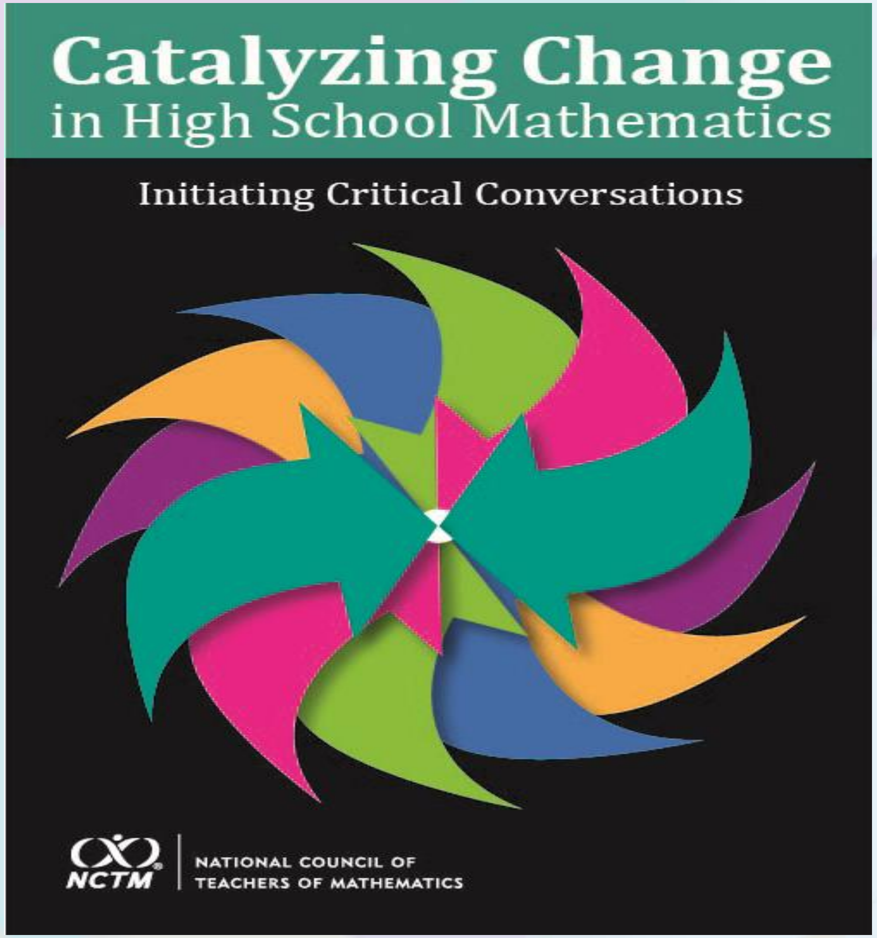
Daniel Teague

NC School of Science and Mathematics

NCTM Annual Meeting

April 28, 2018

- **Broaden the purposes** for teaching high school mathematics.
- **Catalyze discussions** of the **challenges** as well as recommendations for implementing actions to overcome those challenges.
- **Define imperatives** in the areas of structures, instructional practices, curriculum, and pathways for students.
- **Identify Essential Concepts for focus** that all students should learn and understand at a deep level.
- **What to consider for common pathways** of mathematical study. Each pathway includes a **common set** of mathematical study expected of high all school students, followed by **alternate paths of study**, differentiated by postsecondary education and career goals.



What is Mathematics?

Poet Marianne Moore describes poets as

"literalists of the imagination"

and poetry as

"imaginary gardens with real toads in them"

Mathematics has its imaginary gardens and real toads
mathematical theory (pure mathematics)
mathematical modeling (applied mathematics).

Theory and Application

High school students can experience these two connected but distinct aspects of mathematics through their experiences in school with mathematical proof and mathematical modeling.

Both proof and modeling call on the students to use the essential concepts and mathematical tools developed in class in creative and collaborative investigations.

Mathematical Identity

A student's *mathematics identity* comprises the dispositions and deeply held beliefs they develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics in their lives.

A mathematics identity may reflect a sense of oneself as a competent performer who is able to do mathematics or as the kind of person who is unable to do mathematics.

The Impact of Identity in K–8 Mathematics
Aguirre, Mayfield-Ingram, Martin

Ownership

Ownership of the mathematics occurs when students have the flexibility to make decisions about what to solve and how to solve it themselves.

This means they are thinking their way through problems rather than just remembering what they were told to do or repeating the teacher's approach.

How do you do mathematics?

What did the teacher say to do?

By remembering?

What do we think we should do?

By thinking?

Joy, Beauty, Inspiration, Understanding

Students gain confidence in their mathematical identity when they solve problems *by* themselves and in their own way.

In both mathematical proof and in mathematical modeling, students can move from doing mathematics *by* themselves to doing mathematics *for* themselves.

What is Mathematical Modeling?

Mathematical Modeling is when you use mathematics to understand a situation in the real world, and then perhaps use it to take action or even to predict the future, and where both the real world situation and the ensuing mathematics are taken seriously.

“What is Mathematical Modeling” H. Pollak, Teachers College, Columbia University

Real World Taken Seriously?

1. Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

At no point in the problem will the fact that the problem addresses grass clippings play a role in the mathematics required of the students.

Real World Taken Seriously

You and three friends have a box of cookies. There are 10 cookies in the box. How many cookies should each of you get if the box is divided fairly?

"Each of us gets two cookies except Mia. She gets one."

"Michael gets three, the rest of us get two."

Teacher: You have to use up all the cookies

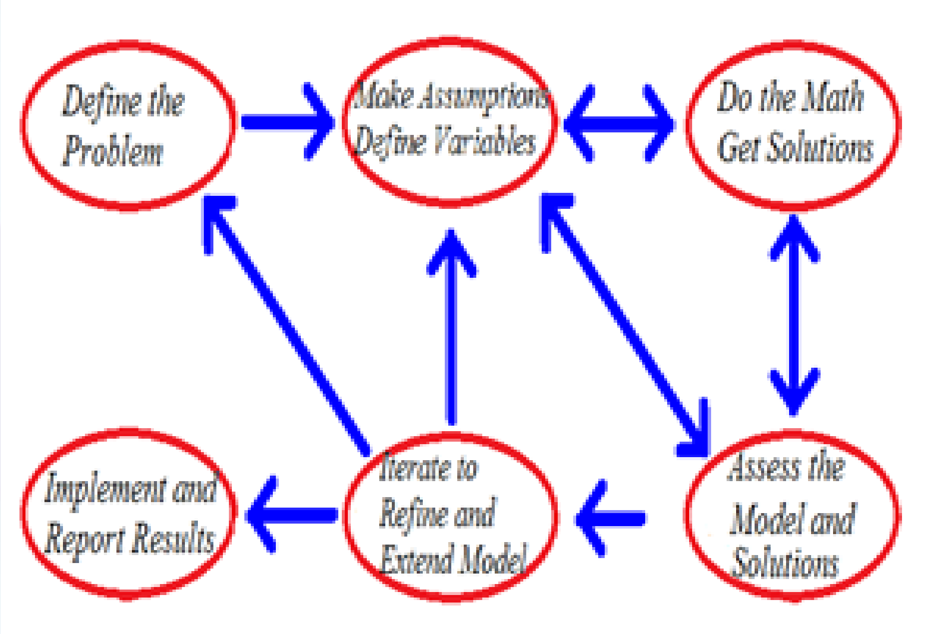
"We each get two and we give the rest to the teacher."

"We each get two then you (the teacher) and Mr. Green (the custodian) can each have one."

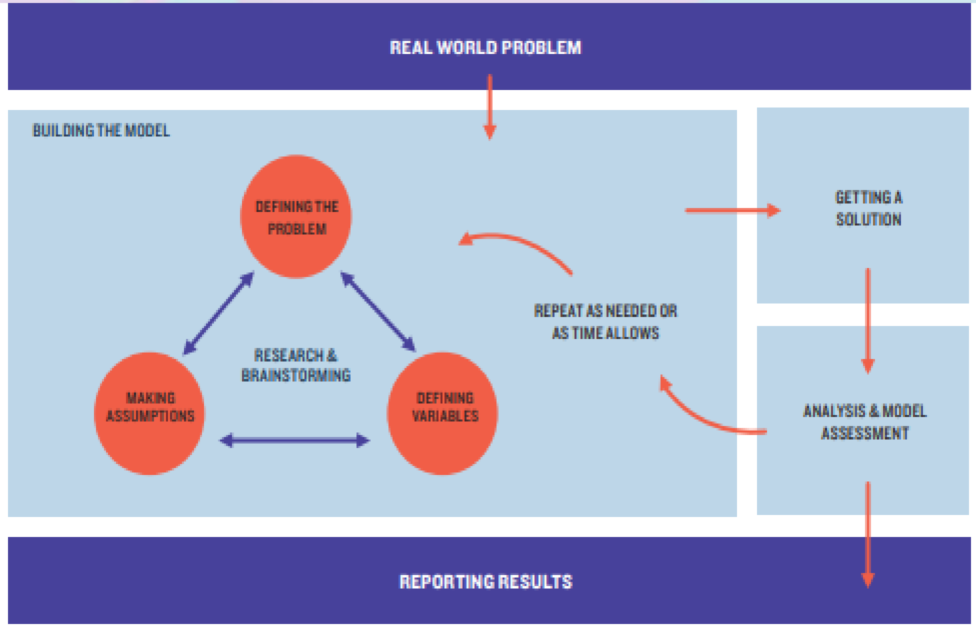
The teacher was trying to force them to see and perhaps to say, "we each get 2 ½ cookies".

In the student's real world, that was not a correct answer. They were modeling, they were taking their real world seriously.

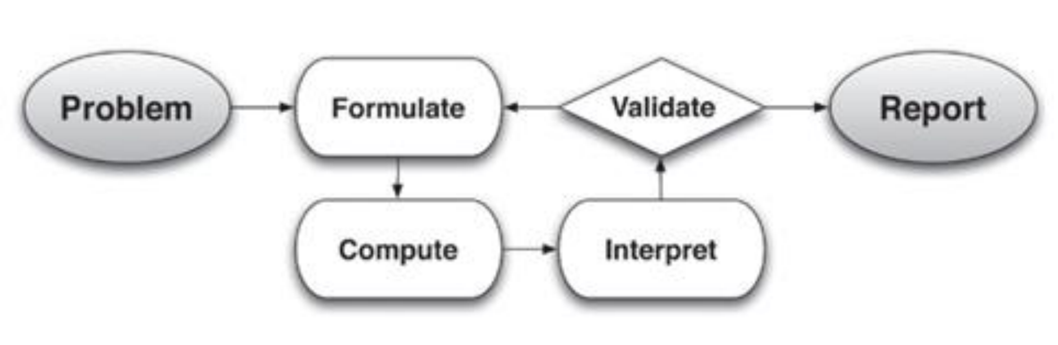
Example Modeling Cycles



GAIMME



Math Modeling: Computing and Communicating



IMMERSION

Keys to Modeling

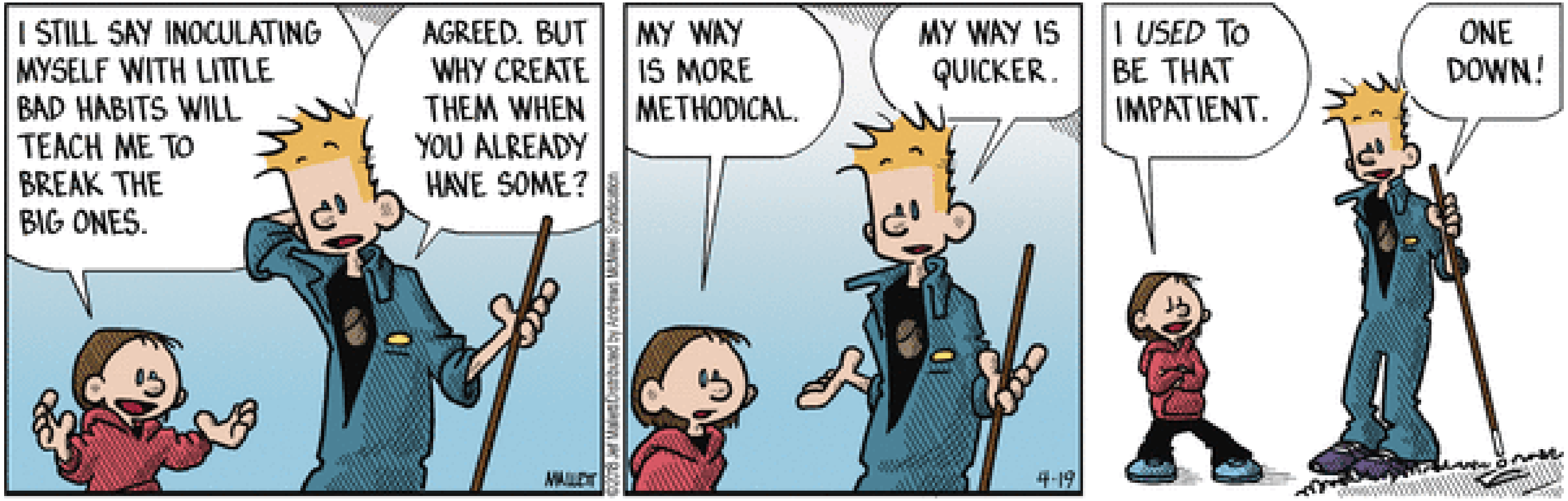
Modeling happens when students make important decisions about what problem to solve, how to proceed, and when to turn back.

Start Small and Don't Stop

Frazz

BY JEF MALLETT

April 19, 2018 ▼ GO



UCLICK.

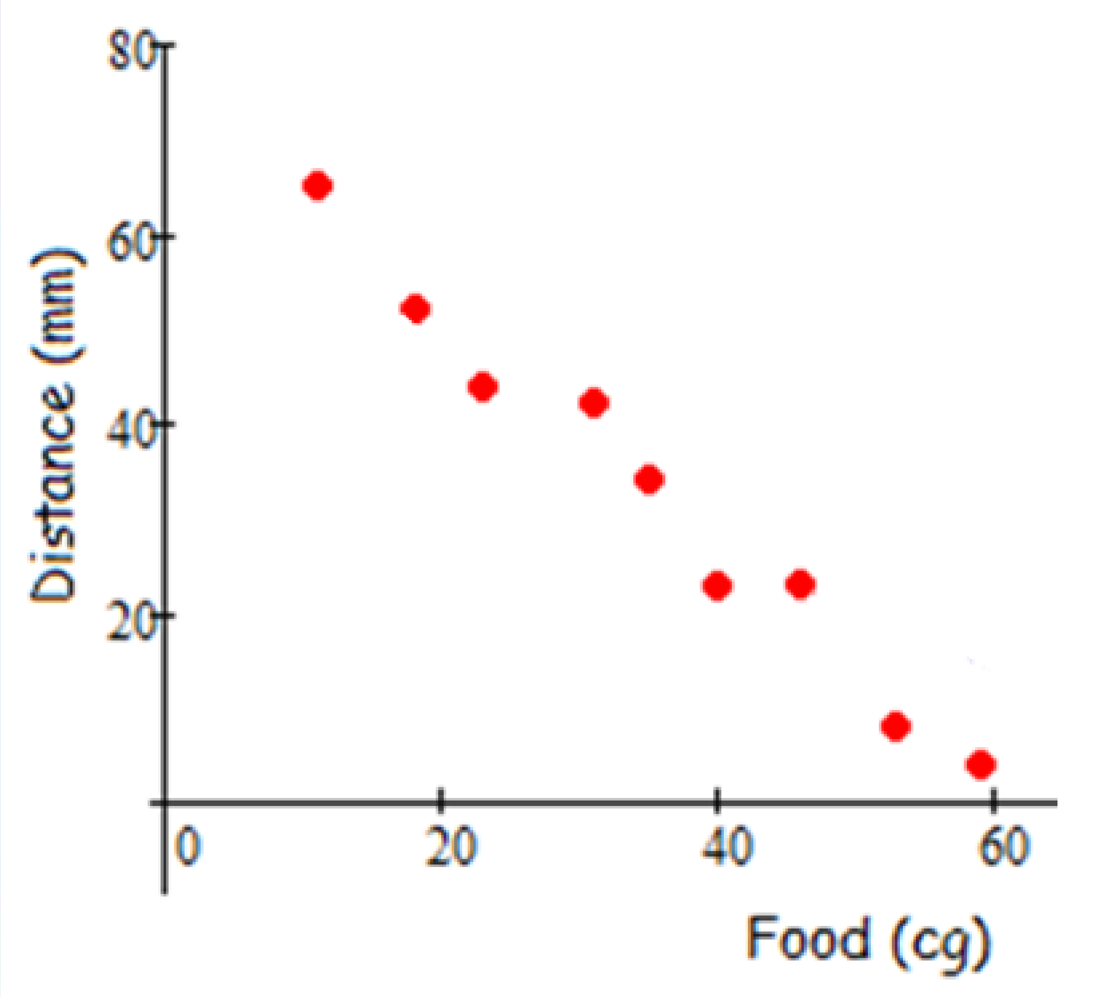
Small Problems: Students Making Decisions

***The Mantid Problem*:** A mantid is a small, crawling insect that closely resembles a cockroach. Mantids are often used in biological studies because they are the insect version of a sloth, that is, they rarely move, so it is easy to keep track of them. However, the mantid will move to seek food. Researchers have been studying the relationship between the distance a mantid will move for food and the amount of food already in the mantid's stomach. The distance is measured in millimeters and the amount of food in centigrams (a hundredth of a gram). In the research, food was placed progressively nearer to a mantid. The point the mantid began to move to the food was defined as the distance in the table below. The amount of food, F, in the mantid's stomach was also measured.. Measurements for 15 mantids are given below:

Food (cg)	11	18	23	31	35	40	46	53	59	66	70	72	75	86	90
Distance (mm)	65	52	44	42	34	23	23	8	4	0	0	0	0	0	0

Based on the data provided, what can you say about the relationship between the amount of food in the mantid's stomach and the distance it will walk to eat?

