NEW LONGITUDINAL AND INSTRUCTIONAL RESEARCH ON FRACTIONS

NCTM Annual Meeting
Boston, MA
April 18, 2015

Russell Gersten, Ph.D.
Director, Instructional Research Group
Professor Emeritus,
University of Oregon

Rob Ochsendorf, Ed.D.
Research Scientist, NCSER
Institute of Education Sciences

Karen S. Karp, Ed.D.
Professor, Mathematics Education
University of Louisville

Instructional Research Group

CENTER FOR IMPROVING THE LEARNING OF FRACTIONS

1. **Strand 1 – Longitudinal Research: Students With Math Difficulties**: Improving At-Risk Learners' Understanding of Fractions (Nancy Jordan)
2. **Strand 2 –** The **Centrality of Fractions to Mathematics Learning** and the **Centrality of Magnitude Comparison** and the **Number Line** to Mathematics Proficiency (Bob Siegler)
3. **Strand 3 – Interventions for Students At-Risk for Failure**: randomized controlled trials (Lynn Fuchs and Robin Schumacher)
4. **Strand 4 – Dissemination and Leadership**

(Note: slides adapted from those developed by Siegler, Jordan and Fuchs and Schumacher)

Instructional Research Group

IN THIS PRESENTATION, WE WILL:

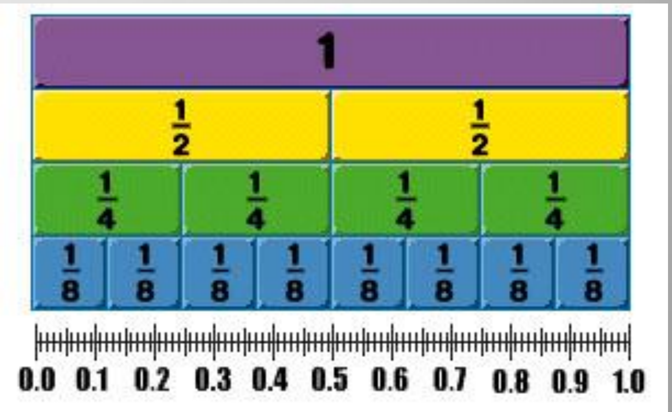
1. Share key findings from longitudinal and descriptive research in terms of:
   - ✓ The types of fractions problems most students solve successfully
   - ✓ The types of fractions problems that are difficult for students
   - ✓ How student understanding develops over time
2. Discuss evidence-based interventions for struggling learners

Contemporary State Standards and Common Core

Instructional Research Group

FRACTIONS ARE HARD!

1. Numbers of the same magnitude can look different (e.g., $\frac{3}{4}$ and $\frac{9}{12}$)



2. Sometimes, when numerals get bigger, the fraction gets smaller ($\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{8}$)

3. Not always the case, however ($\frac{2}{4} < \frac{6}{7}$)

4. Infinite amount of numbers between 2 fractions.

Instructional Research Group

RESEARCH STRAND 1: STUDY SAMPLE

Children placed into one of two groups:

- ✓**Lower Math Achievement** (scored ≤ 35th %ile in 3rd grade). Includes at risk as well as very low performing. Mean 18.5%ile
- ✓**Higher Math Achievement** (scored > 36th %ile in 3rd grade).   Mean 69th %ile

Note: Only children who completed all assessments were included (n = 309)

KEY AREA OF CONFUSION

1. Both groups of students perform similarly on items that require a part-whole understanding of a fraction, BUT… not on linear representation problems

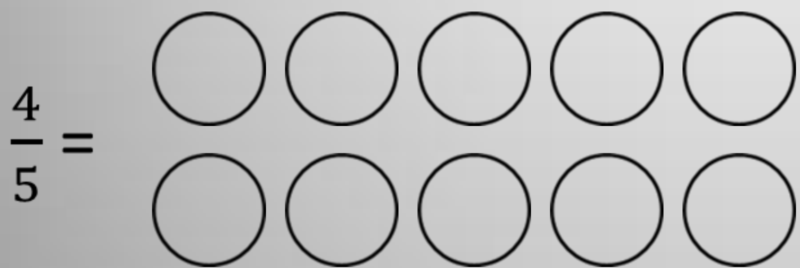
3rd Grade CCSS → **Develop an understanding of fractions as numbers.**

