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Paper Title: Ferris wheels and filling bottles: A case of a student's transfer of covariational reasoning

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Abstract

We share results of a design experiment study in which we used an actor-oriented perspective on transfer to investigate the research question: How might a student's covariational reasoning on Ferris wheel tasks, involving changing attributes measurable with one-dimensional units, influence a student's covariational reasoning on filling bottle tasks, involving changing attributes measurable with one- and three-dimensional units? We discuss implications for research and teaching.

Covariational reasoning, which entails forming and interpreting relationships between changing attributes, is useful for students' study of gatekeeping math concepts such as rate and function (e.g., Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Confrey & Smith, 1995; Ellis, Özgür, Kulow, Williams, & Amidon, 2015; Johnson, 2012, 2015a; Moore, 2014; Saldanha & Thompson, 1998; Thompson, 1994; Thompson & Carlson, in press). However, students may not use covariational reasoning when investigating situations for which covariational reasoning would be appropriate (Johnson, 2015b; Johnson & McClintock, under review). Furthermore, students' impoverished conceptions of the attributes that are changing may impact their covariational reasoning (Moore, 2014; Moore & Carlson, 2012).

Researchers have used dynamic computer environments to provide secondary students opportunities to form and interpret relationships between changing quantities (e.g., Ellis et al., 2015; Johnson, 2015b, 2015c; Kaput & Roschelle, 1999; Kaput & Schorr, 2008; Saldanha & Thompson, 1998; Thompson, 1994), and results identify difficulties that students face. Students' difficulties encompass their images of change and their conceptions of which attributes are changing. Despite interacting with dynamic computer environments representing multiple, dynamically changing attributes, secondary students can have difficulty conceiving of related attributes as changing continually (e.g., Ellis et al., 2015; Johnson, 2015c; Saldanha & Thompson, 1998). Furthermore, secondary students may conceive of only one changing attribute, rather than multiple changing attributes (e.g., Johnson, 2015b).

When students engage in covariational reasoning, it is useful for them to envision variation not only in the direction of change, but also to envision variation in change occurring in a single direction, or variation in unidirectional change (e.g., Carlson et al., 2002; Ellis et al., 2015; Johnson, 2012; Johnson & McClintock, under review; Oehrtman, Carlson, & Thompson, 2008; Saldanha & Thompson, 1998; Stalvey & Vidakovic, 2015). However, even successful university students have difficulty using covariational reasoning to make sense of situations involving variation in change that occurs in a single direction, such as a temperature or volume increasing at a decreasing rate (e.g., Carlson et al., 2002; Oehrtman et al., 2008; Stalvey & Vidakovic, 2015). Furthermore, less is known about how situations involving attributes measurable with different kinds of units (e.g., volume and height, distance and height) might afford or constrain students' covariational reasoning.

An actor-oriented perspective on transfer (an AOT perspective) is useful for studying students' transfer of complex cognitive processes, such as covariational reasoning (Lobato, 2003, 2012; Thompson, 2011). Yet, little is known about how students' covariational reasoning might transfer across situations involving attributes measurable with different kinds of units. In this design experiment study (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) we used an AOT perspective (Lobato, 2003) to investigate the research question: How might a student's covariational reasoning on Ferris wheel tasks, involving changing attributes measurable with one-dimensional units, influence a student's covariational reasoning on Filling bottle tasks, involving changing attributes measurable with one- and three-dimensional units?

Conceptual/Theoretical Perspective

Conceptions of Attributes and Images of Change

We ground our conceptual framework in a cognitive perspective on reasoning, which emphasizes students' conceptualization (e.g., Piaget, 1970, 1985). From this perspective, students' conceptions of the attributes that are changing and students' images of change are central to their covariational reasoning. Drawing on Thompson's research (e.g., Thompson, 1993, 1994, 2011), we make distinctions between ways in which students might conceive of attributes of objects. Like Thompson, we use the term *quantity* to refer to an attribute that a student conceives of being possible to measure. For example, we would say that a student conceives of the height of liquid in a bottle as a quantity if the student demonstrates that she conceives of possibility of measuring the height. In contrast, a student might use the term "height" to refer to the level of water in a bottle, as related to the shape of the bottle, without conceiving of the possibility of measuring that height.

When we use the term image, we refer to the "dynamics of mental operations," (Thompson, 1994, p. 231), which Thompson roots in Piaget's theory. By mental operation, we mean something that an individual can enact in thought (e.g., Piaget, 1985). For example, an individual could conceive of measuring the height of liquid in a bottle without actually engaging in an observable action of measuring that height. By images of change, we mean students' mental operations related to their envisioning of change (Castillo-Garsow, Johnson, & Moore, 2013). Castillo-Garsow et al. (2013) posit two contrasting images of change: chunky and smooth. The images of change do not refer to how change might actually be occurring in the physical world, rather they refer to students' conceptions of change. A smooth image of change involves a conception of change as progressing. A chunky image of change entails a conception of change as having occurred in particular increments. Castillo-Garsow et al. (2013) use the examples of conceiving of a bottle being filled continually, as if from a dispenser, and a bottle being filled incrementally, as if pouring in liquid in cups, to distinguish between smooth and chunky images of change.

Actor-Oriented Transfer (AOT)

Lobato (2003) defined an AOT perspective as "the personal construction of relations of similarity across activities, (i.e., seeing situations as the same)" (p. 20). Researchers using an AOT perspective focus on how individuals make meaning of situations, not assuming that individuals will notice structural similarities between situations that researchers might construe. Furthermore, individuals can still engage in transfer without necessarily noticing structural similarities identified by researchers. Therefore, from an AOT perspective, transfer occurs when an individual construes that she can treat a different situation as an instance of something about which she has already thought (Lobato, 2012).

An AOT perspective is particularly useful for researchers investigating transfer involving individuals' conceptions, rather than an individual's transfer of a procedure or skill (Lobato, 2012). Thompson (2011) addressed the utility of the AOT perspective for investigating students' quantitative reasoning, providing an example from his earlier research findings. From an AOT perspective, students in Thompson's (1994) study transferred the way in which they conceived of distance, time and speed from one situation to another. In this research, we use an AOT perspective to demonstrate how students can transfer covariational reasoning across situations involving attributes measurable with different kinds of units.

Method

Research Methodology: Design Experiment and Case Study

Design experiment studies consist of an iterative process, in which researchers test and refine conjecture, for the purpose of building "humble" theories, closely tied to the data (Cobb et al., 2003). In this design experiment study, we conjectured that situations involving attributes that were measurable with one-dimensional units (e.g., distance and height) might be useful for promoting students' covariational reasoning. We intended to test if a student's covariational reasoning about situations involving attributes measurable with one-dimensional units might transfer to situations involving an attribute measurable with three-dimensional units.

Yin (2006) identifies three steps in designing case studies: defining the case, justifying selection of single or multiple case studies, and explicitly articulating how theoretical perspectives inform (or deliberately do not inform) the case. We report a single case study (Stake, 2005) of a student's (Ana's) transfer of covariational reasoning from a Ferris wheel task to a filling bottle task. We identify this single case as an instrumental case (Stake, 2005), through which we intend to advance what is known about secondary students' quantitative and covariational reasoning. We deliberately used our theoretical perspectives on reasoning and transfer to inform our design and analysis of this case study.