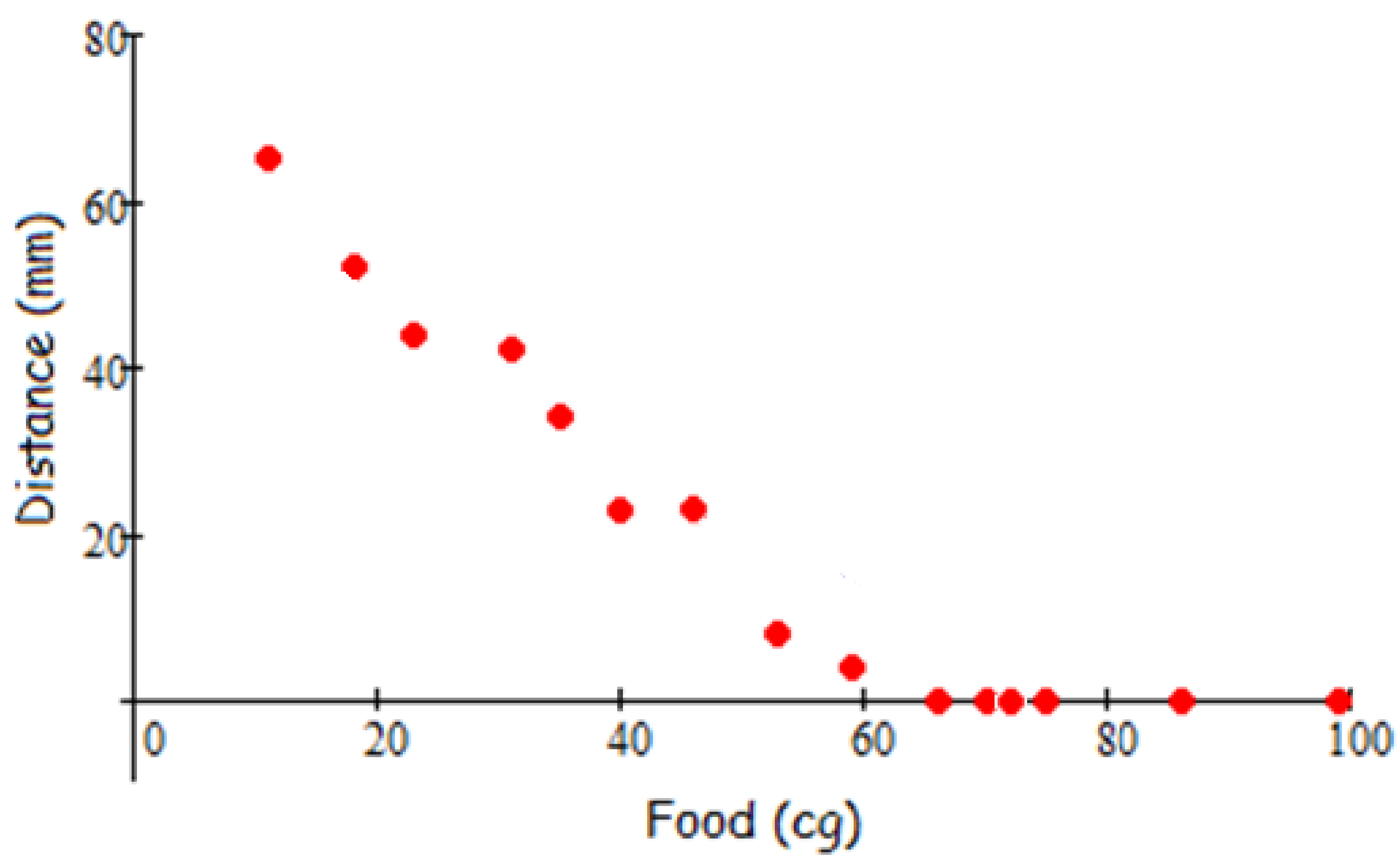
Food (cg)	11	18	23	31	35	40	46	53	59
Distance (mm)	65	52	44	42	34	23	23	8	4

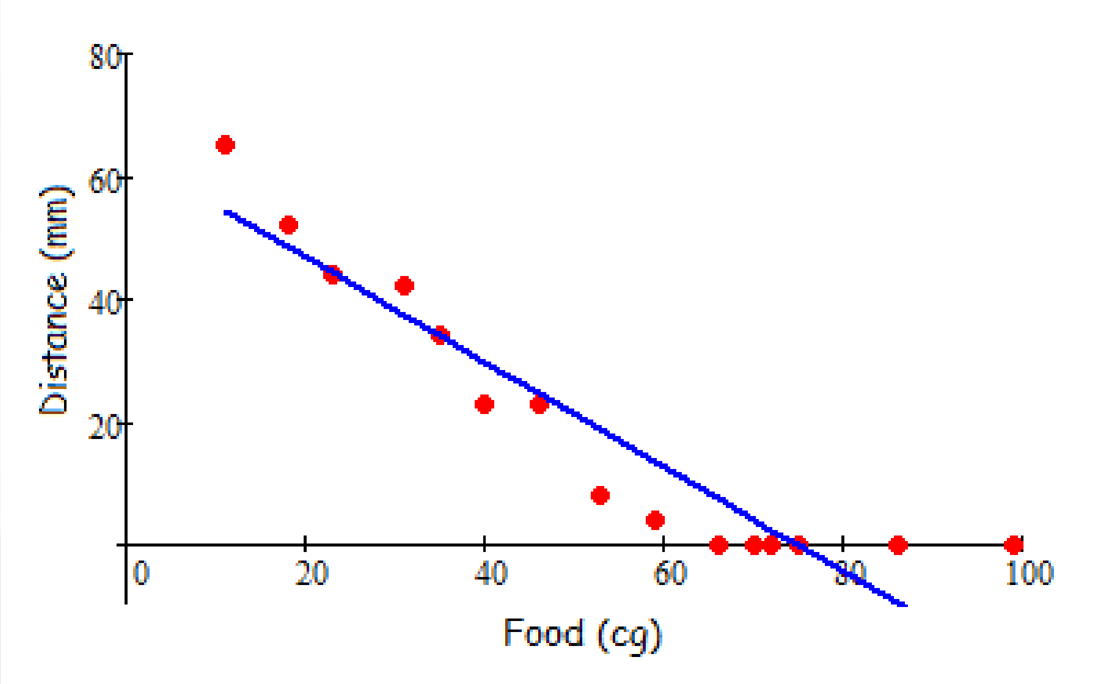


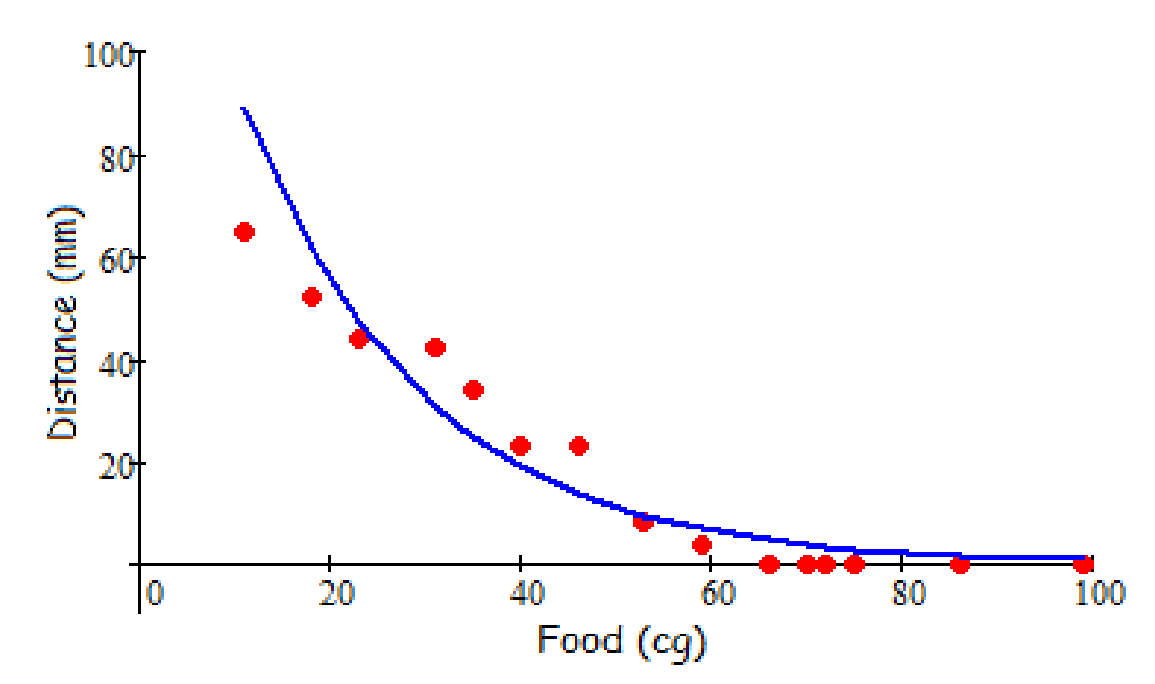
Standard Textbook Problem

This is not modeling, the students are just doing what they have been taught to do.

Food (cg)	11	18	23	31	35	40	46	53	59	66	70	72	75	86	90
Distance (mm)	65	52	44	42	34	23	23	8	4	0	0	0	0	0	0

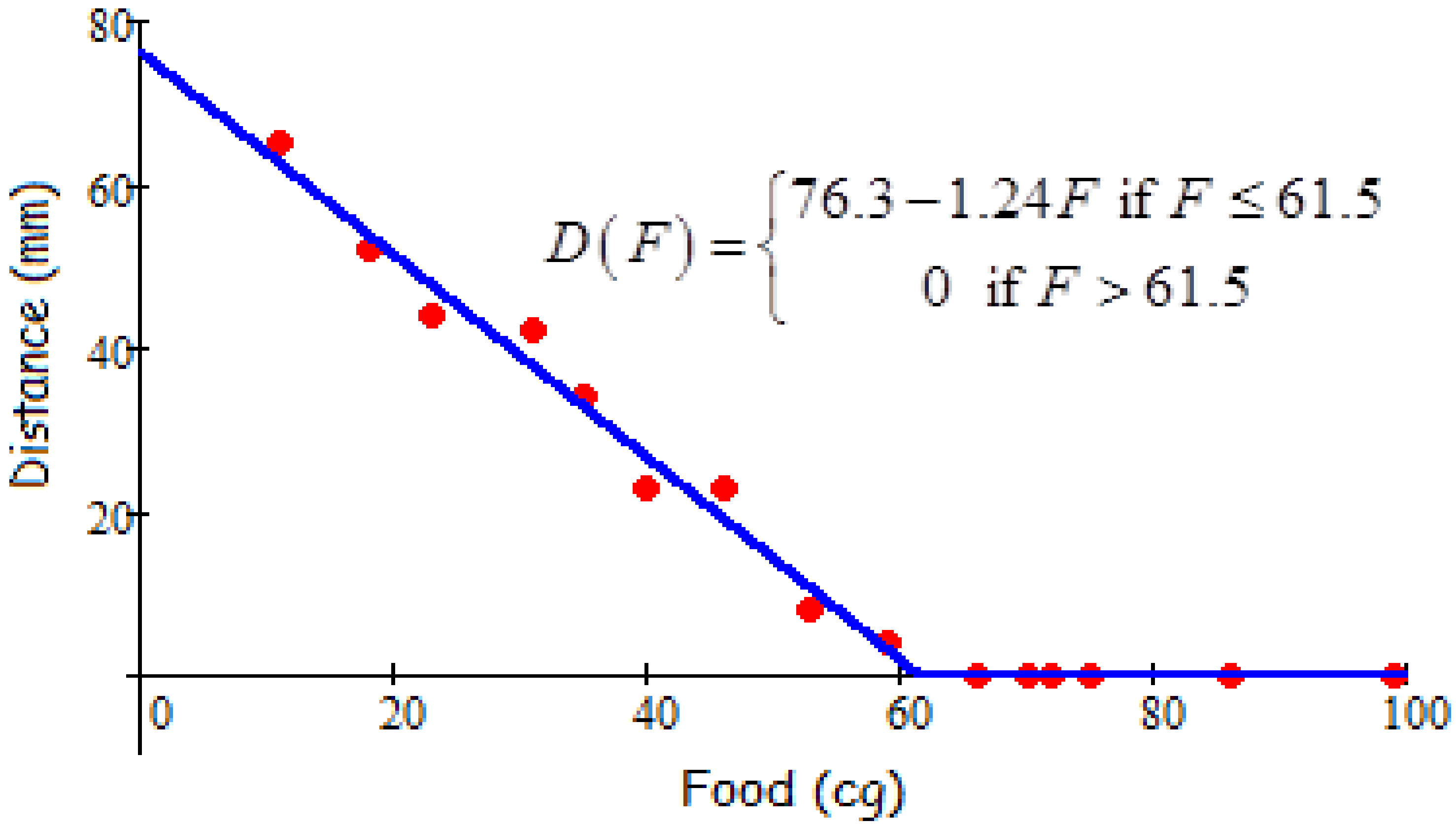






Food (cg)	11	18	23	31	35	40	46	53	59	66	70	72	75	86	90
Distance (mm)	65	52	44	42	34	23	23	8	4	0	0	0	0	0	0

How far will the mantid walk when it walks for food?