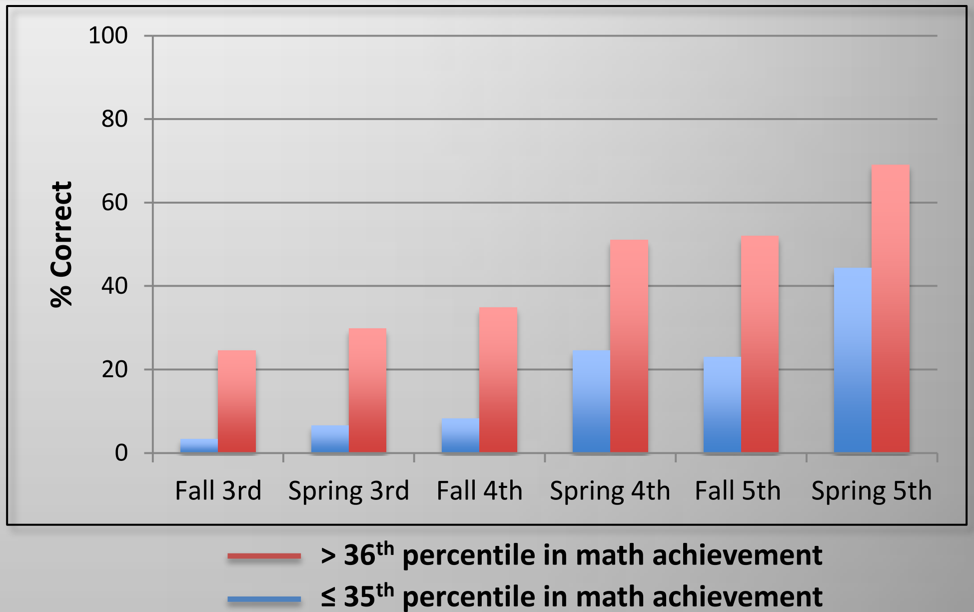
WHAT TYPES OF FRACTIONS CONCEPTS ARE DIFFICULT FOR STUDENTS?

2. Children's prior knowledge of whole numbers interferes with learning when they are asked about more advanced fraction concepts problems.

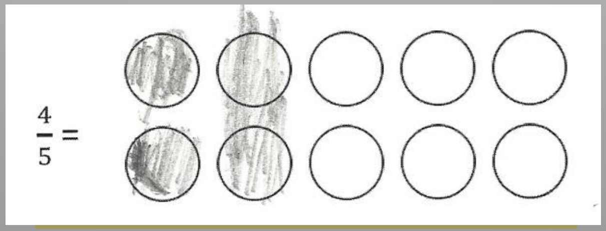
4th Grade CCSS → **Extend understanding of fraction equivalence and ordering.**

FRACTION CONCEPTS

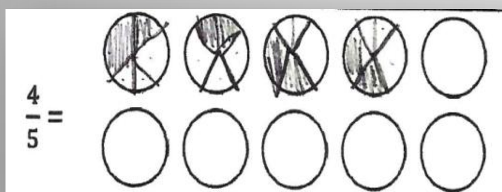
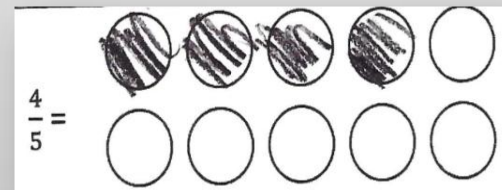
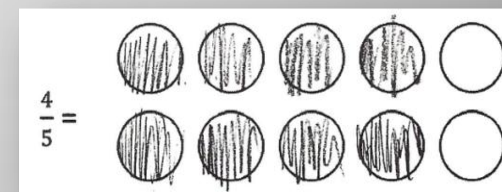
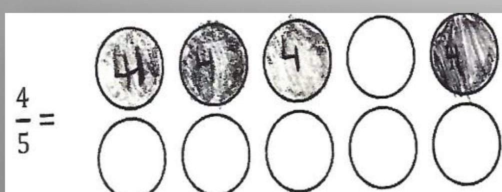
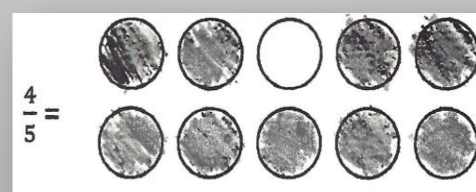