Setting and Participants

The case study we report is part of a larger study (Johnson, 2015b) investigating five ninth grade students' quantitative and covariational reasoning. All five ninth grade students were in the same Algebra I course, taught by the same teacher. Prior to conducting the study, Johnson visited the students' Algebra I classes on a number of occasions so that Johnson could build rapport with the students. We conducted this research in a 6-12 neighborhood school in an industrial region just north of a quickly gentrifying area of a large city. During the time of the study, at the school, 98% of students qualified for free and reduced lunch, and 99% of the students identified as nonwhite.

Interview Schedule

Ana participated in a series of six task-based interviews (Goldin, 2000), for which Johnson served as the interviewer. Johnson conducted one interview per week, beginning in late April, after the students had finished their district mandated testing, and concluding in late May, just prior to the end of the school year. In the first three interviews, Johnson interviewed Ana with other students. In the last three interviews, Johnson interviewed Ana individually. Johnson reorganized the interviews in the midst of the design experiment for two reasons. First, Ana was absent due to illness during the day of the fourth interview in the series. However, the two other students were available for the interview, so Johnson decided to interview the other students, so as to not create an undue delay in the project, because the school year was rapidly drawing to a close. Second, in earlier interviews, Johnson had noticed that the other students would at times defer to Ana, or say that they thought in ways that were the same as or similar to Ana's thinking. Johnson conjectured that by interviewing Ana separately from the other two students, not only would she not need to delay the research, she might learn more about all three students' thinking.

During the interviews, Ana had opportunities to form and interpret relationships between attributes represented in the Ferris wheel and Filling bottle tasks (distance, width, height, and volume). The Ferris wheel tasks involved only attributes that were measurable with one-dimensional units (distance, height, and width). The filling bottle tasks included an attribute that was measurable with three-dimensional units (volume). The interview series consisted of a PreInterview, four Design Experiment Interviews, and a Post Interview. We included filling bottle tasks in the PreInterview and PostInterview. We included Ferris wheel tasks during the Design Experiment Interviews and the PostInterview. As part of each interview, Johnson asked Ana to sketch a Cartesian graph representing a relationship between different changing attributes, which were not always represented on the same axes. Table 1 lists the task situations, attributes, and the axes on which the attributes were represented.

Interview	Task Situation(s)	Attributes	Representation on Cartesian Graph
PreInterview	Filling bottle	Height, volume	Height: horizontal axis Volume: vertical axis
Design Experiment 1	Ferris wheel	Height, distance	Distance: horizontal axis Height: vertical axis
Design Experiment 2	Ferris wheel	Height, distance	Distance: horizontal axis Height: vertical axis
Design Experiment 3	Ferris wheel	Width, distance	Distance: horizontal axis Width: vertical axis
Design Experiment 4	Ferris wheel	Width, distance Height, distance	Height/Width: horizontal axis Distance: vertical axis
Post Interview	Ferris wheel Filling bottle	Height, distance Height, volume	Height: horizontal axis opening left Distance: vertical axis Height: horizontal axis Volume: vertical axis

Table 1. Task situations and attributes, by interview

Tasks

During the PreInterview and PostInterview, Ana viewed video animations of two filling bottles, available at desmos.com. After viewing the video animations, Johnson asked Ana to sketch a graph relating the height and volume of water in the filling bottle. The first bottle looked like an Erlenmeyer flask (Fig. 1, left). The second bottle had a wide, spherical base and a narrow, cylindrical neck (Fig. 1, right).

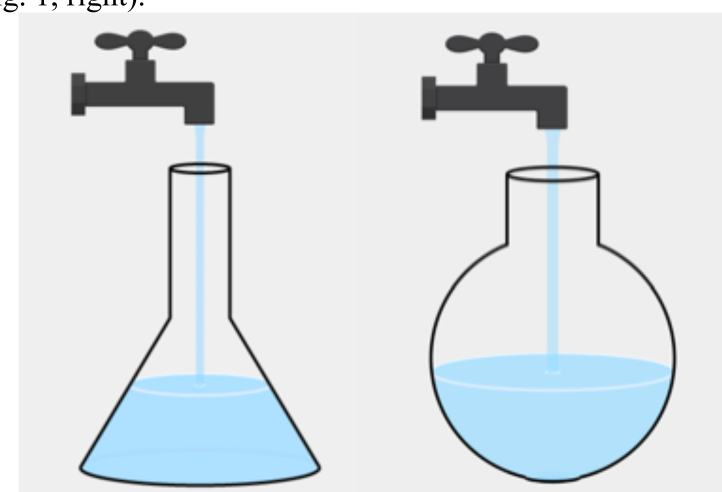

Fig. 1. Filling Bottle animations (https://teacher.desmos.com/waterline/walkthrough#Erlenmeyer; https://teacher.desmos.com/waterline/walkthrough#Evaporation)

During the Design Experiment Interviews, Ana interacted with a dynamic computer environment that linked an animation of a turning Ferris wheel with a dynamic graph relating different attributes, including *distance* traveled around one revolution of the Ferris wheel (circumference), *height* from the ground (vertical distance), and *width* from the center (horizontal distance) shown in Figs. 2 and 3, respectively. To depict a Ferris wheel, the dynamic computer environment used a circle containing an active point, representing a car on the Ferris wheel.