Food (cg)	11	18	23	31	35	40	46	53	59	66	70	72	75	86	90
Distance (mm)	65	52	44	42	34	23	23	8	4	0	0	0	0	0	0

Variety of Assignments

- Group work: 3-4 students working together on a problem set (shorter problems)
- Activities/Labs: Include a short writing component
- Projects: More formal write-ups
- Week-long Investigations/Modeling Problems

3-Act Math Lessons

Dan Meyer
whenmathhappens.com/3-act-math/

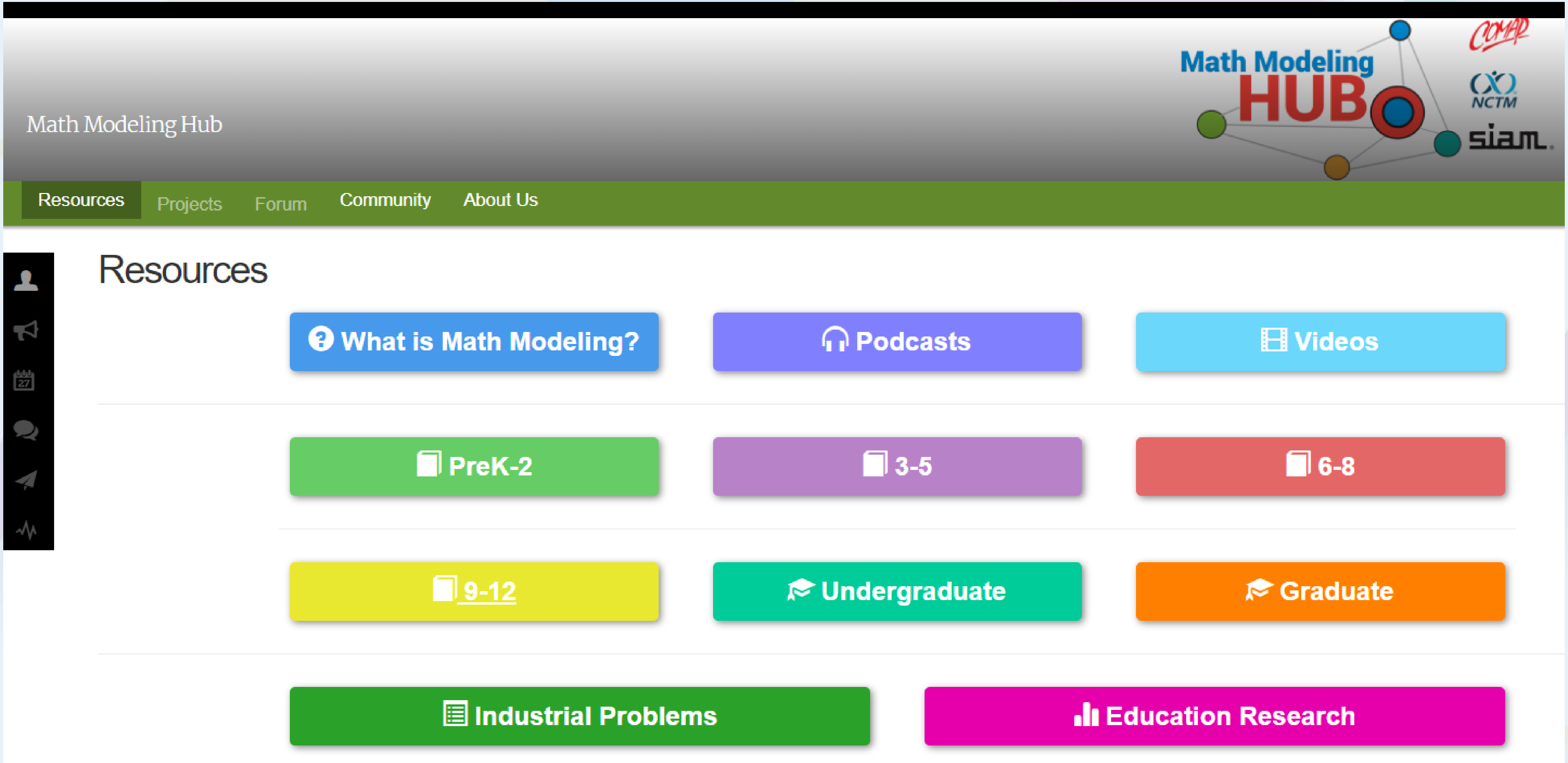
Robert Kaplinski
http://robertkaplinsky.com/lessons/

Graham Fletcher
gfletchy.com/3-act-lessons/

Support Modeling with Access to Material and Examples

Math Modeling Hub

https://qubeshub.org/community/groups/mmfmn



When Up and Running, this is what you will find

Guidelines for Assessment and Instruction in Mathematical Modeling Education

COMAP/SIAM

http://www.comap.com/Free/GAIMME/

Keys to Modeling

Creative modeling happens when students make important decisions about what problem to solve, how to proceed, and when to turn back.

Create the simplest form of the problem that contains the essence of the problem.

Use your basic solution and the iterative process to add more reality to your initial solution.

Pay close attention to errors. Try to understand why, how, and by how much they are wrong.

The Midge Problem: Summer Bridge Version

In 1981, two new varieties of a tiny biting insect called a midge were discovered by biologists W. L. Grogan and W. W. Wirth in the jungles of Brazil. They dubbed one kind of midge an **Apf** midge and the other an **Af** midge. The biologists found out that the **Apf** midge is a carrier of a disease. The other form of the midge, the **Af**, is quite harmless and a valuable pollinator. In an effort to distinguish the two varieties, the biologist took measurements on the midges they caught. The two measurements taken were of wing length and antennae length, both measured in centimeters.

Af Midges

Wing Length (cm)	1.72	1.64	1.74	1.70	1.82	1.82	1.90	1.82	2.08
Antenna Length (cm)	1.24	1.38	1.36	1.40	1.38	1.48	1.38	1.54	1.56

Apf Midges

Wing Length (cm)	1.78	1.86	1.96	2.00	2.00	1.96
Antenna Length (cm)	1.14	1.20	1.30	1.26	1.28	1.18

Is it possible to distinguish an **Af** midge from an **Apf** midge on the basis of wing and antenna length? Write a report that describes to a naturalist in the field how to classify a midge he or she has just captured.

The Midge Problem: Adv. Precal Version

In 1981, two new varieties of a tiny biting insect called a midge were discovered by biologists W. L. Grogan and W. W. Wirth in the jungles of Brazil. They dubbed one kind of midge an **Apf** midge and the other an **Af** midge. The biologists found out that the **Apf** midge is a carrier of a disease. The other form of the midge, the **Af**, is quite harmless and a valuable pollinator. In an effort to distinguish the two varieties, the biologist took measurements on the midges they caught. The two measurements taken were of wing length and antennae length, both measured in centimeters.

Af Midges

Wing Length (cm)	1.72	1.64	1.74	1.70	1.82	1.82	1.90	1.82	2.08
Antenna Length (cm)	1.24	1.38	1.36	1.40	1.38	1.48	1.38	1.54	1.56

Apf Midges

Wing Length (cm)	1.78	1.86	1.96	2.00	2.00	1.96
Antenna Length (cm)	1.14	1.20	1.30	1.26	1.28	1.18

Is it possible to distinguish an **Af** midge from an **Apf** midge on the basis of wing and antenna length? Write a report that describes to a naturalist in the field how to classify a midge he or she has just captured.

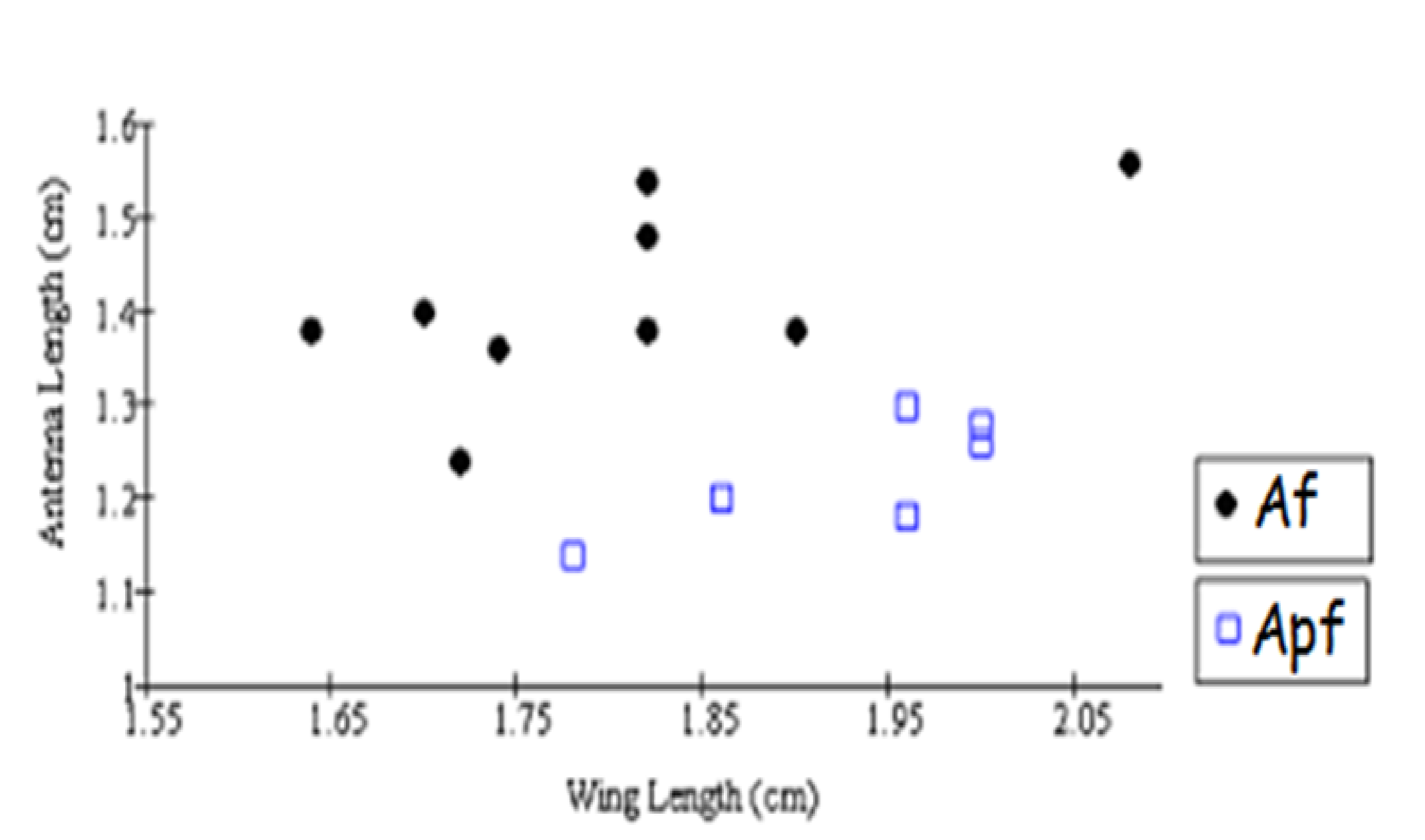
The Midge Problem: Post Calculus Version

In the early 1990's, two new varieties of a large biting fly were discovered by biologists in the jungles of Guatemala. The biologists found that one of the flies, which in the field they called the **Ax** fly, was a carrier of disease, while the other, called the **Aa** fly, was quite harmless (the names were later changed once the biologists were able to determine the appropriate genus). In an effort to distinguish the two varieties when captured in the field, the biologists took simple measurements on the flies they caught. The easiest measurements were the wing length, the abdomen length, and antennae length (all measured in centimeters).

Aa Biting Fly										
Wing Length (cm)	2.10	3.31	3.83	3.11	1.65	1.73	1.84	2.49	2.53	2.99
Abdomen Length (cm)	1.72	1.94	1.74	1.70	1.82	1.83	1.90	1.82	1.88	1.87
Antenna Length (cm)	1.34	1.58	1.64	1.45	1.38	1.48	1.49	1.54	1.56	1.65

Ax Biting Fly							
Wing Length (cm)	2.87	4.02	3.18	3.51	3.74	3.95	3.28
Abdomen Length (cm)	1.78	1.86	1.96	2.00	2.10	1.96	1.87
Antenna Length (cm)	1.14	1.29	1.30	1.26	1.39	1.28	1.28

Find a way using the of wing, abdomen, and antenna lengths to distinguish an **Aa** from an **Ax** fly.



Antenna Length (cm)
1.6
1.5
1.4
1.3
1.2
1.1
1
1.55
1.65
1.75
1.85
1.95
2.05
Wing Length (cm)
Af
Apf