Most common incorrect response was to color 4 of the 10 circles, indicating that students were only attending to the numerator.



EXAMPLES OF STUDENT PROGRESS

3rd Grade	4th Grade	5th Grade
		
		

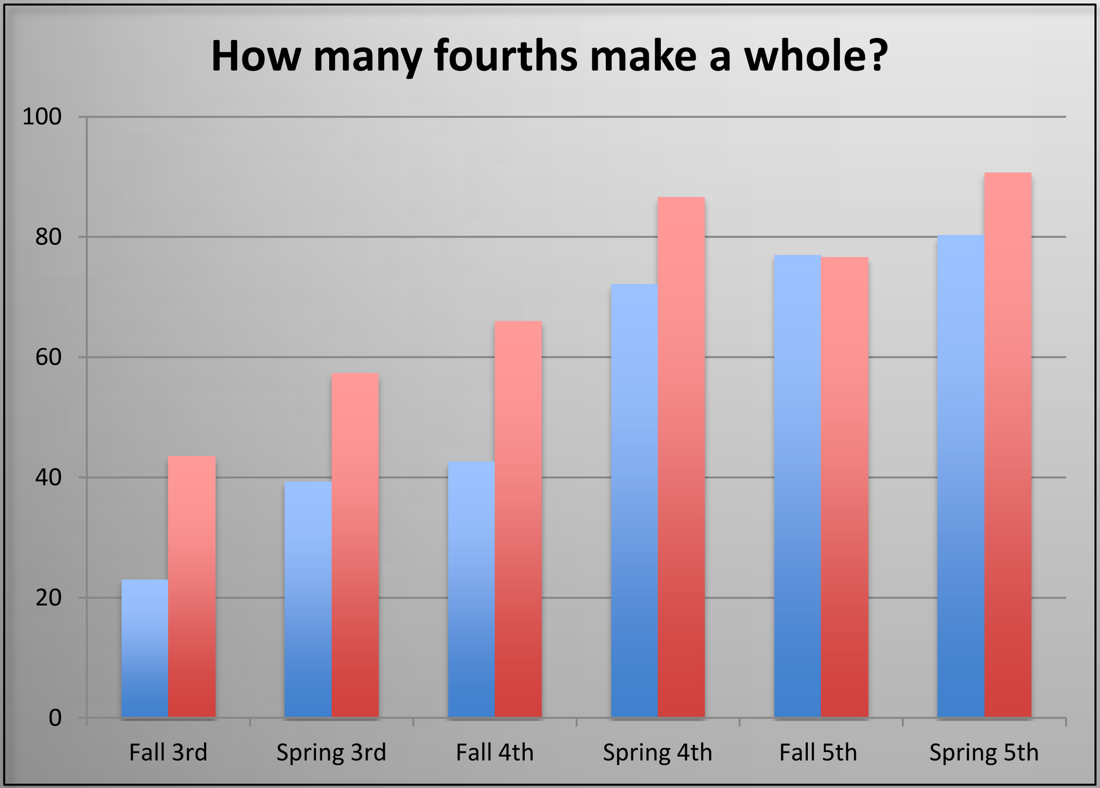
WHAT TYPES OF FRACTIONS CONCEPTS ARE DIFFICULT FOR STUDENTS?