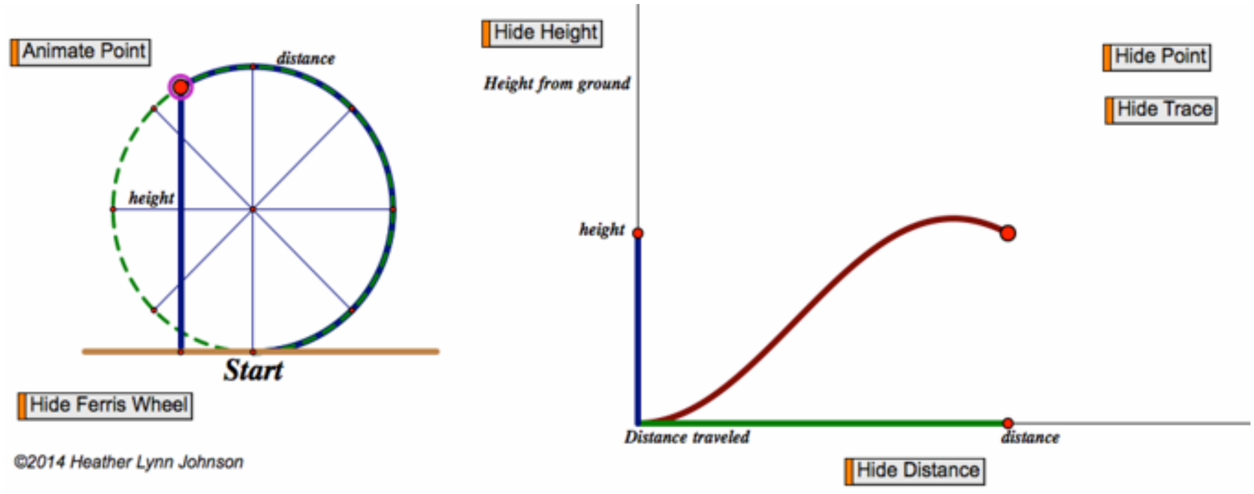

Fig. 2. Dynamic Ferris wheel computer environment, distance and height

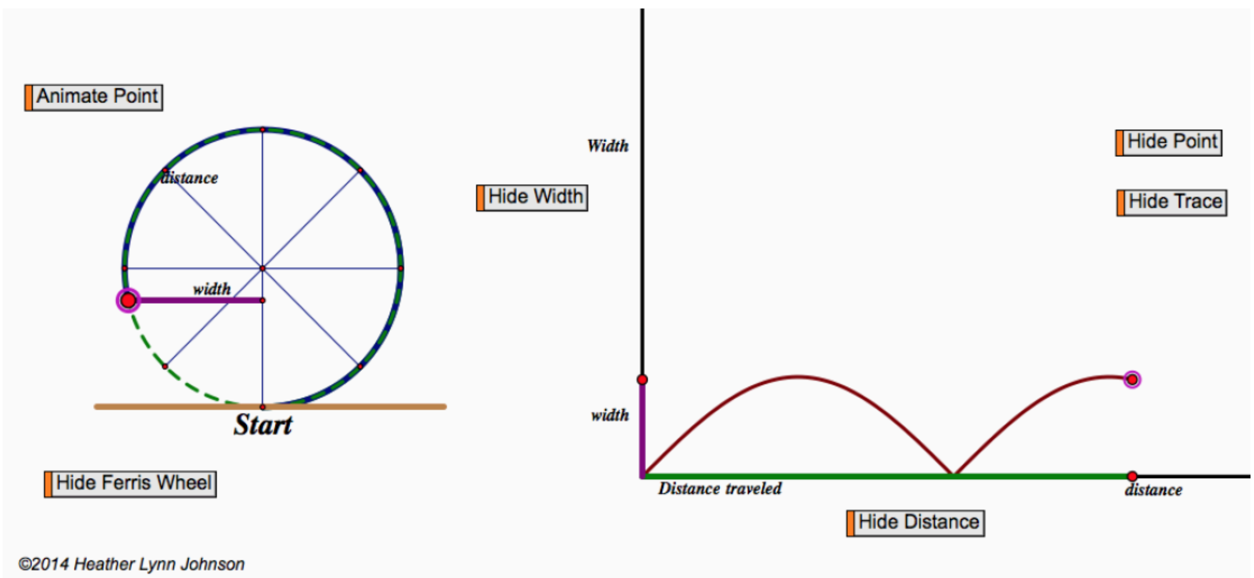

Fig. 3. Dynamic Ferris wheel computer environment, width and height

Key design elements of the Ferris wheel environment included graphs with dynamic segments on the vertical and horizontal axes to represent change in individual quantities and quantities measurable with one-dimensional units (Johnson, 2015b). Students could click and drag the dynamic segments to predict and represent changing quantities. In addition, students could observe how the dynamic segments changed as they clicked and dragged the active point on the Ferris wheel animation. Important, the dynamic segments provided students opportunities to predict, interpret and represent change in individual, then multiple quantities.

The dynamic graphs in Figs. 2 and 3 represent the distance on the horizontal axis and the height or width on the vertical axis. During the design experiment interviews, Ana also interacted with dynamic graphs representing distance on the horizontal axis and the height or width on the vertical axis. Prior to interacting with the dynamic graphs, Johnson asked Ana to sketch graphs representing the attributes of distance and height or distance and width. After sketching her initial graphs, Ana interacted with the dynamic animation and graph. In addition, she had the opportunity to revise her initial graphs if she chose to do so.

During the PostInterview, Johnson presented Ana with a pair of coordinate axes relating height and distance that opened to the left, rather than to the right, as is typical (Fig. 4). In making this design decision, Johnson drew on Moore's "breaking of conventions" to investigate preservice secondary mathematics teachers' quantitative and covariational reasoning (e.g., Moore, Paoletti, & Musgrave, 2013; Moore, Silverman, Paoletti, & LaForest, 2014). In addition, Johnson did not create a dynamic Ferris wheel environment relating a dynamic animation and graph for this situation. By not providing students a dynamic graph with which they could test and refine conjectures, Johnson intended to investigate the extent to which students could engage in covariational reasoning about attributes that they needed to envision, rather than view, as changing.

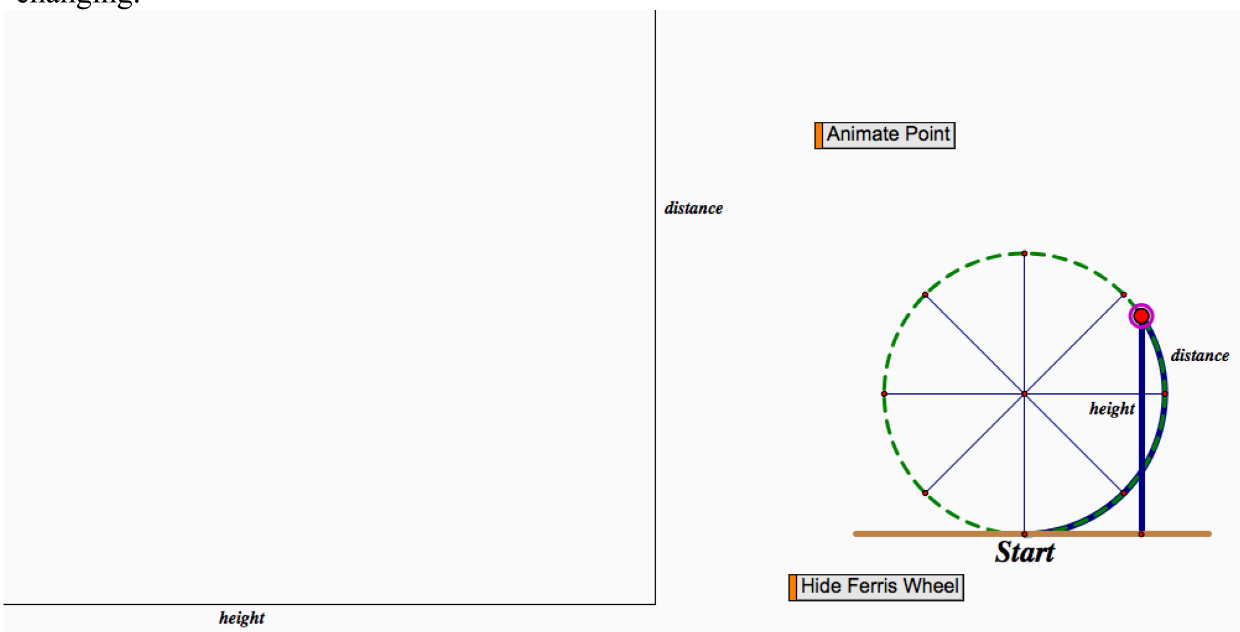

Fig. 4. Ferris Wheel coordinate plane for PostInterview

Data Analysis

Lobato (2008) identified four elements to address to make claims regarding students' transfer: Change in students' reasoning from PreInterviews to PostInterviews; Students' demonstration of limited reasoning during PreInterview tasks; Viable relationships between students' work during PostInterview and design experiment; and Attribution of students' different reasoning to the design experiment rather than to spontaneous activity. Our data analysis methods worked to address each of these elements.