Fit Regression Models

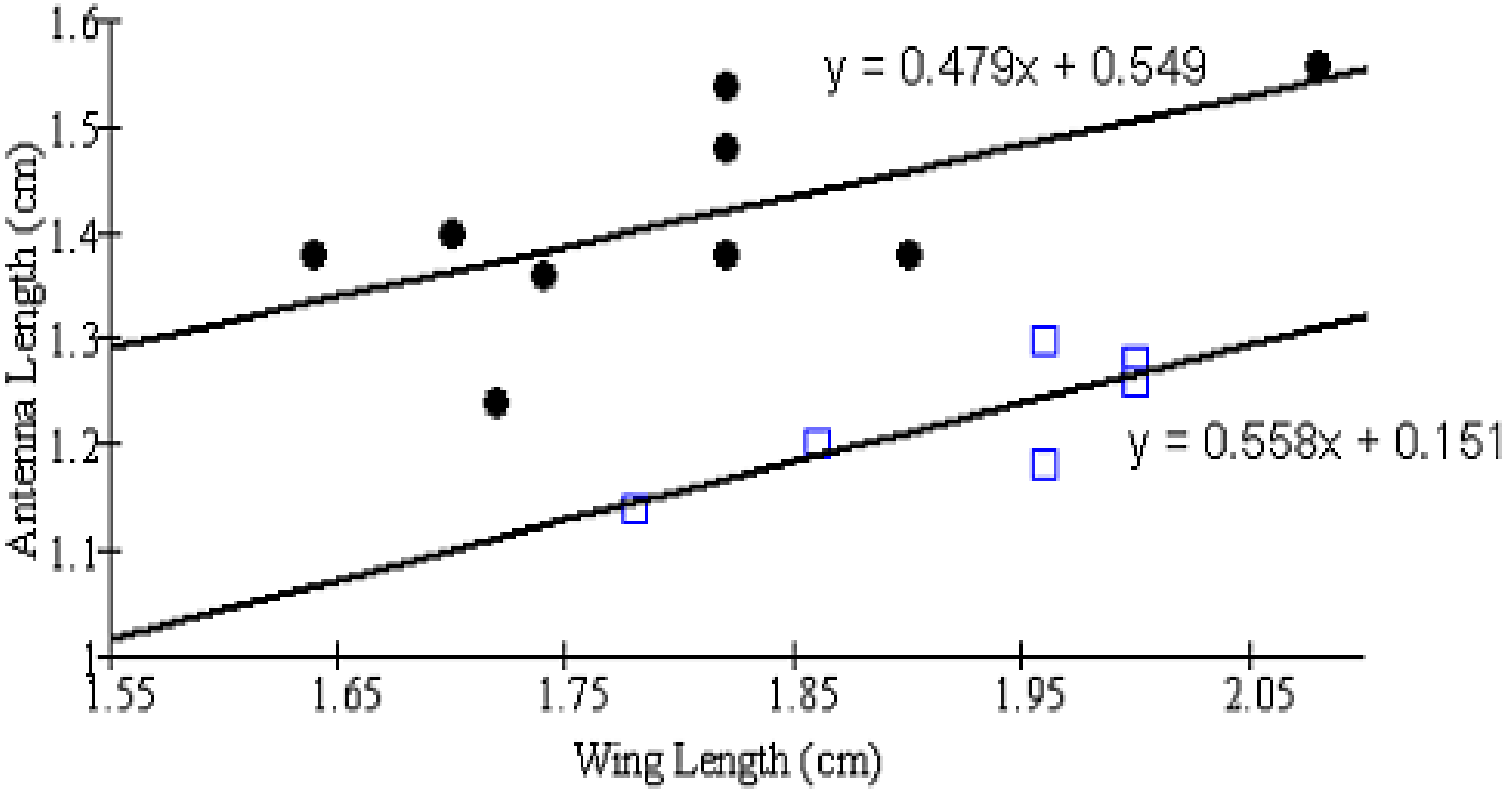


Figure 2: Linear Least-Squares Line Fit to Each Data Set

Average or Mid-Line

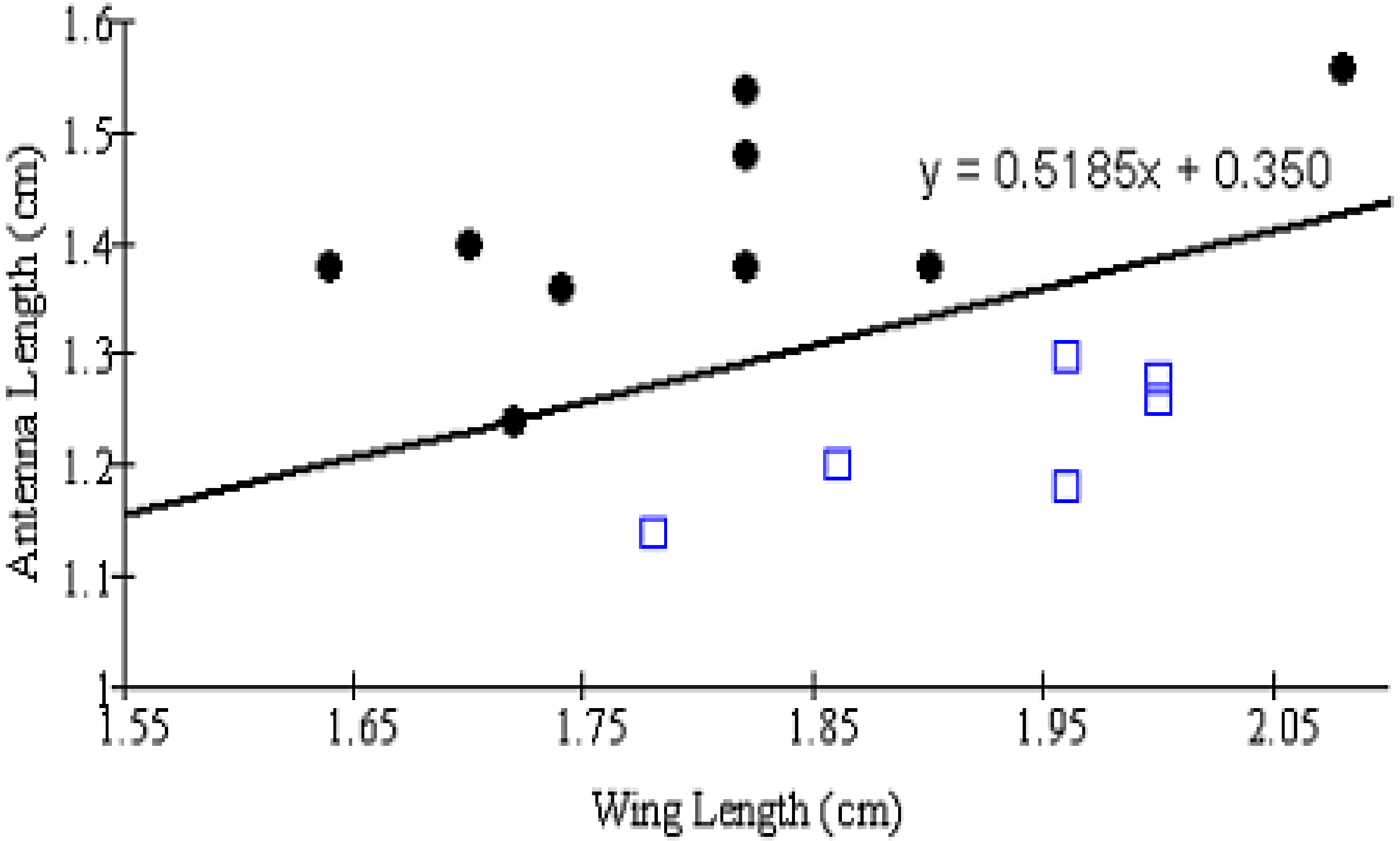
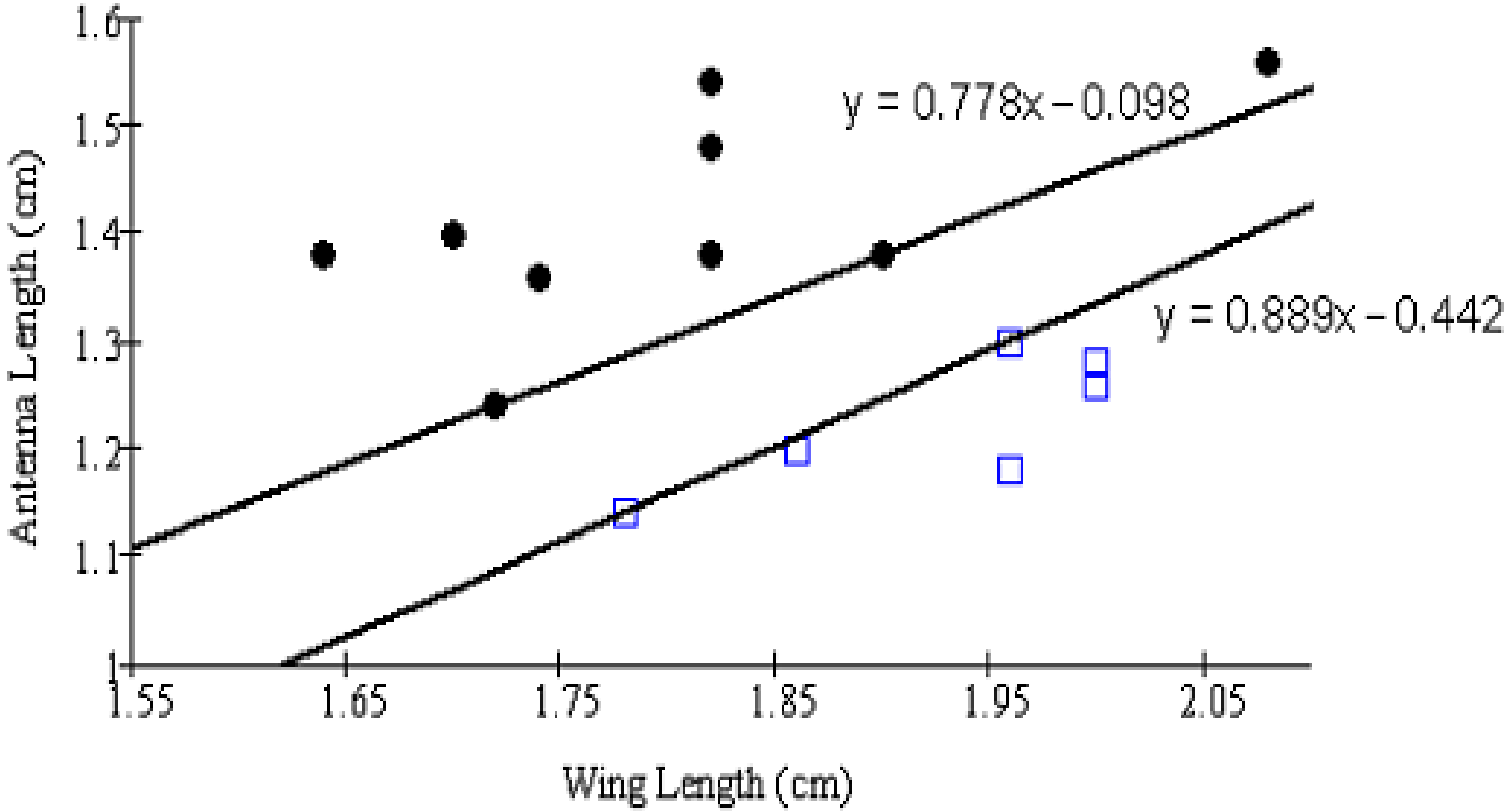


Figure 3: Least Squares Mid-Line



Outer-Most Midge Lines

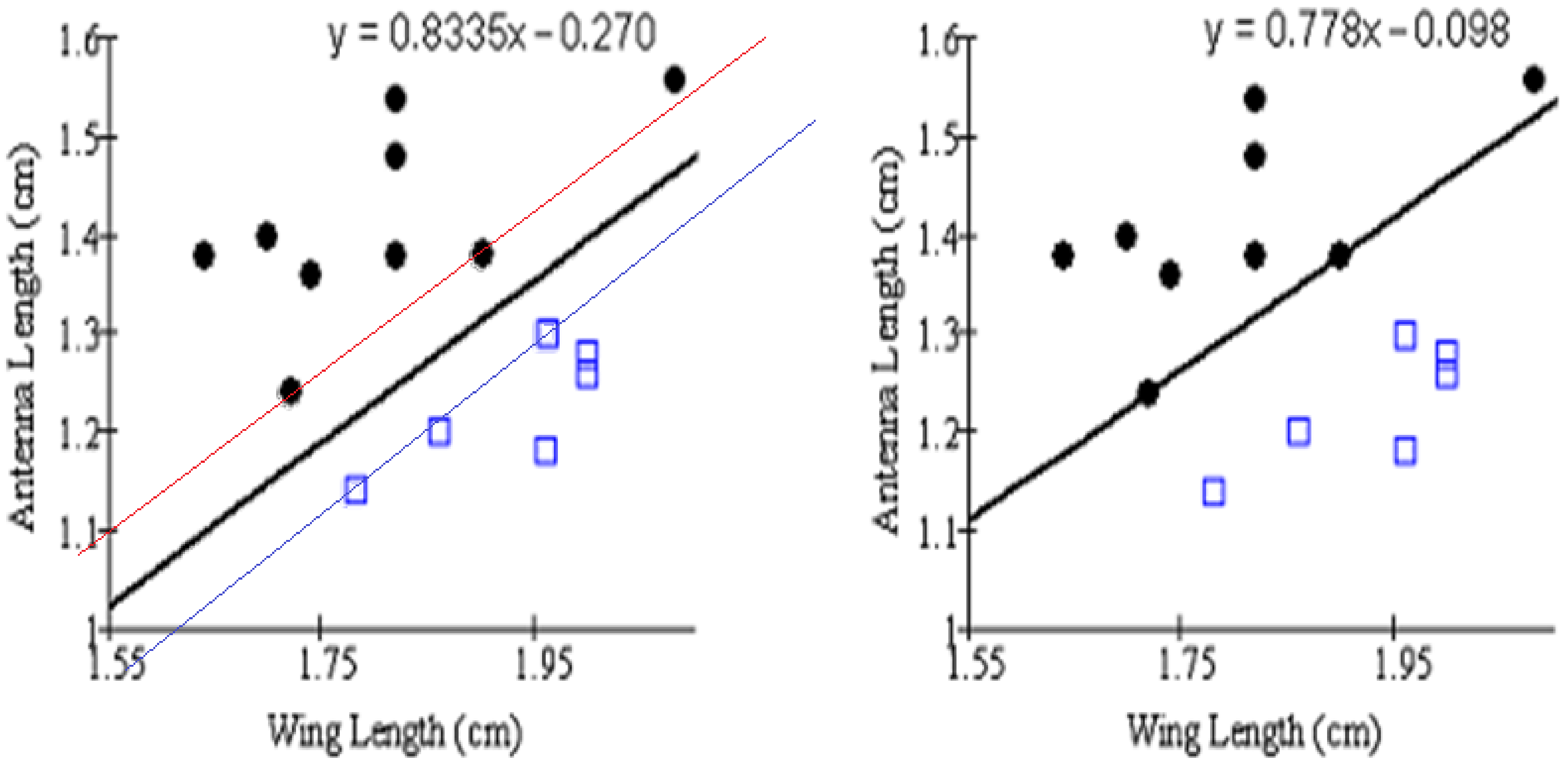
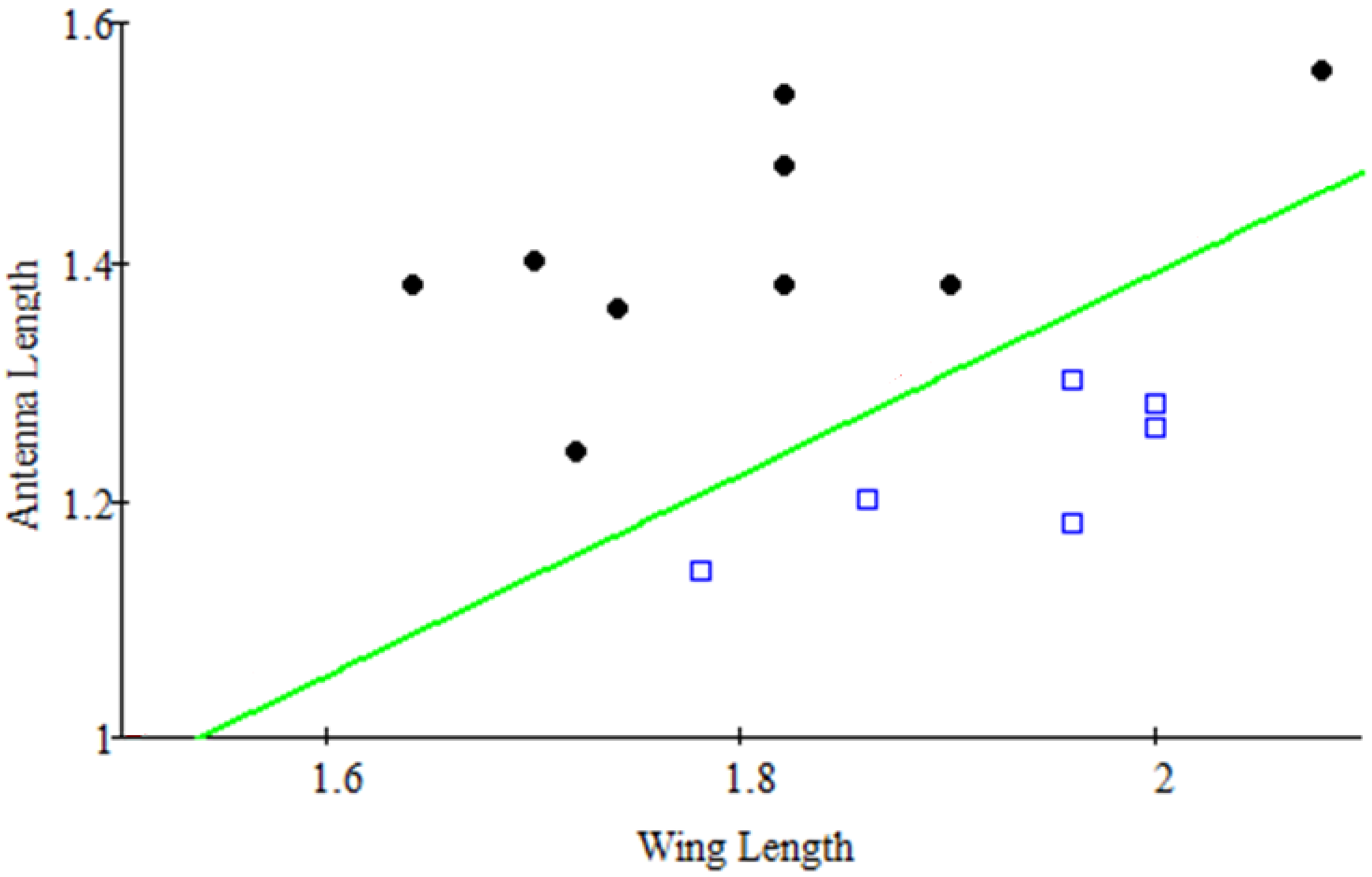