3. Students do not have a strong understanding of the meaning of the denominator.

3rd Grade CCSS

Develop an understanding of fractions as numbers.

MEANING OF THE DENOMINATOR
WHAT DO STUDENTS UNDERSTAND?



> 36th percentile in math achievement

≤ 35th percentile in math achievement

Most students can state how many fourths make a whole by the end of 5th grade, BUT…

Instructional Research Group

WHAT TYPES OF FRACTIONS CONCEPTS ARE DIFFICULT FOR STUDENTS?

4. Students have difficulty identifying where fractions go on number lines.

3rd Grade CCSS

Understand a fraction as a number on a number line.

WHAT TYPES OF FRACTIONS PROCEDURES ARE DIFFICULT FOR STUDENTS?

5. Children's prior knowledge of whole numbers interferes with learning when they first learn fraction procedure problems.

4th Grade CCSS

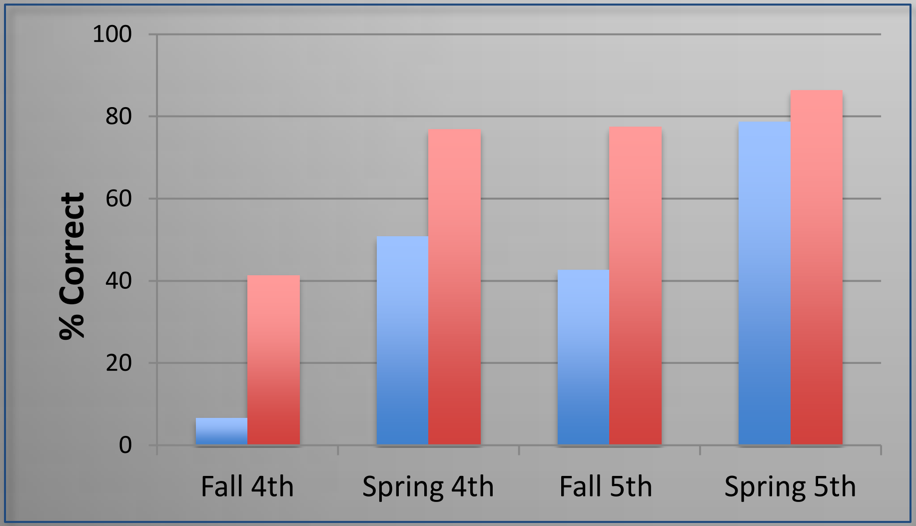
Solve problems involving addition and subtraction of fractions referring to the same whole and having like denominators

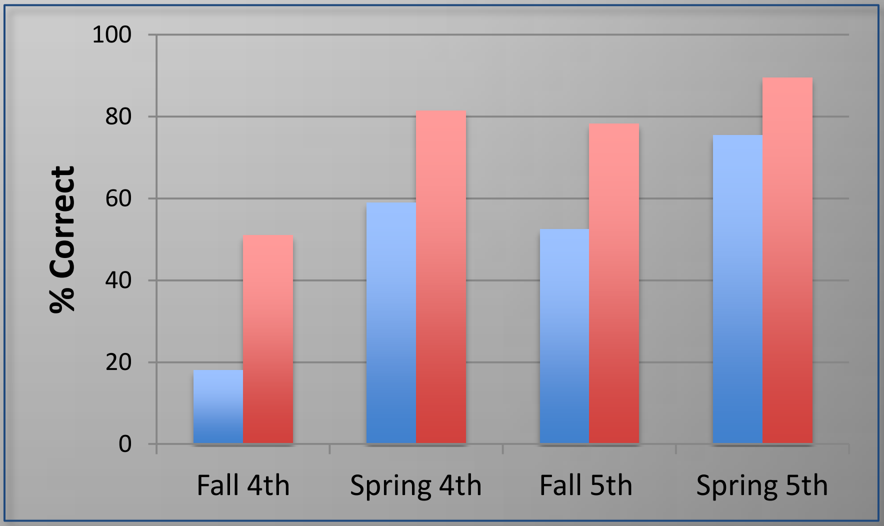
FRACTION PROCEDURES

1. By the end of 5th grade, most students successfully solved problems with like denominators.
2. But -- students in the lower math achievement group took an extra year to learn to solve these problems.