In the first pass of data analysis, we addressed students' reasoning. We employed a modified version of open coding (Corbin & Strauss, 2008), to identify and code portions of data across four dimensions, shown in Table 2. We modified the process of open coding by not entering into coding in a manner that was totally open. Within each dimension, we coded for two or three concepts. Despite beginning with codes in mind, our analysis incorporated a process of qualifying what each code meant in terms of Ana's conception, image, or envisioning. We view this process of qualifying what we mean by codes to be consistent with the qualifying that Corbin and Strauss (2008) identify as part of the process of open coding. For example, when we coded Ana's conception of an attribute as being "measurable," we described the way in which she conceived of measuring that attribute, along with evidence supporting such a conception. In the case of "distance traveled around the Ferris wheel," we described Ana's conception of distance as a length that she could measure with a linear unit, such as meters or feet. In our analysis, we did not go into the coding process assuming a singular meaning for each of the codes. Furthermore, we did not assume that the codes with which we began necessarily would be sufficient to address the totality of the dimensions. For example, initially we began only with the codes "chunky" and "smooth" for images of change, and we added the code "smooth chunks," which Johnson (2012) reported. We used the code "smooth chunks" to indicate when a student conceived of change as occurring in distinct sections, but envisioned change as progressing within a section.

Dimension	**Codes**
Conceptions of attributes	Measurable Not measurable
Images of change	Chunky Smooth "Smooth chunks"
Envisioning change in attribute(s)	Variation (one changing attribute) Covariation (attributes changing together)
Envisioning variation in change	Variation in the direction of change (e.g., increase vs. decrease) Variation in unidirectional change (e.g., variation in increases)

Table 2. Dimensions and codes

In our analysis, we nested the "Conceptions and Images" dimensions within the "Envisioning" dimensions. This is to say that when we coded for variation or covariation, we necessarily included codes for students' conceptions of attributes and images of change. Furthermore, we nested the variation/covariation codes within the dimension entailing the envisioning of variation in change. For example, when we coded that Ana's envisioning variation in change as "variation in unidirectional change," we necessarily coded whether Ana was envisioning variation in one or more changing attributes (variation/covariation), her corresponding images of change for each changing attribute (chunky/smooth/smooth chunks), and her conceptions of each changing attribute (measurable/not measurable).

In subsequent passes of analysis, we used comparative analysis, examining portions of data when Ana appeared to be engaging in compatible forms of reasoning, then looking across the entirety of tasks to trace changes in Ana's reasoning. We began by using the dimension, envisioning variation in change, to compare Ana's reasoning on the Filling Bottle tasks in the PreInterview and PostInterview. We selected this dimension because it encompassed all others. Then we compared Ana's reasoning on the PostInterview task to her reasoning across the set of Ferris wheel tasks, again looking across each of the four dimensions. We worked from the descriptions qualifying what we meant by each code to identify relationships between Ana's work during the Design Experiment Interviews and the PostInterview. For example, we coded that Ana envisioned variation in unidirectional change when working on both the Ferris wheel and filling bottle tasks. We then looked at Ana's conceptions of attributes, images of change, and envisioning variation across those instances in which she envisioned variation in unidirectional change. Building from the relationships we identified, we worked to attribute Ana's changes in covariational reasoning to the design experiment.

Results

We report data from Ana's work on the PreInterview and PostInterview. First we present Ana's work on the PreInterview filling bottle task, to demonstrate her initial reasoning. Second, we present Ana's work on the PostInterview Ferris wheel task, to demonstrate the extent of her covariational reasoning on the Ferris wheel tasks. Third, we present Ana's work on the PostInterview filling bottle task, to demonstrate her change from her initial reasoning.

PreInterview: Filling Bottle Task

After viewing the video of the Filling bottle shaped like an Erlenmeyer flask (Fig. 5, left), Johnson asked Ana to sketch a graph relating the height and volume of the liquid in the filling bottle. Johnson provided Ana with a large pair of coordinate axes, with height labeled on the horizontal axis and volume labeled on the vertical axis. Prior to sketching the graph, Johnson asked Ana to state what each axis represented, and she correctly identified the horizontal axis as representing the height in inches and the vertical axis as representing the volume in ounces. When graphing, Ana spontaneously made connections to distance-time graphs, that she had drawn earlier in the year as part of her work for her Algebra I class. Rather than using the axes provided by Johnson, Ana flipped the paper over to the back, which was blank. She then drew her own axes and sketched two graphs (Fig. 5, right). The excerpt below begins when Ana flips over her paper and concludes when she had sketched the two graphs.

Ana: Because when, *[Flips paper over, sketches a small pair of coordinate axes, Fig. 5 upper right]* like we did it. Because he could be running fast, so it could be like very steep *[Beginning at origin, sketches a line segment very close to the vertical axis, Fig. 5 upper right]*. And he could be like slowing down, so it could go like this *[Beginning at the endpoint of the first line segment, sketches a much less steep line segment, Fig. 5 upper right.]*. And it goes with the shape, because *[Sketches a new pair of coordinate axes, Fig. 5 lower right]* when the glass is like in the bottom, it's filling up really slowly. *[Beginning at origin, sketches a line segment very close to the horizontal axis, Fig. 5 lower right.]* But then it's getting higher, *[Extends the original line segment, Fig. 5 lower right.]* and then it's like- voom *[Beginning at the endpoint of the first line segment, sketches a much steeper line segment.]* because of, this is like the same. *[Points to neck of bottle, Fig. 5 left.]*

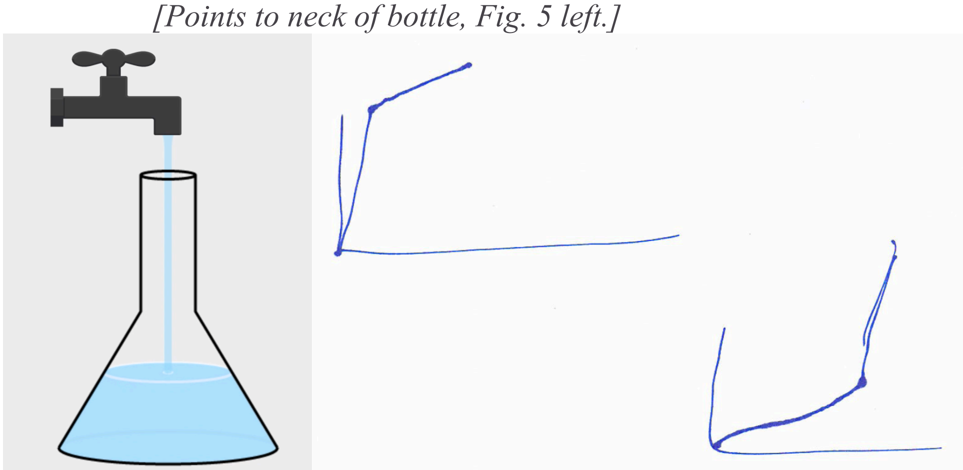

Fig. 5 Ana's graphs for the PreInterview filling bottle task

Ana's second graph demonstrated that she envisioned variation in unidirectional change, because she made distinctions between slower and faster increases (e.g., "in the bottom, it's filling up really slowly," "and then it's like- voom"), which she represented using two line segments having different steepness. Along the other dimensions for which we analyzed, in this episode we coded Ana's envisioning change in attribute(s) as variation, her images of change as "smooth chunks", and her conception of the changing attribute as not measurable. We coded for variation rather than covariation, because Ana envisioned only one attribute as varying, the "it" that "goes with the shape." We interpret the "it" to be the level of the changing liquid, which extended across the bottle. Although we could label the horizontal axis as passing time, we argue that Ana was not envisioning variation in individual attributes, then forming and interpreting relationships between those attributes, which forms the essence of covariational reasoning. Rather, Ana was envisioning an attribute changing along with passing time, and representing variation in the unidirectional change of that attribute. We coded Ana's images of change as "smooth chunks," because she conceived of two distinct sections of change, while envisioning change as progressing within each of those sections. We coded Ana's conception of the changing attribute as not measurable, because she used the shape of the bottle to define the "it" that she was representing, rather than separating the "it" from the bottle, then conceiving of the possibility of measuring that "it."