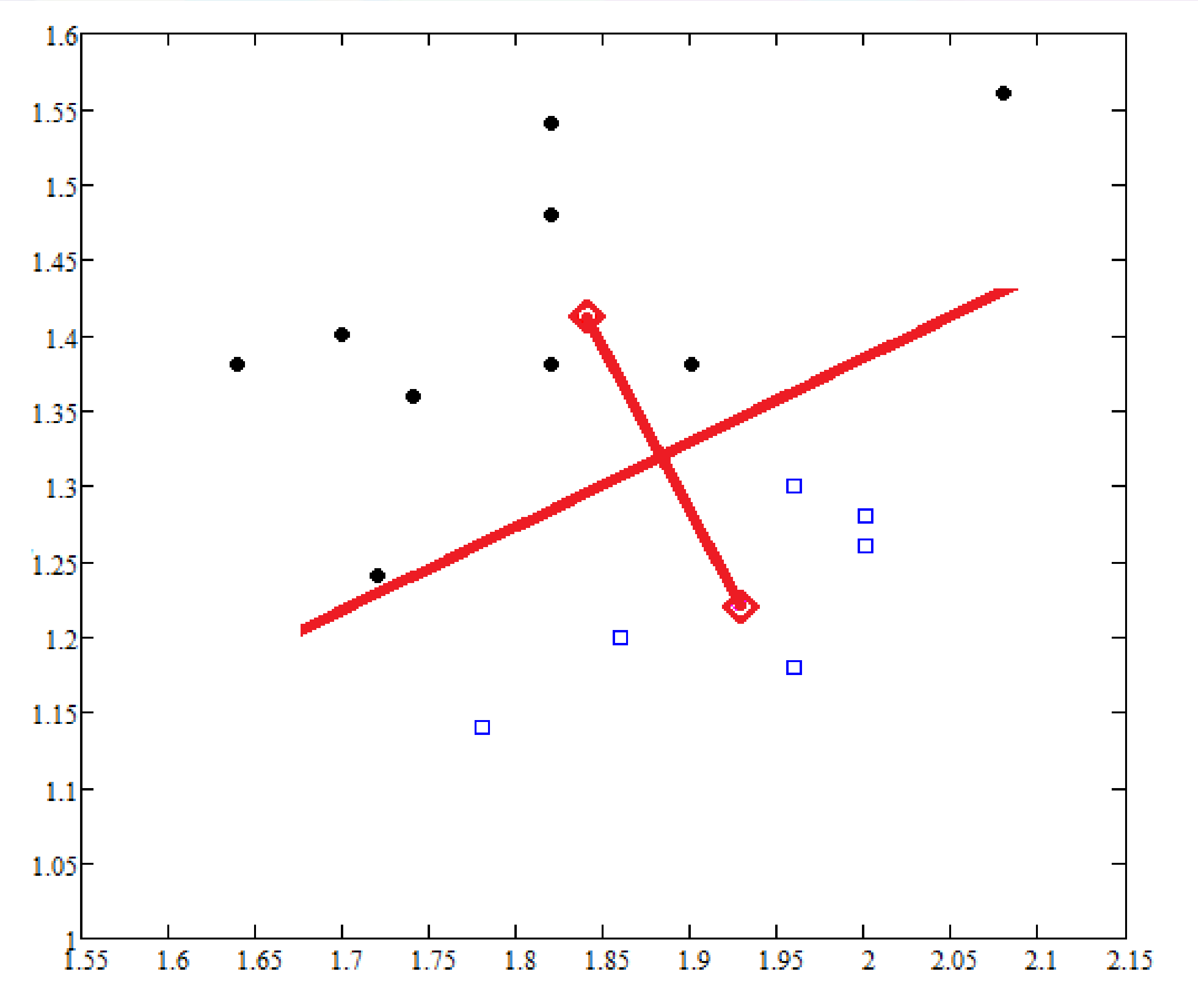
Figure 5: Lines for "Accuracy" and "Safety"

Weight the lines since 9 of the 15 are Af

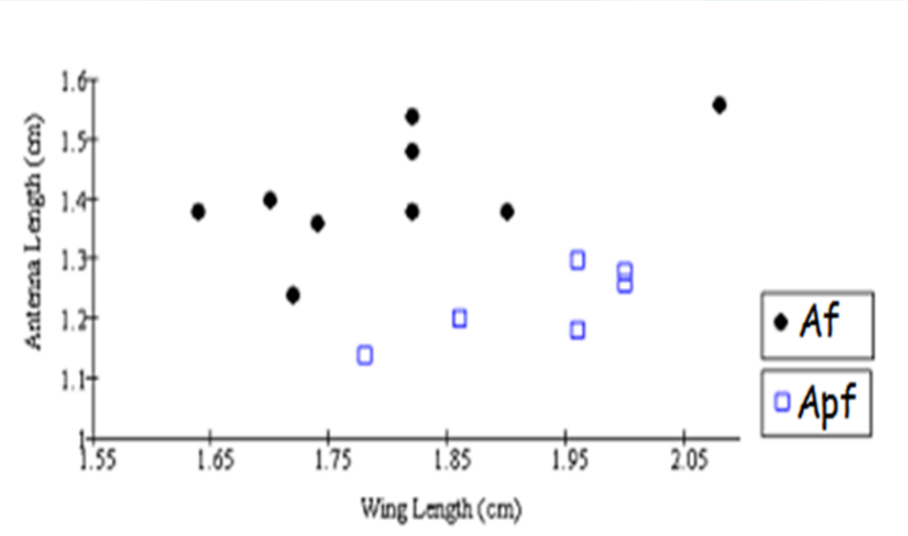


3/5 of flies are Af, so (2/5 Af line) + (3/5 Apf line)

Distance from Average Values

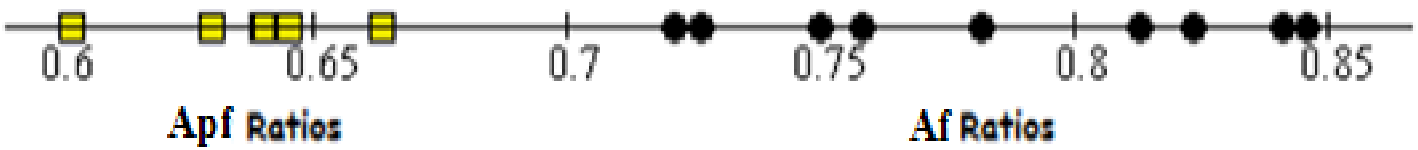


The Af's have slightly larger Antenna and slightly smaller Wings.

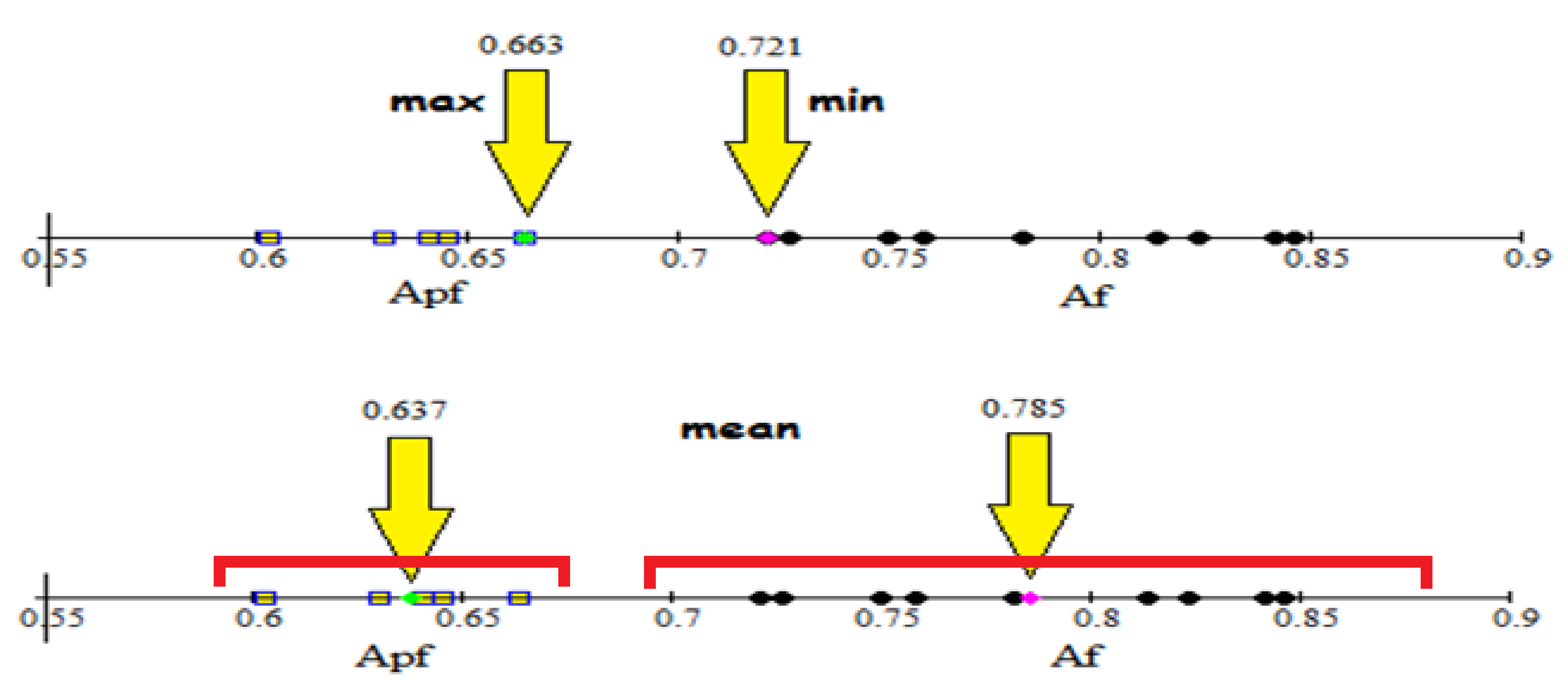


Would ratios help?

Af	.721	.841	.782	.824	.758	.813	.726	.846	.750
Apf	.640	.645	.663	.630	.640	.602			

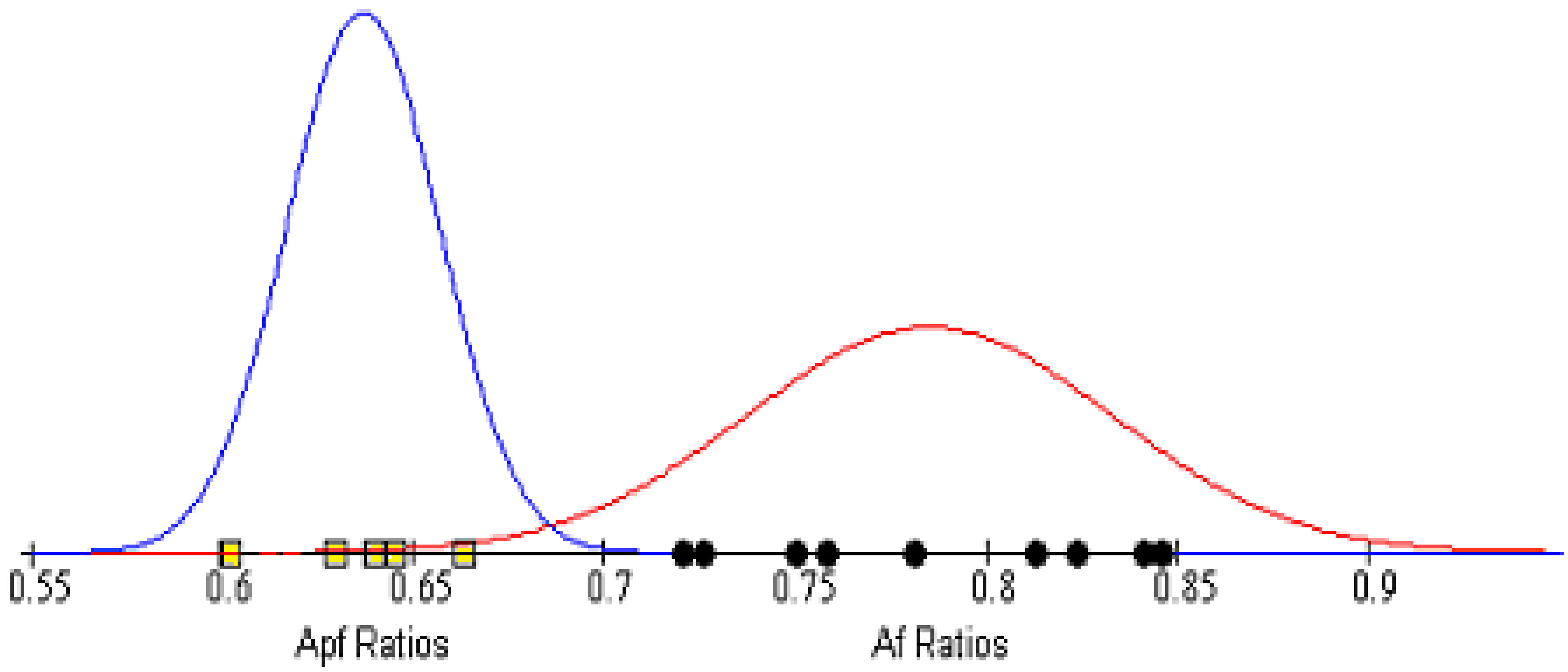


Number Line with Antenna to Wing Length Ratios



0.663
0.721
max
min
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
Apf
Af
0.637
mean
0.785
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
Apf
Af

- Use z-scores?

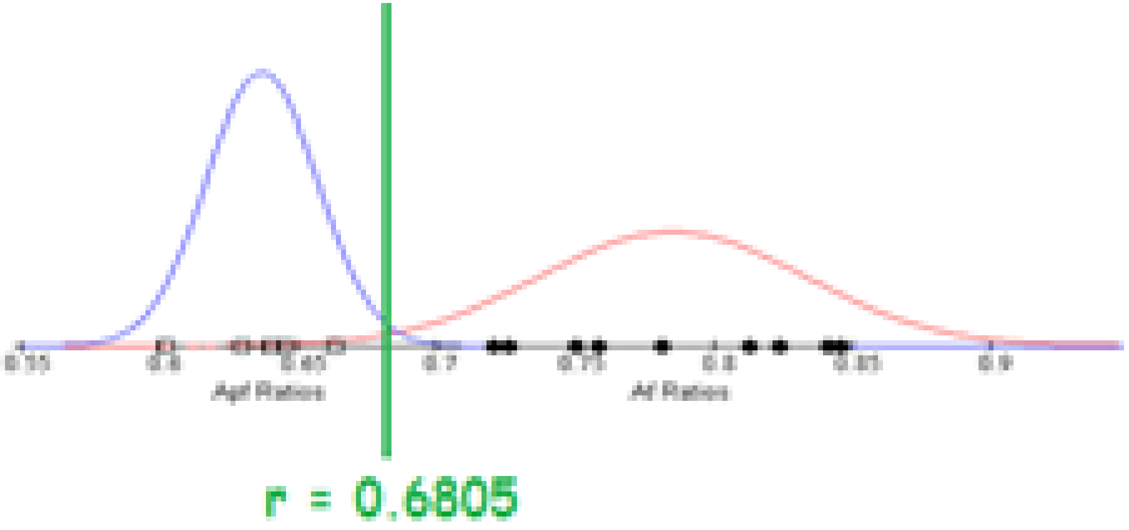


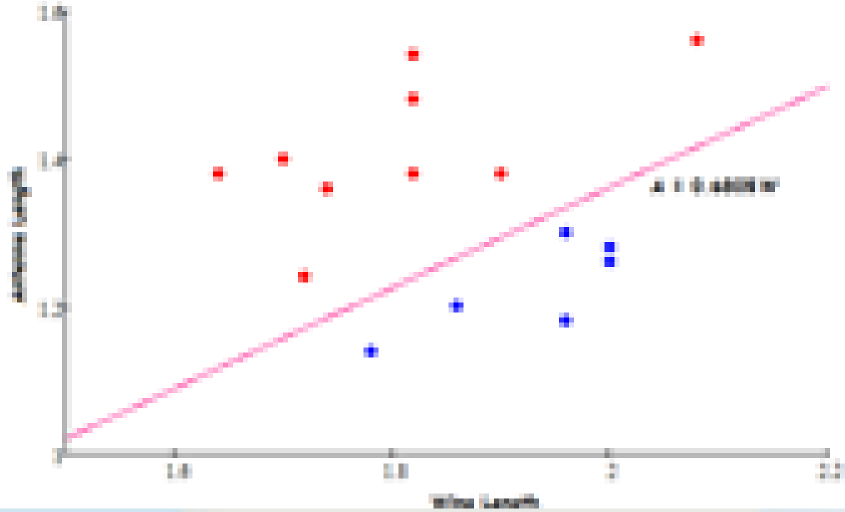
Approximate Normal Distributions for Ratios

Equalize z-scores

$$\frac{r^* - 0.637}{0.020} = \frac{0.785 - r^*}{0.048} \quad \text{we find that } r^* = 0.6805.$$

If $\dfrac{A}{W} = 0.6805$ is used as a boundary on the number line, then $A = 0.6805W$ is an equivalent boundary in the plane.





linear fit for the Aa fly is

$$A = 0.479W + 0.549$$

while for the Ax fly is

$$A = 0.558W + 0.151.$$

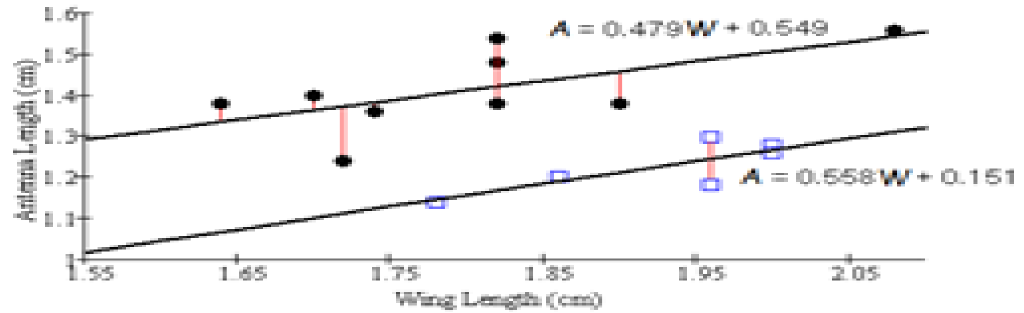


Figure 10: Least-Squares Line Fit to Each Data Set with Residuals

An equi-unusual boundary can be found by equating the lines 1, 1.5, 2, 2.5, and 3 standard deviation above the Ax line and below the Aa line. This boundary is linear, and has the equation $A = 0.537W + 0.284$. The figure below illustrates this equi-unusual boundary. Any fly below this equi-unusual boundary line is considered to be a dangerous Ax fly while those above the line will be classified as Aa. In this situation, observations on the boundary are considered to be the more dangerous Ax.