$$\frac{3}{6} + \frac{1}{6} =$$

$$\frac{3}{4} - \frac{1}{4} =$$





> 36th percentile in math achievement

≤ 35th percentile in math achievement

Instructional Research Group

INCORRECT STRATEGIES USED

There were two main whole number strategies students used to solve problems with like denominators.

Strategy	Example
Independent whole numbers	$\frac{3}{6} + \frac{1}{6} = \frac{4}{12}$ $\quad$ $\frac{3}{4} - \frac{1}{4} = \frac{2}{0}$
“Add all”	$\frac{3}{6} + \frac{1}{6} = \underline{16}$

By the end of fifth grade, most students successfully solved addition and subtraction problems with like denominators.

It took students in the lower-achieving group about one additional year to catch up to their higher achieving peers.

Although students successfully solve problems with like denominators by the end of 5th grade, they still are not successful with problems with unlike denominators.

Students revert back to their whole number strategies when confronted with novel problems.

WHAT PREREQUISITE SKILLS DO STUDENTS NEED BEFORE THEY ENCOUNTER FRACTIONS?

6. Students should have fluent fact mastery so that they can execute fraction procedures correctly.

2^{nd} and 3^{rd} Grade CCCS

Fluently add and subtract within 20 by the end of 2^{nd} grade.

Fluently multiply within 100 by the end of 3^{rd} grade.

SUMMARY POINTS

1. Most students have a basic understanding of the part-whole meaning of a fraction but many struggle when it comes to items requiring knowledge of **equivalence and ordering**.
2. Struggling learners do not have a strong understanding of the meaning of a denominator.
3. Even by the end of fifth grade, less than half of the students in both groups can correctly locate fractions on the number line.

SUMMARY POINTS (CONT.)

4. By the end of 5^{th} grade, most students successfully solve addition and subtraction problems with like denominators but it took students with lower achievement an additional year to catch up to their higher achieving peers.
5. Most students struggle with addition and subtraction problems with unlike denominators.

RESEARCH STRAND 2: THE CENTRALITY OF FRACTION MAGNITUDES

Adapted from Siegler, R. (2014). *An Integrated Theory of Numerical Development: Magnitude*

and

Siegler, Thompson, & Schneider, (2011).

DEVELOPMENT OF NUMERICAL MAGNITUDE REPRESENTATIONS: A USEFUL UNIFYING THEME

1. **History:** Robbie Case indicated that mathematical development can be captured by the increased sophistication of a person's **mental number line.**
2. **Contemporary research supports this position**
   - ✓ Linearity of whole number magnitude representations correlates positively and quite strongly with:
     - Numerical magnitude comparison (Laski & Siegler, 2007)
     - Arithmetic proficiency (Gilmore, et al., 2007; Halberda, et al., 2008; Holloway & Ansari, 2008; Schneider et al., 2009)
     - Standardized achievement tests (Booth & Siegler, 2006; 2008; Geary et al., 2007; 2011)