Design Experiment Interviews & PostInterview: Ferris Wheel

A key aspect of Ana's work with the Ferris wheel involved isolating attributes that were measurable with one-dimensional units, envisioning each of those attributes as varying individually, then envisioning those attributes as changing together. When working with the Ferris wheel tasks, on three occasions, across three different interviews, Ana spontaneously mentioned how she thought the Ferris wheel was like the filling bottle. On the first two occasions, which occurred during the third and fourth design experiment interviews, she drew relationships between the curved shape of the Ferris wheel and a curved shape of a filling water bottle. She used the curved shape to account for variation in unidirectional change that she noticed to be occurring in the dynamic animations.

On the final occasion, which occurred during the first part of the PostInterview, before she encountered the second filling bottle task, Ana attempted to make an analogy between an attribute of the filling bottle and an attribute of the Ferris wheel. Prior to attempting to make this analogy, Ana had drawn the graph relating distance and height shown in Fig. 6. In this episode, we coded Ana as envisioning variation in the direction of change, her envisioning change in attribute(s) as covariation, her images of change as smooth, and her conceptions of attributes as measurable. We coded for variation in direction of change rather than variation in unidirectional change, because Ana focused on whether the attributes of height and distance were increasing or decreasing, not on variation within those increasing or decreasing sections. We coded for covariation because Ana envisioned two attributes as varying individually, then represented a relationship between those varying attributes. She drew the marks to the right and below the vertical and horizontal axes, respectively, when representing the change in the individual attributes of distance and height. We coded Ana's images of change as smooth, because she envisioned change in height and distance as progressing during one revolution of the Ferris wheel. We coded Ana's conception of the attributes of distance and height as measurable, because she demonstrated that she could separate each attribute from the shape of the Ferris wheel, and she also showed she could use a one-dimensional unit to measure each of those attributes.

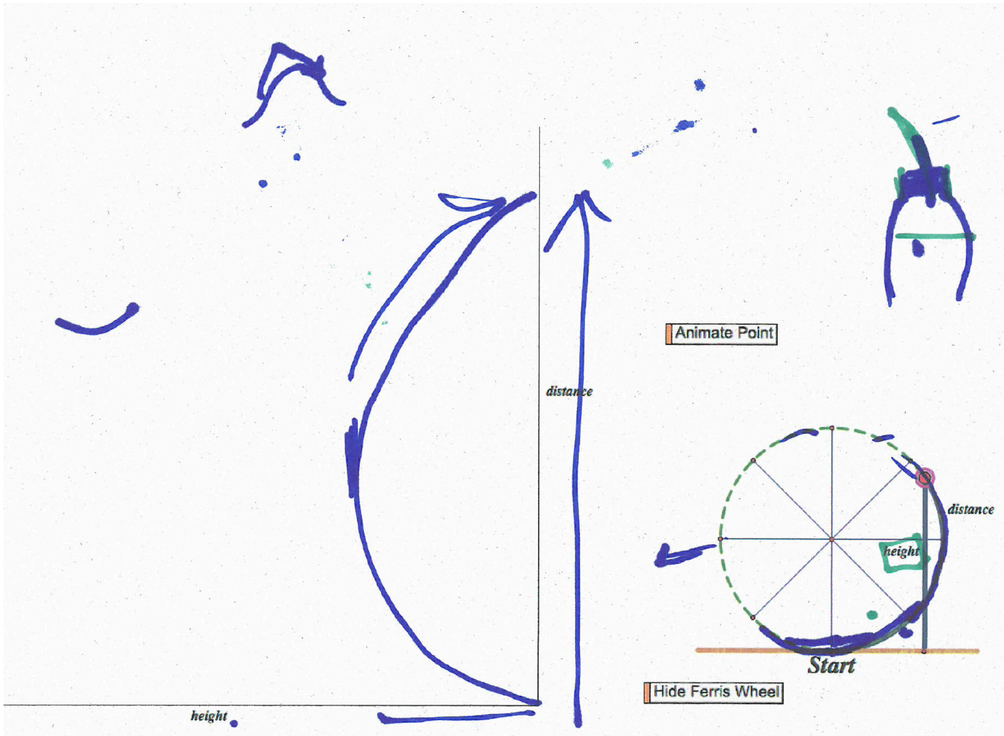

Fig. 6 Ana's graph for the PostInterview Ferris wheel task

After Ana sketched the graph shown in Fig. 6, Johnson showed Ana a dynamic graph from a task that occurred during the fourth Design Experiment interview, in which Ana represented height on the horizontal axis and distance on the vertical axis, for a pair of coordinate axes opening to the right. Johnson asked Ana how she thought about the graphs "as similar or different in terms of their shape," then followed up by asking Ana how she thought about "these parts," pointing to the concave up portions. Ana sketched a small concave up curve to the left of the graph she drew in Fig. 6, and Johnson followed up by asking Ana how she made sense of that. Initially Ana appealed to the shape of the Ferris wheel, again mentioning that it could be likened to the shape of the water bottles, stating that one could see the dot representing the car of the Ferris wheel "slowly rising."

Johnson decided to follow up to investigate if Ana might be able to separate attributes of height and distance from her perceived shape of the Ferris wheel, to envision variation in unidirectional change. Johnson asked Ana if the speed at which she might drag the point representing the Ferris wheel mattered: "would it still be slowly rising if I made if go really quick, if I just pulled it *[the dot representing the car, Fig. 6]* really fast?" The excerpt begins with Ana's attempt to relate distance and height to respond to Johnson's question.

Ana: Like you could be in the- perhaps you could say you're here, or well here *[Makes a dot on Ferris wheel to represent the car, Fig. 7 right]*, but that, it kind of seems like the same level above ground *[Darkens the horizontal segment under the Ferris wheel, Fig. 7, right]*. So the cart has already begun, like the little, like the spin, and you're here – you have already moved, but you're level, you're distance, or well height, it's still like the same as you started.

Johnson: So if you wanted to show that on the graph, where the distance went around, but the height was almost the same as you started, how would you show that?

Ana: Like this–you started to go up. *[Sketches leftmost portion of graph, contained within the dotted oval, Fig. 7 left.]*

Johnson: Why like that?

Ana: Because distance is going, but the height is still like zero. *[Darkens horizontal, then vertical axis, marking on the axis, with marks contained within the dotted oval, Fig. 7 left.]*