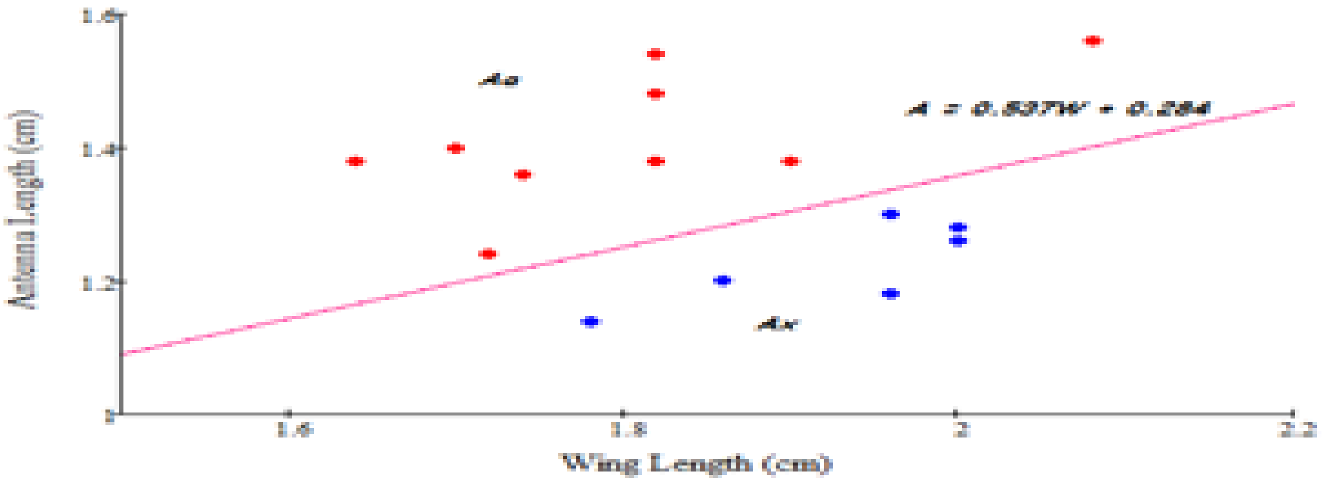


Figure 11: Equi-Unusual Boundary based on Residuals

With this boundary we can classify the three unknown flies:(1.80, 1.24) Ax, (1.84, 1.28) Aa, and (2.04, 1.40) Aa.

Δ–Correlation

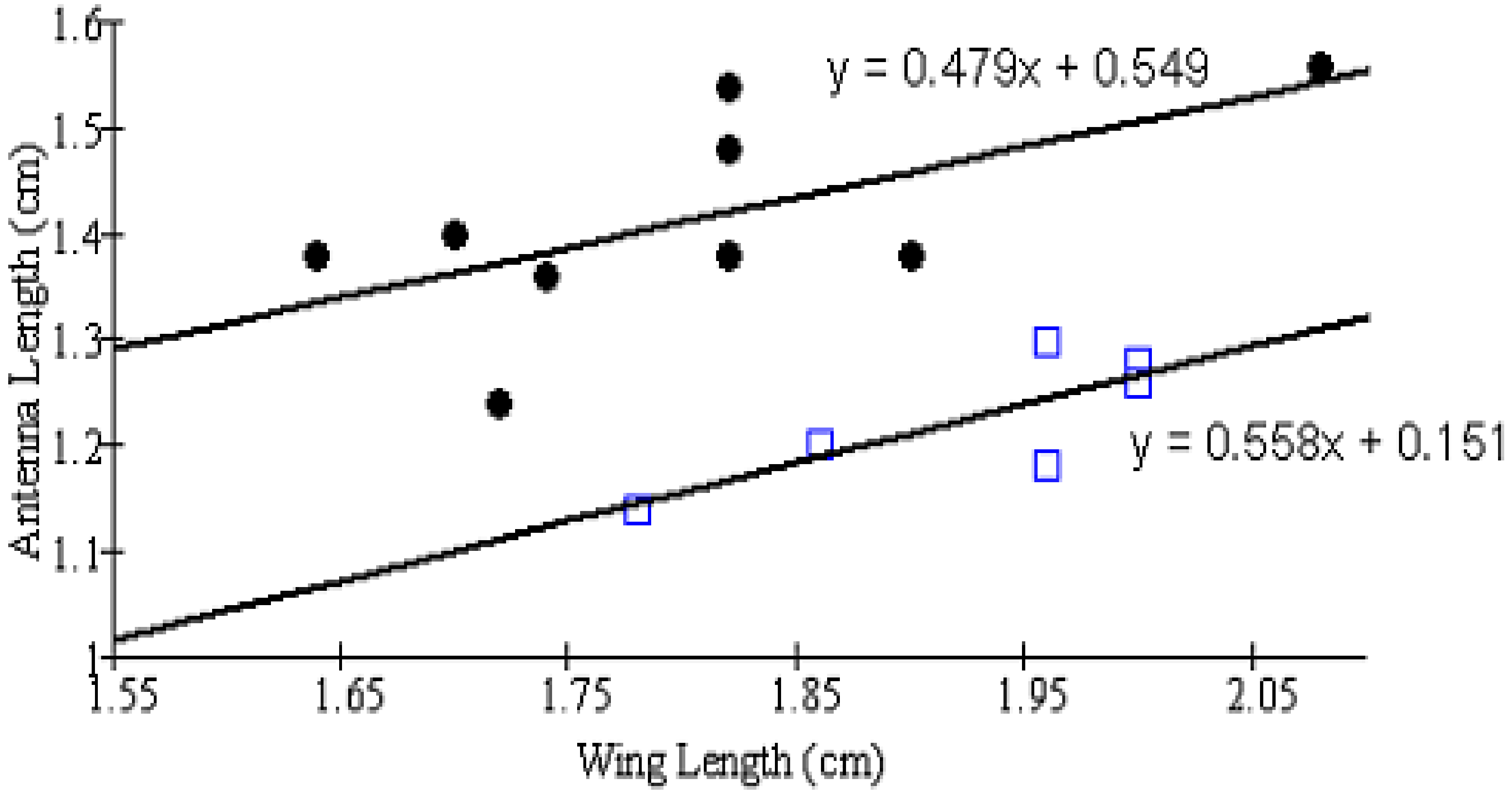


Figure 2: Linear Least-Squares Line Fit to Each Data Set

STATISTICS
SECTION II

Question 6

2001

Directions: Show all your work. Indicate clearly the methods you use, because you will be graded on the correctness of your methods as well as on the accuracy of your results and explanation.

6. The statistics department at a large university is trying to determine if it is possible to predict whether an applicant will successfully complete the Ph.D. program or will leave before completing the program. The department is considering whether GPA (grade point average) in undergraduate statistics and mathematics courses (a measure of performance) and mean number of credit hours per semester (a measure of workload) would be helpful measures. To gather data, a random sample of 20 entering students from the past 5 years is taken. The data are given below.

Successfully Completed Ph.D. Program

Student	A	B	C	D	E	F	G	H	I	J	K	L	M
GPA	3.8	3.5	4.0	3.9	2.9	3.5	3.5	4.0	3.9	3.0	3.4	3.7	3.6
Credit hours	12.7	13.1	12.5	13.0	15.0	14.7	14.5	12.0	13.1	15.3	14.6	12.5	14.0

Did Not Complete Ph.D. Program

Student	N	O	P	Q	R	S	T
GPA	3.6	2.9	3.1	3.5	3.9	3.6	3.3
Credit hours	11.1	14.5	14.0	10.9	11.5	12.1	12.0

The regression output at the top of the next page resulted from fitting a line to the data in each group. The residual plots (not shown) indicated no unusual patterns, and the assumptions necessary for inference were judged to be reasonable.

AP Statistics Teachers Have Some Experience

Helping students make the transition to a more student-centered classroom

Where the teacher isn't the only source of intellectual authority

Where it's safe to make mistakes – Better than safe…

Where students are asked to share their ideas with each other and the whole class

Where students experience productive struggle

Benefits for Students

Their ideas are valued.

They learn to become independent learners.

They feel a greater sense of accomplishment.

They discover the math through the exploration of their own ideas.

They get to use their own creativity in problem solving.

The Imaginary Garden: Mathematical Proof

Science has theories.

Mathematics has theorems.

They are not at all the same.

Hy Bass in *Mathematics for Teaching*

Deductive proof accounts for a fundamental contrast between mathematics and the scientific disciplines…

Mathematical knowledge tends much more to be cumulative. New mathematics builds on, but does not discard, what came before. … In science, by contrast, new observations or discoveries can invalidate previous models, which then lose their scientific currency.

What is Mathematical Reasoning?

The American Mathematical Society Resource Group, noted,

"The most important thing to emphasize about mathematical reasoning is that it exists — more, that it is the heart of the subject, that mathematics is a coherent subject, and that mathematical reasoning is what makes it so. … Mathematics should simply be taught as a subject where things make sense and where you can figure out why they are the way they are."

Proof Trajectories of Bass and Mac Lane

Mathematical work generally progresses through a trajectory

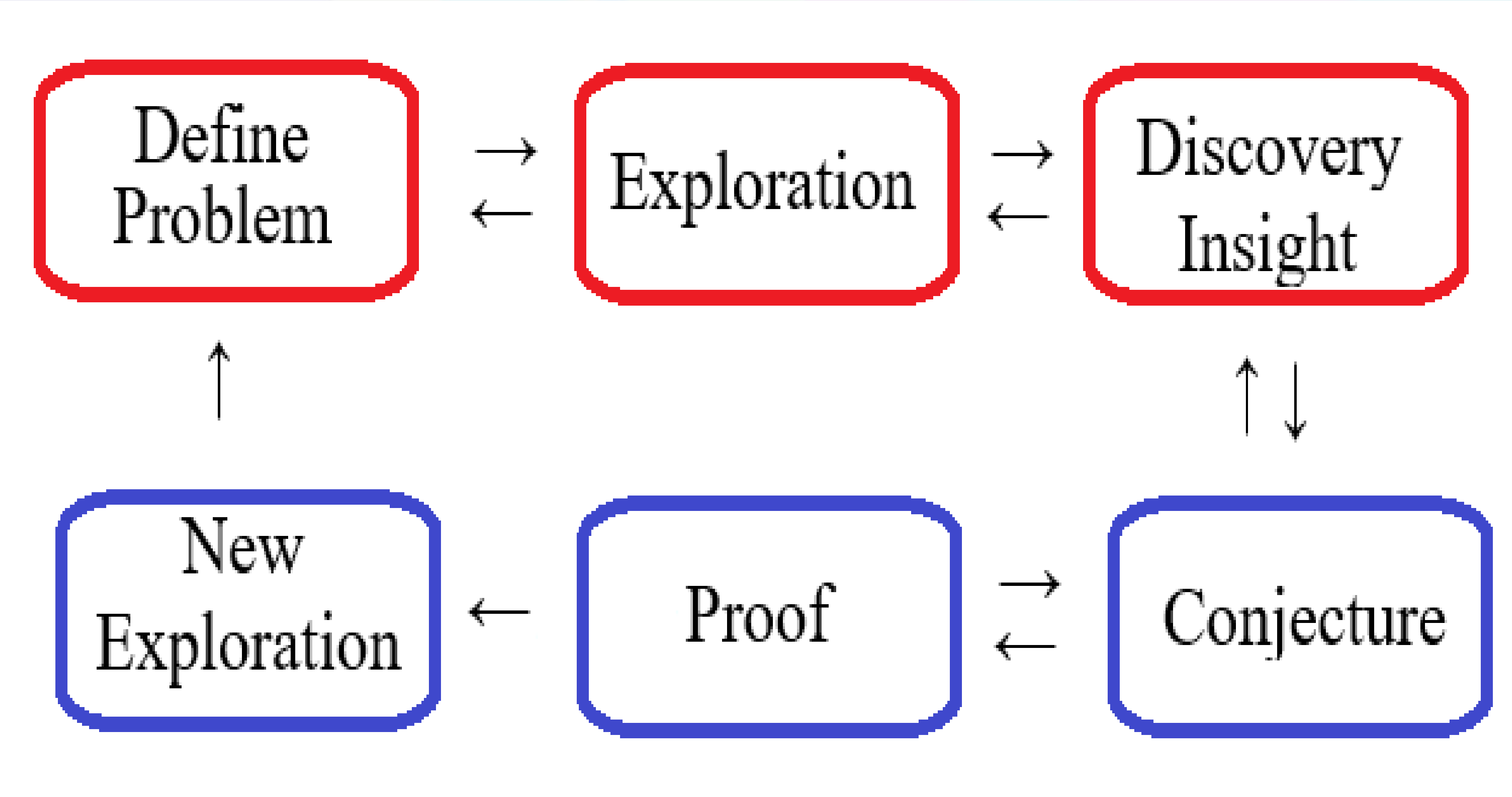
Exploration → Discovery → Conjecture → Proof → Certification

Mathematical understanding progresses from:

Intuition — Trial — Error — Speculation — Conjecture — Proof

In both, the first phases form the **essential reasoning of inquiry**, of **fiddling around** with the mathematical ideas to gain insight into the problem being studied, and of **speculating on possible truths** that have been discovered. And in both, the last components represent the **reasoning of validation and justification**.

The Proof Cycle



Modeling and Proof

"So often in mathematics, we say 'prove the following theorem' or 'solve the following problem'.

When we start at this point, we are ignoring the fact that finding the theorem or the right problem was a large part of the battle.

By emphasizing problem finding, mathematical modeling brings back to mathematics education this aspect of our subject, and greatly reinforces the unity of the total mathematical experience." Henry Pollak

Modeling and Proof

"So often in mathematics, we say 'prove the following theorem' or 'solve the following problem'.

When we start at this point, we are ignoring the fact that finding the theorem or the right problem was a large part of the battle.