KEY MEASURE: NUMBER LINE ESTIMATION

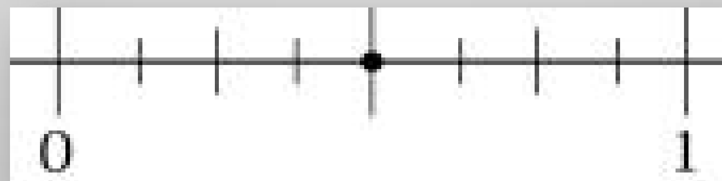
The Number Line Task

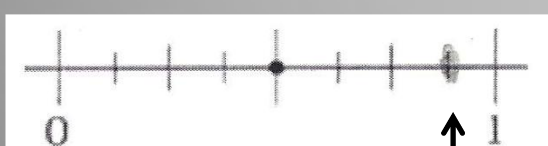
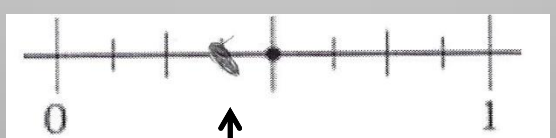
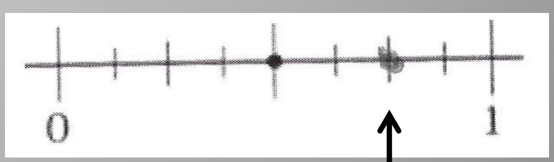
"Where does 87 go?"

0 ———————————————————————— 1000

STUDENT PROGRESS

On the portion of the number line below, a dot shows where $\frac{1}{2}$ is. Use another dot to show where $\frac{3}{4}$ is.



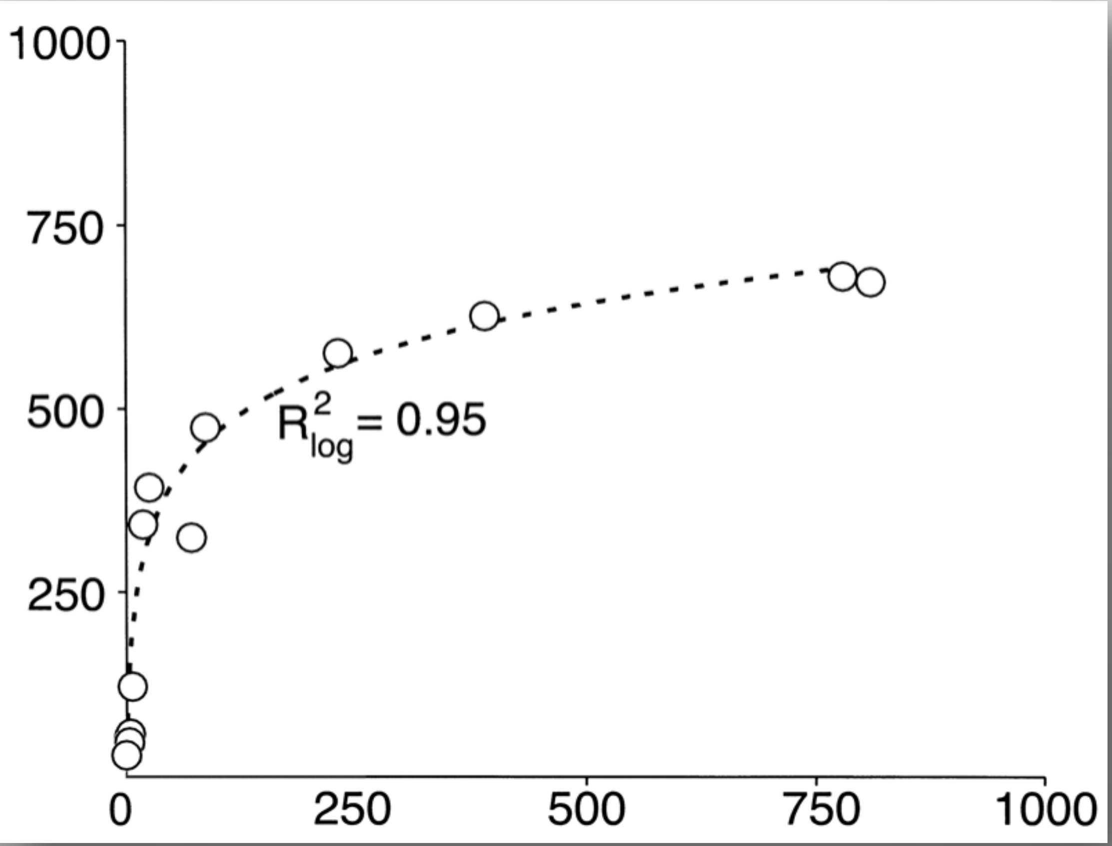
Spring 3rd Grade	Spring 4th Grade	Spring 5th Grade
		