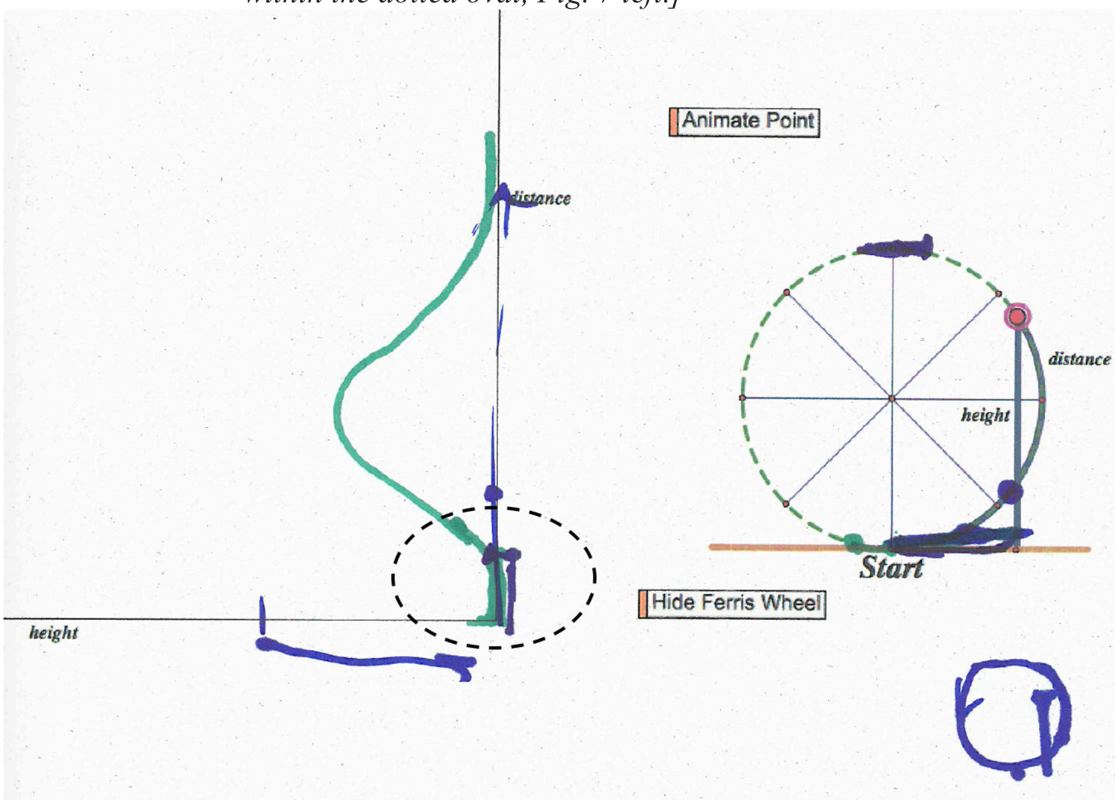

Fig. 7 Ana's revised graph for the PostInterview Ferris wheel task

Fig. 7 shows the completed graph that Ana drew. After this excerpt, Ana had only made the markings shown in green. She added the blue markings later, to represent changes in the distance and height for one revolution of the Ferris wheel. In this episode, we coded Ana as envisioning variation in unidirectional change, her envisioning change in attribute(s) as covariation, her images of change as smooth, and her conceptions of attributes as measurable. We coded her envisioning variation in unidirectional change, because she attempted to represent a relationship between the attributes of distance and height, such that distance changed by a large amount, but the height did not change by very much ("you have already moved, but you're level, you're distance, or well height, it's still like the same as you started.") Important, this was the first time that Ana had represented variation in unidirectional change within an interval, rather than identifying different intervals in which she noticed different kinds of unidirectional change (e.g., faster and slower increases), then comparing those intervals. After Ana drew the entire graph shown in Fig. 7, she attempted to draw an analogy between the water pouring into the bottle and an attribute of the Ferris wheel.

Ana:	It could also be like the water bottle again, the water keeps pouring, but then, I'm not sure what the water would represent in here. *[Points to the Ferris wheel]*
Johnson:	Oh like the water pouring-
Ana:	Like at the same rate.
Johnson:	If you had to guess?
Ana:	It would be like the height. The height or the speed of how fast?
Johnson:	What do you think? Well what are the things that are changing in the water bottle?
Ana:	The amount of water getting filled up, and the shape. I think the water would represent speed.
Johnson:	And what does speed mean for this Ferris wheel?
Ana:	Like how fast the ride is going.

Ana's response demonstrates that she conceived of additional attributes of the Ferris wheel and the filling bottle, the water that "keeps pouring" and the "speed" of the Ferris wheel, which she had not directly represented on one of the coordinate axes. Furthermore, she found the speed of the Ferris wheel to be analogous to the water pouring into the filling bottle. Generally, Ana conceived that different situations can have analogous measurable attributes, and she was able to identify analogous attributes in the Ferris wheel and filling bottle situations. The next excerpt continues with Johnson following up by asking Ana if changing how "fast the ride is going" would change the graph that she had drawn for the Ferris wheel.

Johnson:	If I change the speed of the ride, if I change how fast the ride is going, would it change the graph?
Ana:	Mmm, I don't think so, because we're not graphing the amount of time that it takes.
Johnson:	What are we graphing?
Ana:	The height and the distance, and it doesn't change, because if they had a change, the, the, how big the Ferris wheel would change as well.
Johnson:	Ah, so can you give me an example?
Ana:	So like the distance and the height would only change if it was a smaller Ferris wheel, because if it was a smaller Ferris wheel, the distance would be smaller as well, and the height would be like smaller as well because it wouldn't have to go as tall or long. *[Draws Ferris wheel shape, Fig. 7.]*

By claiming that the Ferris wheel itself would need to change size to result in a change in her graph (e.g., "if it was a smaller Ferris wheel), Ana demonstrated that she can separate the attribute of the speed of the Ferris wheel from the attributes of distance and height, which she represented in her graphs. Notably, only after envisioning variation in unidirectional change in height and distance, did she separate the speed at which the Ferris wheel was turning from the attributes of changing height and distance.

It was not until the PostInterview task that Ana envisioned variation in unidirectional change while engaging in covariational reasoning about attributes that she conceived of as measurable, and drawing on smooth images of change. Prior to this task, when Ana envisioned variation in unidirectional change, she was engaging reasoning about attributes that she conceived of as physical rather than measurable, and her images of change were "smooth chunks." For example, in the filling bottle task in the PreInterview, Ana determined sections that had different kinds of unidirectional change (e.g., slower or faster increases), then envisioned change as occurring within each of those sections. Furthermore, the attribute she envisioned as varying was the "shape" of the bottle, rather than an attribute that she could separate from the bottle itself.

PostInterview: Filling Bottle

After viewing the video of the filling bottle with a seemingly spherical base (Fig. 7), Johnson asked Ana to sketch a graph relating the height and volume of the liquid in the filling bottle. Prior to sketching the graph, Johnson asked Ana what kinds of units she might use to measure the height and volume of the water in the filling bottle. Ana selected centimeters to measure the height, which she described as "how tall it is," and milliliters to measure the volume, which she described as "the amount of water that's inside." In the excerpt the follows, Ana explained her thinking while sketching a graph.

Johnson:	What I was wondering, if you had to sketch me a graph that would relate the volume of water in the bottle and height of the water that's in the bottle-
Ana:	I think it'd be like a linear graph, linear
Johnson:	Show me
Ana:	Because the height, if we were talking about like individuals, the height would still increase *[Sketches segment on height axis, extending from origin, Fig. 8]*, unless you drank it, you'd have to go bathroom *[Smiles]*, and the volume would still increase *[Sketches segment on volume axis, extending from origin, Fig. 8]* because it's like getting filled up, so you want more, so that would be this *[Sketches seemingly linear, monotonically increasing graph, Fig. 8]*
Johnson:	And can you tell me what this graph means in terms of the bottle?
Ana:	The amount of water that's in it-