By emphasizing problem finding, mathematical proof brings back to mathematics education this aspect of our subject, and greatly reinforces the unity of the total mathematical experience." Henry Pollak

Keys to Proof

Creative proof happens when students make important decisions about what problem to solve, how to proceed, and when to turn back.

Create the simplest form of the problem that contains the essence of the problem.

Use your basic solution and the iterative process to add more reality to your initial solution.

Pay close attention to errors. Try to understand why, how, and by how much they are wrong.

A Walk Through the Proof Cycle

What can you say about the family of lines

$$ax + by = c$$

if a, b, and c form an arithmetic progression?

Exploration → Discovery → Conjecture → Proof → New Problem

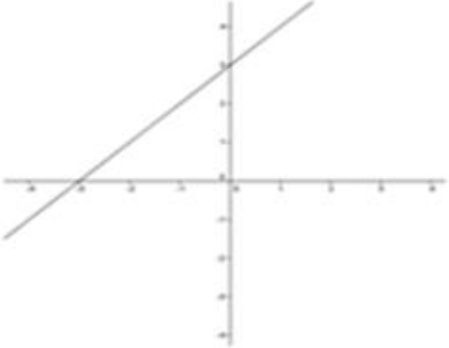
Exploration → Discovery → Conjecture → Insight → Proof → New Exploration

Exploration: Students begin by exploring the graphs of these equations.

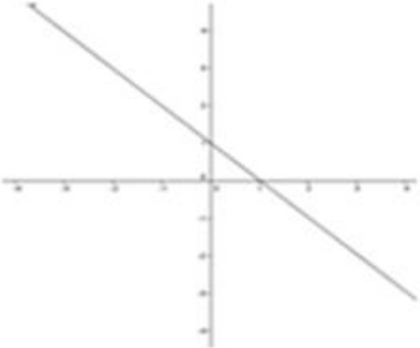
$x+2y=3$

$5x+3y=1$

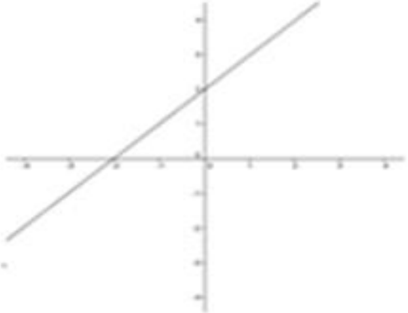
$0.5x+1y=1.5$



$x-2y=-5$



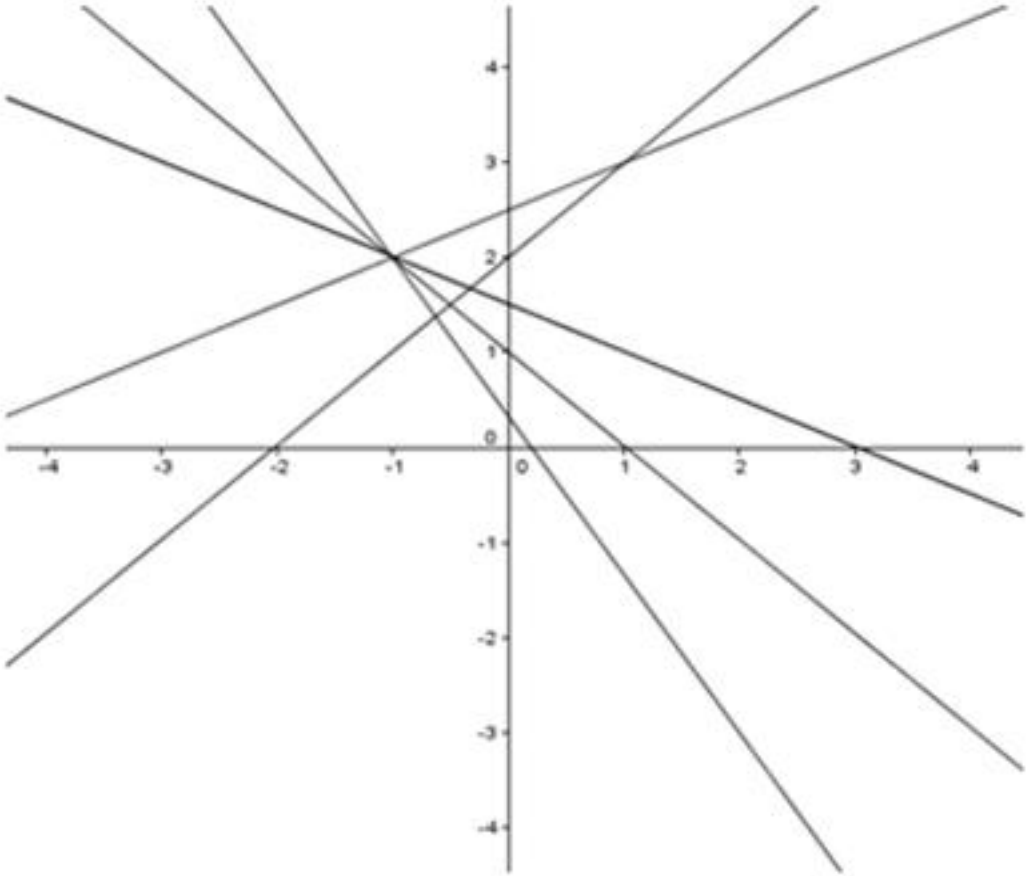
$100x+101y=102$



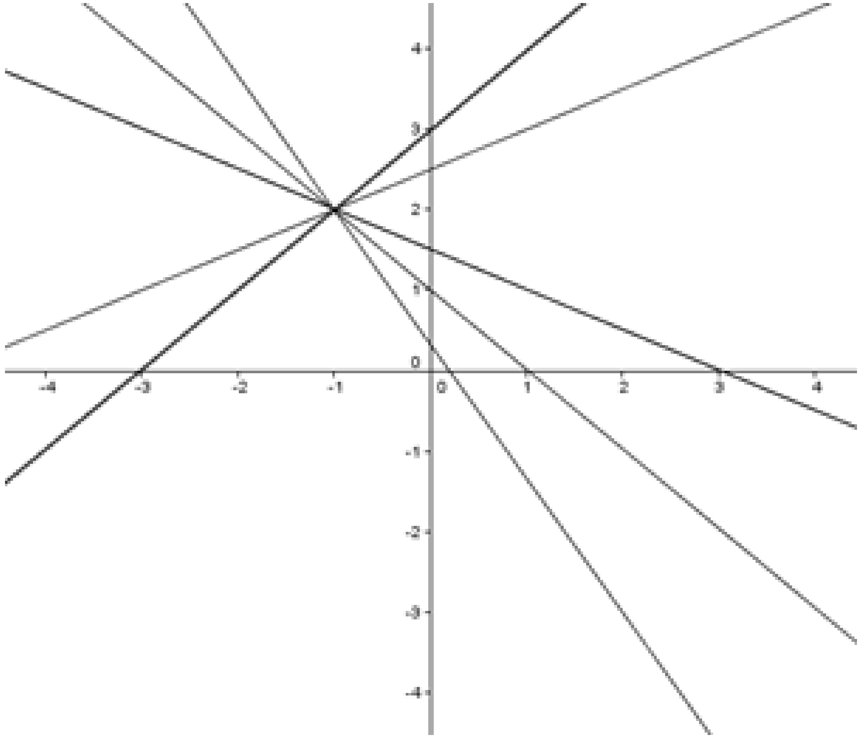
$100x-101y=-202$

Exploration → **Discovery** → Conjecture → Insight→ Proof → New Problem

Discovery: If students plot all their lines in one plane, patterns emerge.



Exploration → Discovery → **Conjecture** → Proof → New Problem



All lines have correct arithmetic progressions

Conjecture: Every line whose equation is in general form $ax + by = c$ where a, b, and c form an arithmetic progression contains the point $(-1, 2)$.

Exploration → Discovery → Conjecture → **Insight** → Proof → New Problem

Insight: One student notices that the integers 1, 0 , –1 are in arithmetic progression. This represents the equation $1x + 0y = -1$, so $x = -1$.

Another adds that 0, 1, 2 are also in arithmetic progression, so $0x + 1y = 2$ and $y = 2$ satisfies our requirement.

Is this a proof that all such line must pass through $(-1, 2)$? If not, what, if anything, does this prove?

Exploration → Discovery → Conjecture → **Insight** → Proof → New Problem

More Insight: For different values of a and k, they all have the form: $ax+(a+k)y=a+2k$. Suppose we write two such equations and solve the system?

Exploration → Discovery → Conjecture → Insight → **Proof** → New Problem

$$\begin{array}{l} ax+(a+k)y=a+2k \\ bx+(b+n)y=b+2n \end{array}, \quad \text{then} \quad \begin{array}{r} bax+b(a+k)y=b(a+2k) \\ -[abx+a(b+n)y=a(b+2n)] \end{array}$$

and

$$[b(a+k)-a(b+n)]y=b(a+2k)-a(b+2n),$$

so $[bk-an]y=2(bk-an)$ and $y=2$.

Consequently, $ax+(a+k)2=a+2k$.

Simplifying we find, $x+2=1$ and $x=-1$.

Exploration → Discovery → Conjecture → Insight → Proof → New Problem

Final Insight: If all the equations must have the form

$$ax + (a+k)y = a + 2k\,,$$

then the left side of the equation,

$$ax + (a+k)y\,,$$

must be the same as the right

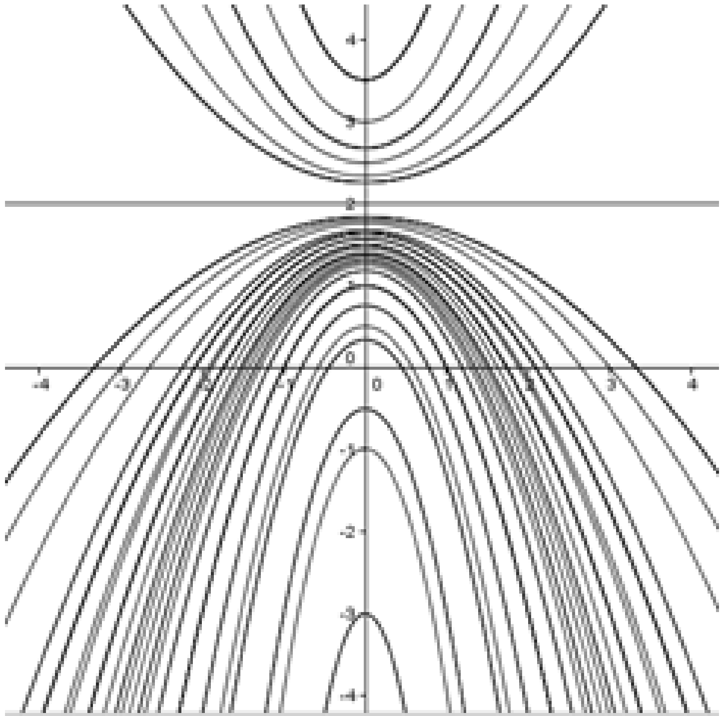
$$a + 2k\,.$$

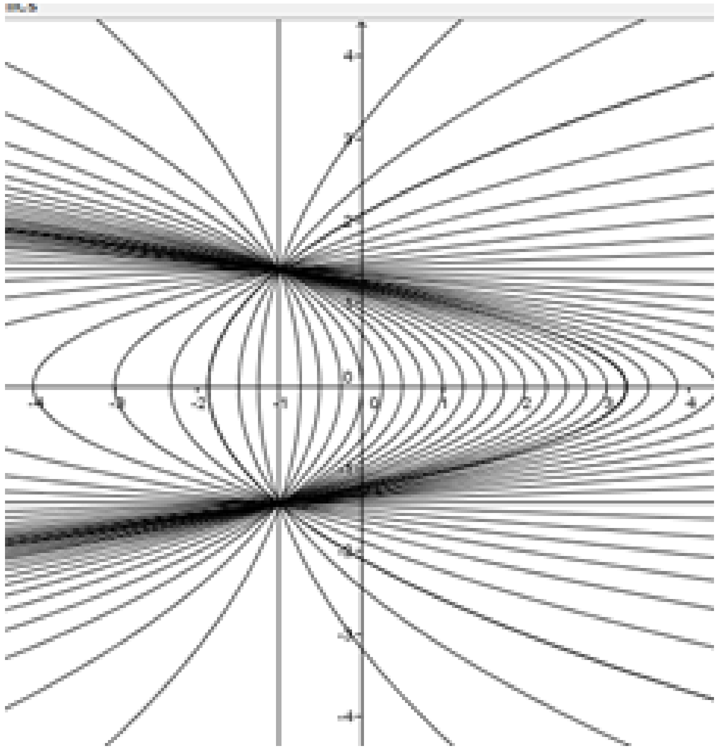
Exploration → Discovery → Conjecture → Insight → **Proof** → New Problem

$$ax + (a + k)y = a + 2k$$

Exploration → Discovery → Conjecture → Insight → Proof → **New Problem**

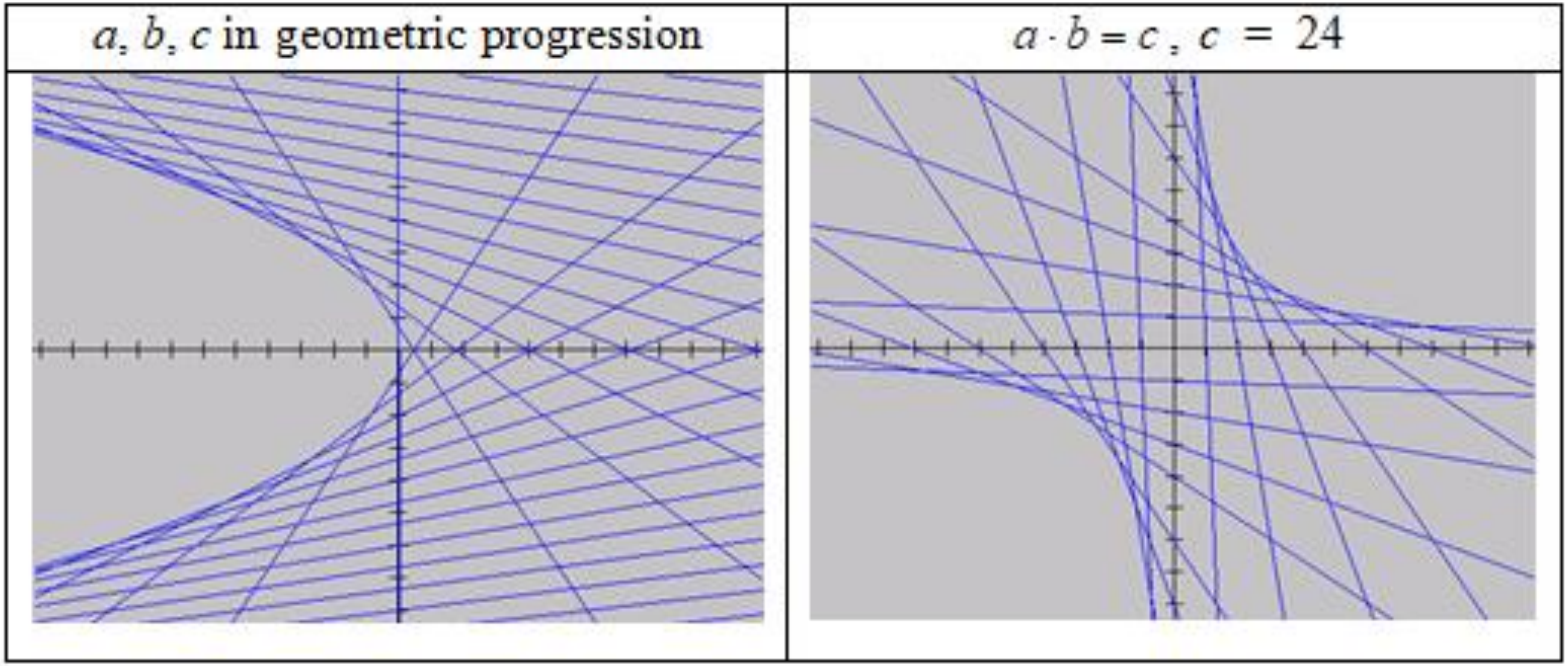
New Exploration: Now that we have this theorem about lines, can we create similar theorems about quadratics?