$\frac{7}{8}$ $\frac{3}{8}$ $\frac{3}{4}$

Instructional Research Group

SECOND GRADERS TEND TO USE A LOGARITHMIC NUMBER LINE

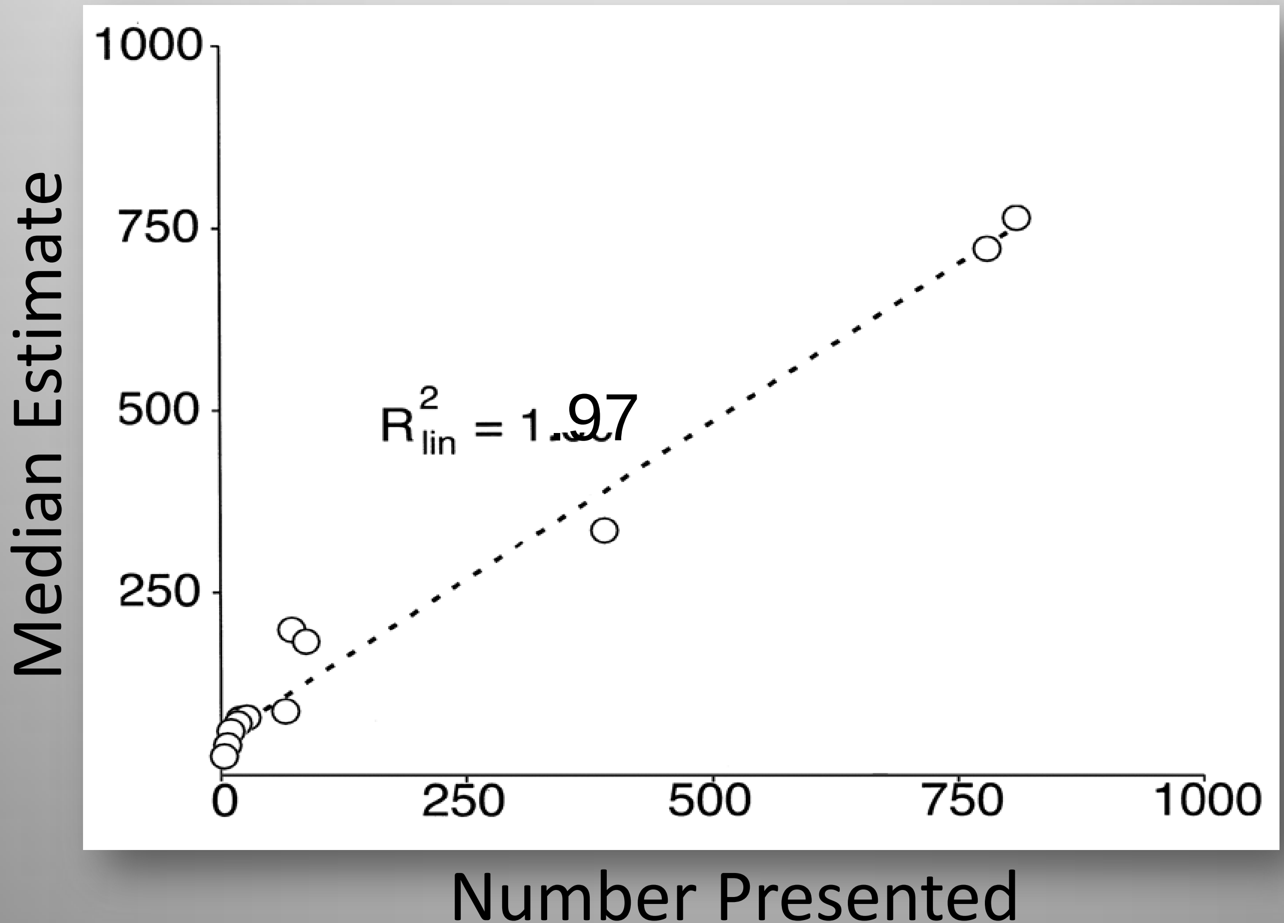
Median Estimate



Number Presented

Instructional Research Group

SIXTH GRADERS ON AVERAGE USE A LINEAR NUMBER LINE



Instructional Research Group

RELATIONS BETWEEN FRACTION MAGNITUDE REPRESENTATIONS AND MATH ACHIEVEMENT SCORES: 8TH GRADERS

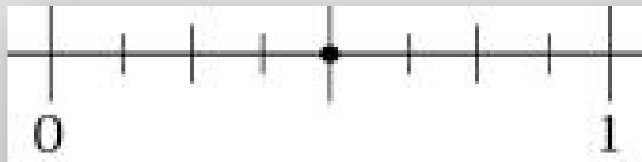
Measure of Magnitude	Math Achievement
✧ Number line 0-1 PAE	**-.63****
✧ **Number line 0-5 PAE**	**-.86****
✧ Magnitude Comparison Accuracy	**.62****