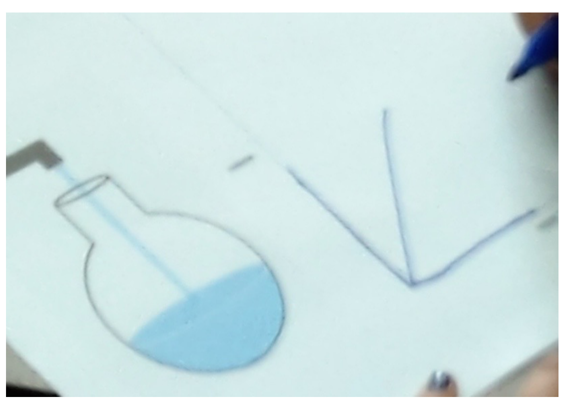

Fig. 8 Ana's graph for the new filling bottle

In this episode, we coded Ana's reasoning as covariation, her images of change as smooth, and her conceptions of attributes as measurable. Because the height and volume were both increasing in the filling bottle situation, Ana did not have an opportunity to envision variation in the direction of change. Sketching a linear graph with a seemingly constant rate of change, Ana did not demonstrate evidence of envisioning variation in unidirectional change. We coded for covariation because Ana envisioned both height and volume as varying individually, then represented a relationship between those varying attributes. She drew segments along the horizontal and vertical axes, respectively, when representing the change in the individual attributes of volume and height. We coded Ana's images of change as smooth, because she envisioned change in height and volume as progressing during one fill of the bottle. We coded Ana's conception of the attributes of volume and height as measurable, because she described what each attribute could measure, and identified an appropriate unit that she could measure.

In the PreInterview, Ana did not envision the attributes of height and volume as varying individually. In her work on the Ferris wheel tasks during the Design Experiment interviews and the PostInterview, Ana envisioned the attributes of distance and height or distance and width as each varying individually, then represented a relationship between those varying attributes. In Ana's work during the PostInterview, she envisioned change in each of the individual attributes of height and volume, then represented a relationship between those changing attributes. Yet, Ana did not demonstrate envisioning variation in unidirectional change, as she had demonstrated in her work on the Ferris wheel. Generally, the covariational reasoning that Ana transferred involved envisioning change as progressing in individual attributes, then forming a relationship between those attributes (covariational reasoning involving attributes she conceived of as measurable in conjunction with smooth images of change.)

Johnson decided to follow up to investigate if Ana might be able to use the attributes of height and volume to envision variation in unidirectional change for the Filling bottle. However, when Johnson asked how her graph showed the "faster" and "slower," Ana shifted from focusing on change in the individual quantities of height and volume, and went back to the "shape" that she had discussed in earlier interviews.

Johnson:	And so, can you show me in your graph, like when we have the water that's changing, can you show me where on the graph you see the faster and the slower? And how you know?
Ana:	Ooh, it could also be one of them graphs where where they're like, he was walking slowly and then faster, and then he stopped *[Sketches small graph with unlabeled axes, Fig. 9 upper middle]*.
Johnson:	So what would that be? So tell me when you say it's going to be like one of-
Ana:	The graphs we were practicing on, the water is filling up very fast, and then it's like slower, and then it's really fast. *[Sketches piecewise linear graph in upper right corner, associating segments of different steepness with faster and slower, Fig. 9 upper right.]* Really fast would happen here *[Puts a box around the neck of the bottle, Fig. 9 lower left]*, and kind of fast here *[Puts a box around the bottom of the bottle, Fig. 9 lower left]*, and slow here *[Writes "slow" on widest part of the bottle, Fig. 9 lower left.]*.
Johnson:	If you had to label the axes, what would you label them?
Ana:	*[Writes volume on vertical axis and height on horizontal axis, Fig. 9 upper right.]* Well because they're both increasing. *[Pause]*
Johnson:	So can you have two different graphs for the same thing? *[Points to graph in upper right corner and graph on given axes, Fig. 9.]*

Ana: *[Pause]* I'm not sure.

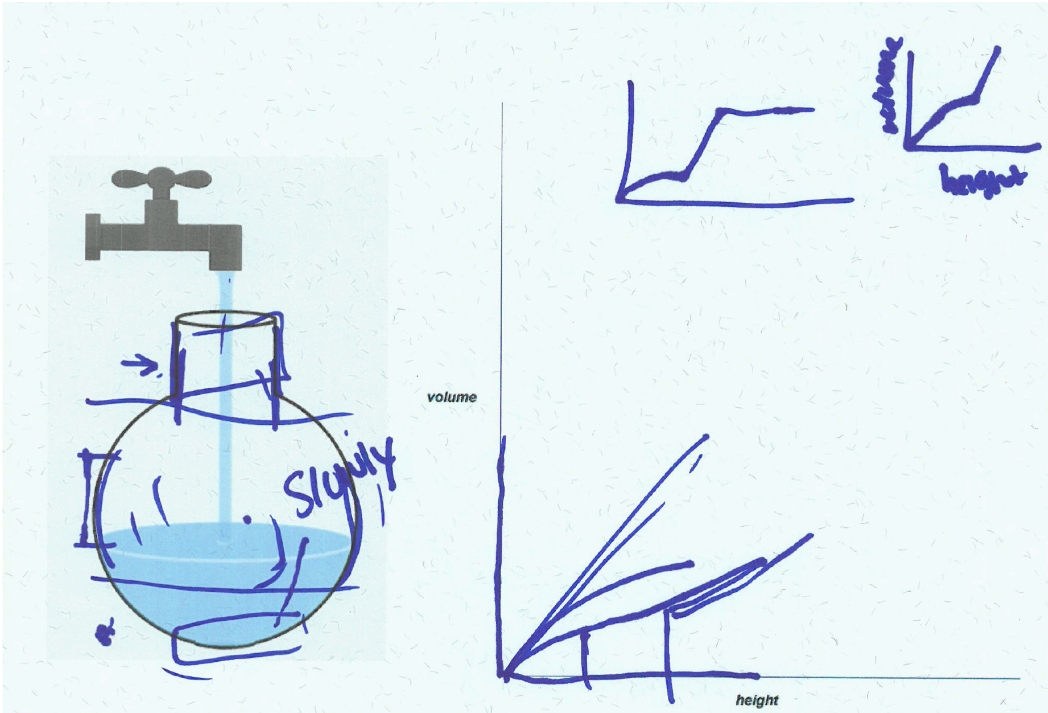

Fig. 9 Ana's continued work for the filling bottle, when asked to show faster and slower increases

After this excerpt, Ana's graph on the given axes was still a single line, as shown in Fig. 8. Ana added the additional markings later. When Ana attempted to represent variation in unidirectional change, she no longer demonstrated evidence that she was conceiving of change in the individual attributes of volume and height. Rather than talking about how the height and volume were changing, as she had just done previously, Ana referred to how the "water" was changing, stating "the water is filling up very fast, and then it's like slower, and then it's really fast." Ana labeled the axes "volume" and "height," which is not what she had done during the PreInterview. However, she labeled them after sketching the graph, rather than envisioning change in each attribute, then representing a relationship between the changing attributes. Although Ana transferred covariational reasoning involving attributes she conceived of as measurable in conjunction with smooth images of change, when attempting to envision variation in unidirectional change, she reverted back to an earlier form of reasoning.

Next, Johnson pursued a second follow up to investigate if Ana might demonstrate that she could conceive of variation in unidirectional change in a single attribute measurable with one-dimensional units—the height of liquid in the bottle. Johnson provided Ana with a dynamic computer environment including a pair of coordinate axes containing manipulable segments and a linked point, which Ana could manipulate individually or in conjunction.

Johnson: So, I'll hide the volume, and if you pull the height the whole way back to the start, show me how the height would change as you fill the bottle.

Ana: So—so when we're starting here, it's like kind of fast, and then it's like slow, *[Dragging segment on horizontal axis while talking, Fig. 10]* but then it goes like- the volume?