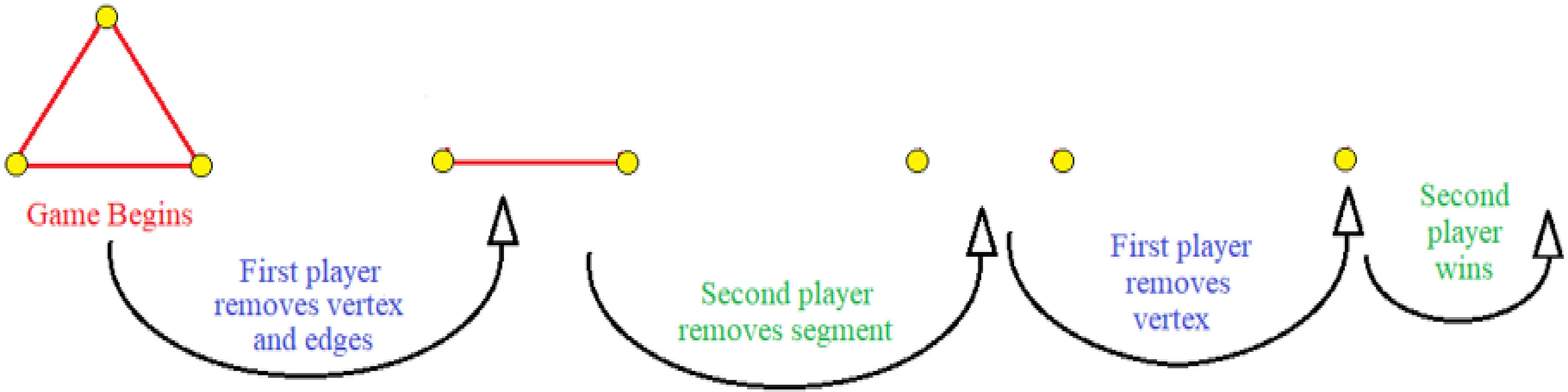
Which is the plot of $ax^2+(a+k)y=a+2k$ and which is $ax+(a+k)y^2=a+2k$?

Exploration → Discovery → Conjecture → Insight → Proof → **New Problem**



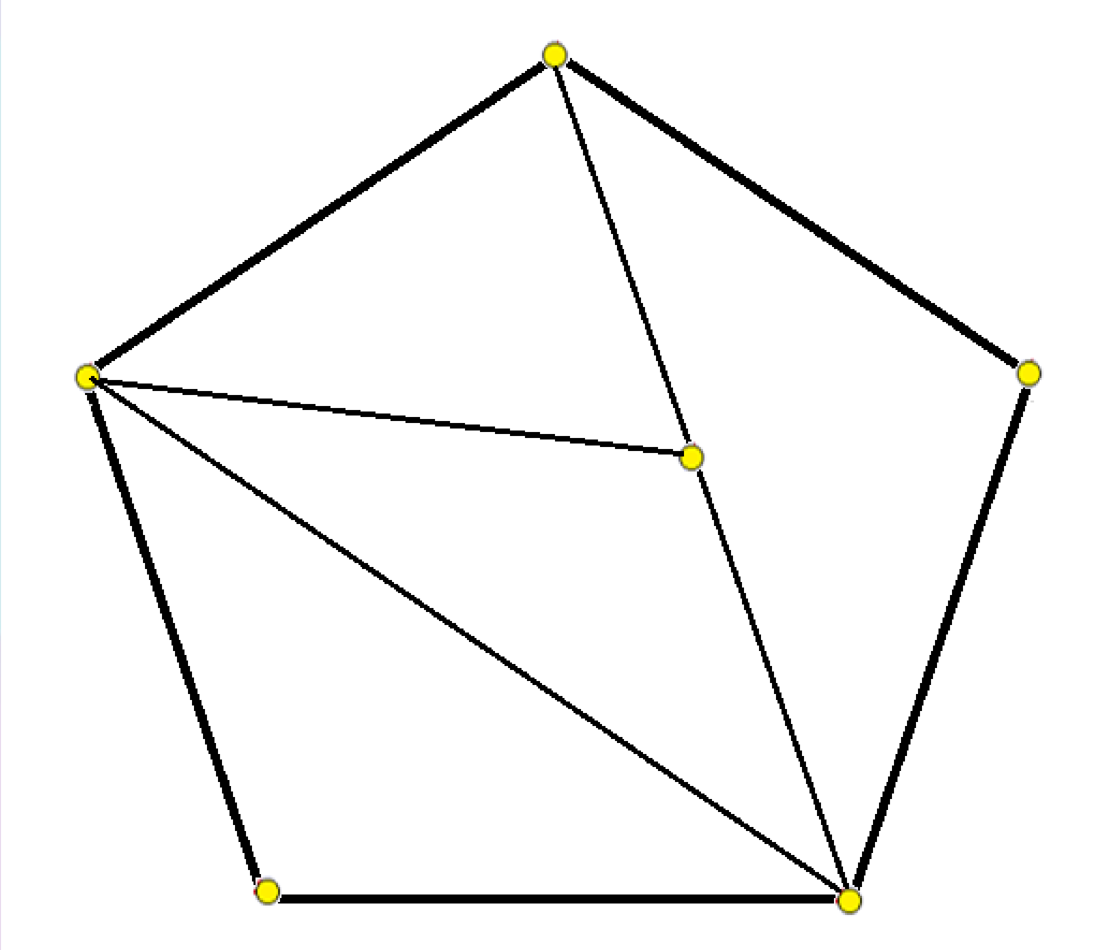
Games and Proof: Chomp the Graph

Given a graph *G*, players alternate turns. On each turn a player may remove a single edge or a single vertex with all of the edges connected to the vertex removed. The player to remove the last vertex wins. In the simple game below on a triangle, the second player wins by removing the last vertex. Could the first player have won using a different strategy?

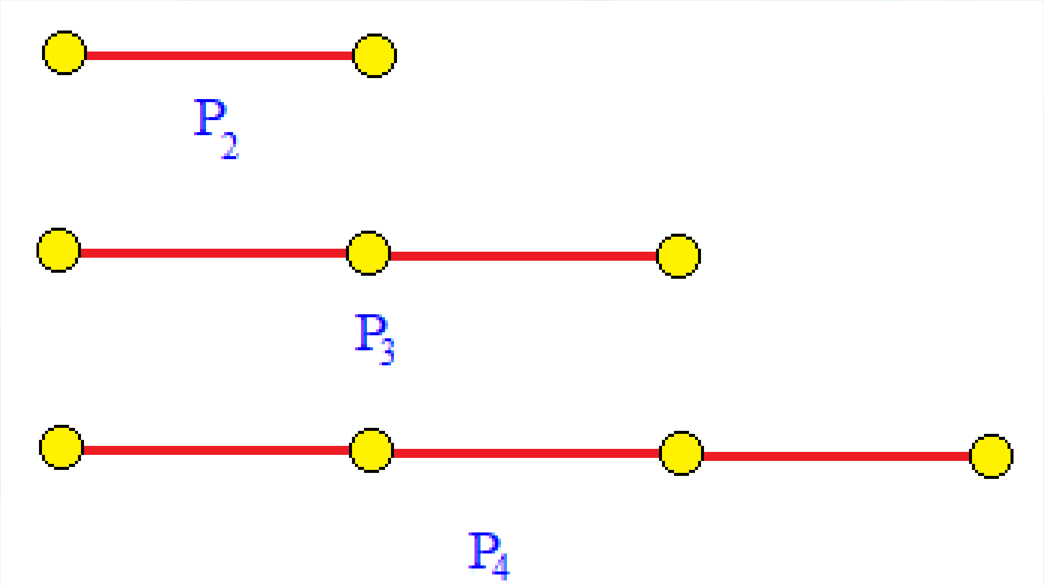


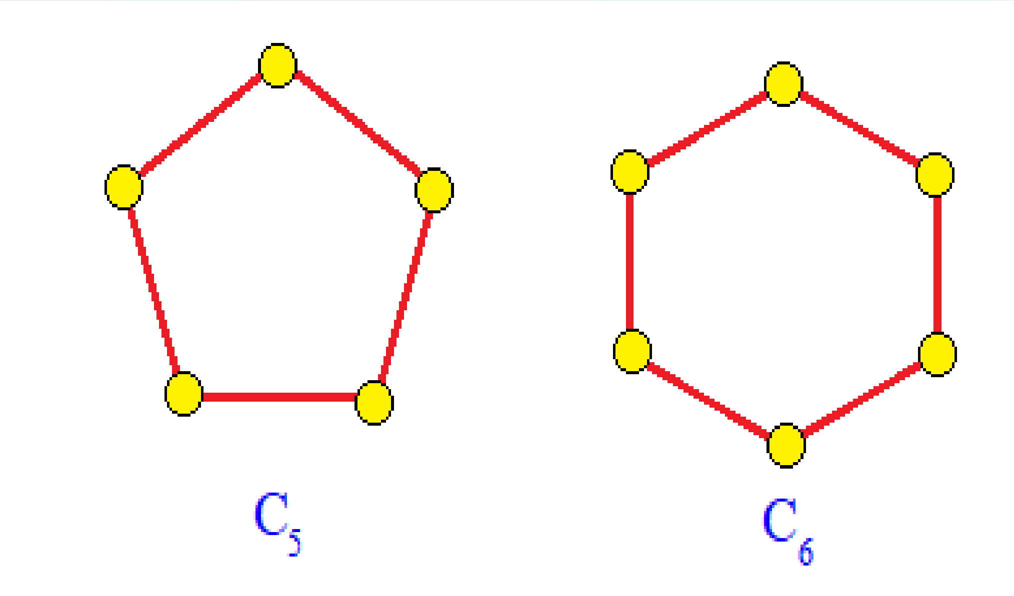
Develop a theory of playing Chomp for a variety of graphs.

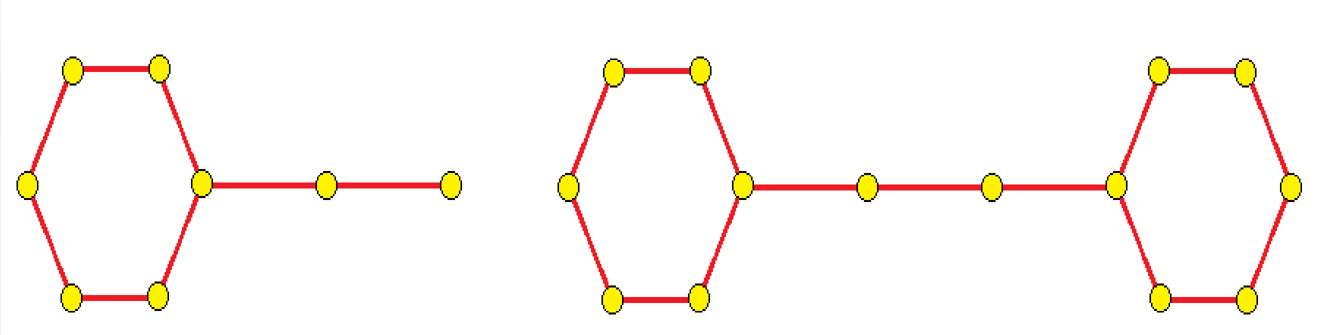
Who Wins this Game?



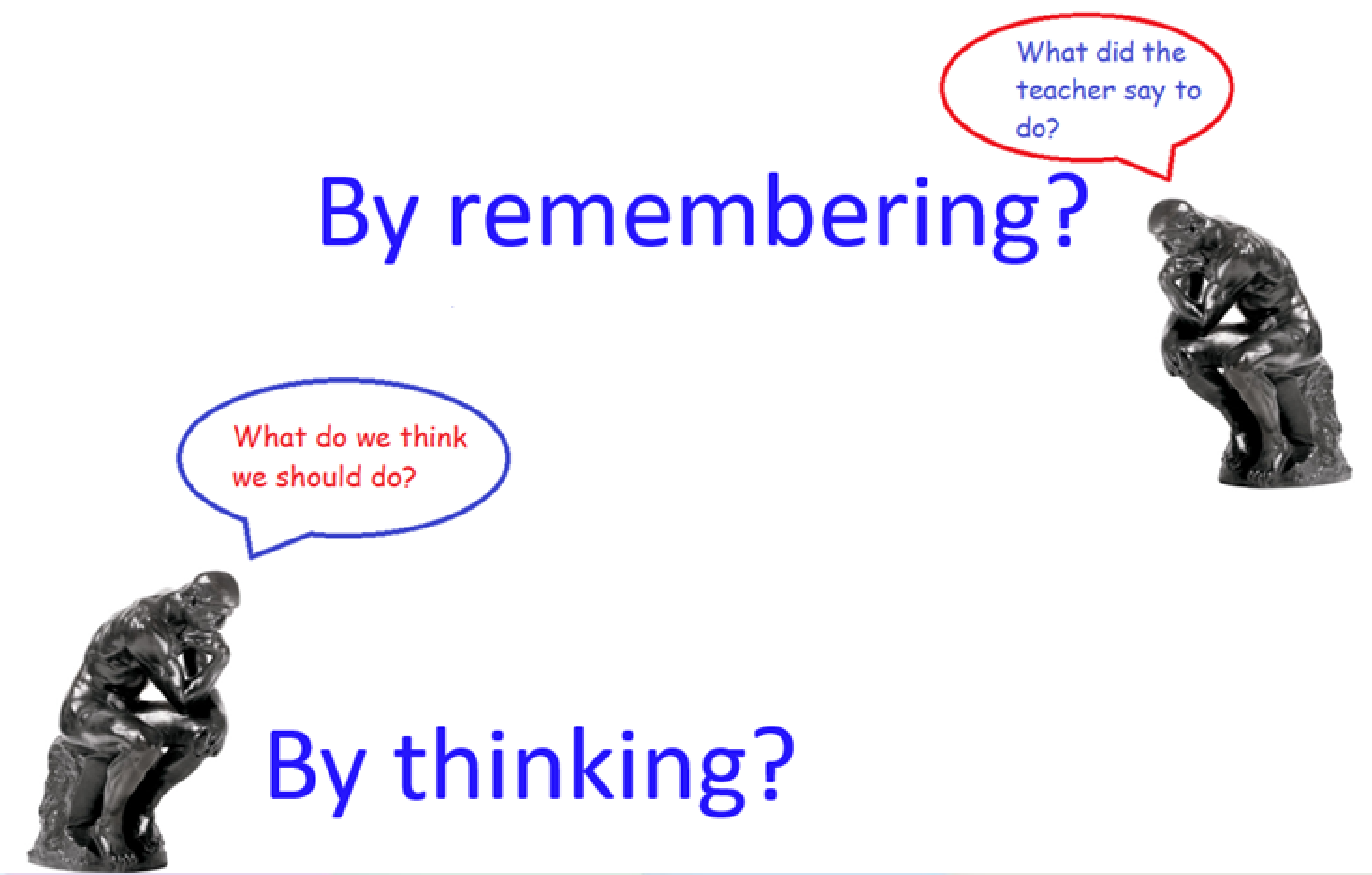
Begin with the simplest form of the problem that contains the essense of the problem







How do you do mathematics?



Modeling and Proof: Assessment

Value the process as well as the product

Group activities with student sharing ideas, skills, and talents

Value communication as well as mathematics

Modeling and Proof: Assessment

Value creativity and originality and authenticity

Value struggle and success and failure

Value excitement and frustration and overcoming and declaring victory and moving on

The Imaginary Garden and the Real Toads

Should be for our students, in some significant ways and for some significant time, the demanding, exciting, challenging, joyful, frustrating, play-box that it is for mathematicians.

Students can see mathematics in themselves and themselves in mathematics.

Modeling for Interest; Proof for Power

Dan Teague

NC School of Science and Mathematics

teague@ncssm.edu