Johnson: The height?

Ana: I think the height will just be like, *[Pause, segment still extended on horizontal axis],* hmm, *[moves segment back to origin],* hmm, it will be a short time, to get like, oh, I'm just confusing my own self

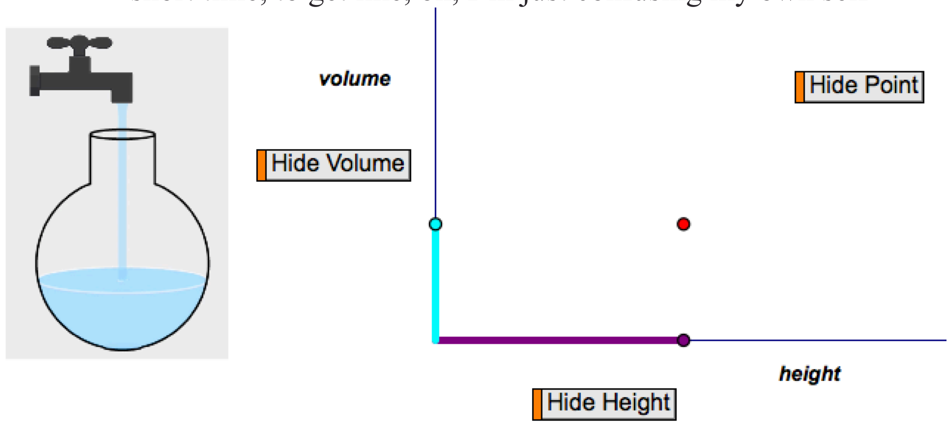

Fig. 10 Dynamic segments and point, which Ana could manipulate individually, or in conjunction

Pulling the height segment across the horizontal axis, Ana represented faster, then slower increases, until she reached the widest portion of the bottle. At that point, she had difficulty separating the height from the volume, and attempted to talk about volume to describe variation in unidirectional change. As a final follow up, Johnson hid the height and volume segments, showed only a single point, then asked Ana what the point on the graph could mean in terms of the bottle. The point was in roughly the same location as shown in Fig. 10. Rather than interpreting the point to be a hypothetical point representing volume and height, Ana interpreted the point to be an actual point on a graph representing volume and height for this bottle.

Johnson: And so, if I show a point on this graph, what does that point mean? *[Shows a point on the computer graph, Fig. 10.]*

Ana: That you have like, your height is kind of like in the middle *[moves cursor along horizontal axis, starting from the origin, then extending up to the point, Fig. 10]*, or in the beginning if you go all the way over here *[moves cursor to the rightmost edge of the horizontal axis, Fig. 10.]* and the volume is still kind of low *[moves cursor up and down along vertical axis, Fig. 10]*

Johnson: And so with your graph, with this line graph that you drew, does that show the height as going like different speeds, or does that show the, does this graph that you drew right here show the height going at different speeds, or does it just show the height going, or can't you tell?

Ana: I think this graph *[points to computer screen, which looks like Fig. 10.]* is going to be like more, cause mine is pretty steep, but that one looks like it's going to be kind of slowly curving, increasing *[sketches two monotonically increasing curves on her graph, both less steep than her original line graph, Fig. 9.]*

Johnson: How would that make sense in terms of the bottle?

Ana: That the height keeps increasing *[Moves pen along horizontal axis.]* but the volume is slowly filling through here *[Moves pen vertically, beginning at the box she drew near the bottom of the bottle in Fig. 9, and continuing* *to the widest portion of the bottle, then makes a circling motion in the widest portion of the bottle, leaving the arced pen marks shown in Fig. 9.]*

When interpreting the point, Ana specified the attributes of height and volume which she had done when transferring her covariational reasoning, rather than talking more generally about "water," as she had done when envisioning variation in unidirectional change. When Ana stated that the "volume is slowly filling through here," she drew a vertical segment, rather than talking about changes in the "amount of water that's inside," which is how she had described volume earlier in the interview. In fact, the height would be increasing slowly in the portion of the bottle she identified, because the wider portions of the bottle would require larger amounts of volume to result in changes to the height of the water.

Although Ana was able to transfer covariational reasoning involving attributes she conceived of as measurable in conjunction with smooth images of change, she did not demonstrate transfer of reasoning when asked to envision variation in unidirectional change. Notably, when Ana reverted to earlier forms of reasoning, she no longer demonstrated envisioning change in individual attributes that she conceived of as measurable. We posit that volume is a more complex attribute for students to envision measuring, and this complexity mitigated Ana's transfer of her covariational reasoning. Furthermore, Ana's difficulties suggest that when students encounter new situations with attributes measurable with different kinds of units, they may need to re-progress through forms of reasoning that they already have demonstrated.

Discussion/Implications

Ana's reasoning provides further evidence to support the utility of smooth images of change for fostering students' quantitative and covariational reasoning (Castillo-Garsow et al., 2013; Johnson, 2012; Moore, 2014; Thompson & Carlson, in press). Specifically, her reasoning illustrates both the challenge and utility of envisioning a quantity varying within, rather than across intervals. When students compare different kinds of increases across different intervals, they do not necessarily need to reason about variation in multiple quantities to make such comparisons. For instance, a student may notice that a bottle fills at different speeds, but that student may just be attending to variation in an attribute that she viewed in the physical world. We recommend that students have opportunities to attend to variation in unidirectional change in related attributes after they demonstrate that they can separate the attributes from the objects, conceive of the possibility of measuring those attributes, and envision those attributes as changing continually. Furthermore, in a recent study, Johnson and McClintock (under review) found that students not conceiving of attributes as measurable did not engage in variational reasoning even when interacting with a dynamic computer environment representing multiple changing quantities. Drawing on Thompson's conceptual framework surrounding quantitative reasoning (e.g., Thompson, 1993; Thompson, 1994, 2011), and building from the results of our research, we argue that envisioning change in attributes students conceive of as measurable, or engaging in quantitative reasoning about "things" that can vary, should form a foundation for students' covariational reasoning.

Thompson (2011) argued that the AOT perspective could prove to be a valuable lens for investigating students' quantitative and covariational reasoning, and we concur. One valuable insight involved students' conceptions of analogous attributes across different task situations. We did not explicitly design our study to investigate whether students might conceive of analogous attributes across situations such as the Ferris wheel and filling bottle. However, Ana's spontaneous response provided us with insights into how students might construe such relationships. Notably, Ana did not identify the speed of the Ferris wheel to be analogous to the flow of water into the filling bottle until after conceiving of variation in unidirectional change when forming relationships between the quantities of distance and height. Recently, Stalvey and Vidakovic posited that filling bottle tasks should incorporate variation in the rate of flow (Stalvey & Vidakovic, 2015). Ana's work suggests that it would be useful for tasks incorporating variation in the rate of flow to come after students can separate the flow rate from the other attributes in the situation. We think that future research should investigate secondary students' covariational reasoning in situations involving filling (or emptying) bottles with variable flow rates, Ferris wheels with variable speeds, or other analogous settings.

We recommend that introductory tasks designed to provide students opportunities to engage in covariational reasoning involve situations consisting of attributes measurable with one-dimensional units. When students conceive of individual attributes as measurable, and then use smooth images of change to envision variation in each attribute, they can form relationships between those attributes. Because attributes such as volume can be difficult for students to envision measuring, it is useful to implement tasks involving such attributes after students have had opportunities to form and interpret relationships between attributes they can more readily conceive of measuring.

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