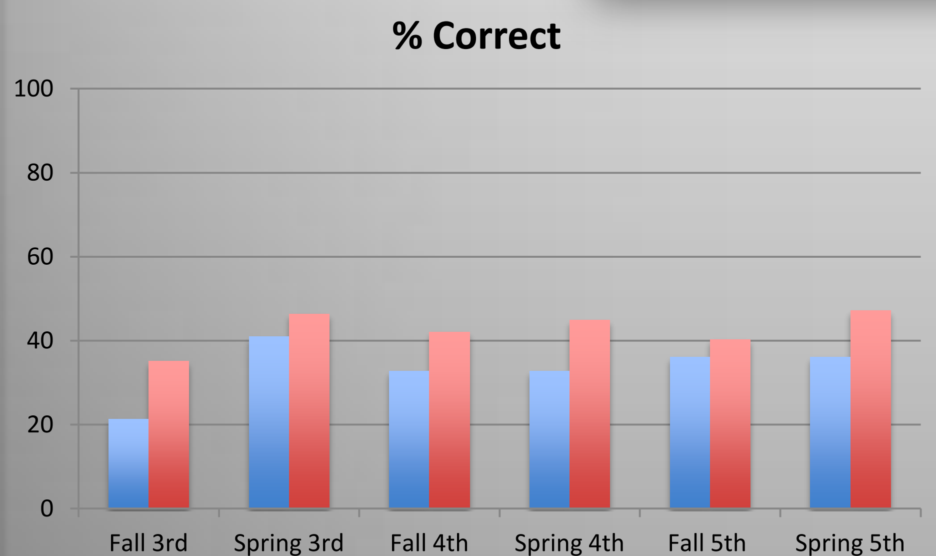
**$p < .01$

Instructional Research Group

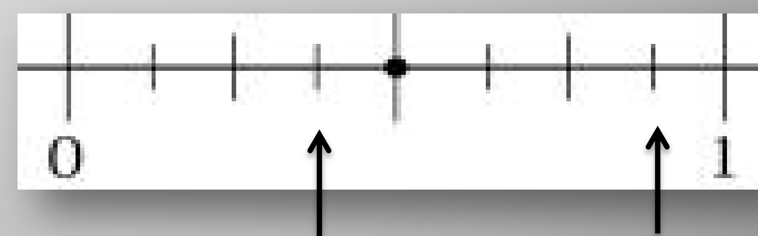
FRACTIONS ON THE NUMBER LINE

On the portion of the number line below, a dot shows where $\frac{1}{2}$ is. Use another dot to show where $\frac{3}{4}$ is.





Common Errors



~10-15% of kids in both groups in 4th and 5th grade

~20-30% of kids in both groups in 4th and 5th grade

> 36th percentile in math achievement

≤ 35th percentile in math achievement

Instructional Research Group

ORDERING FRACTIONS: NAEP ITEM

Correct Response:

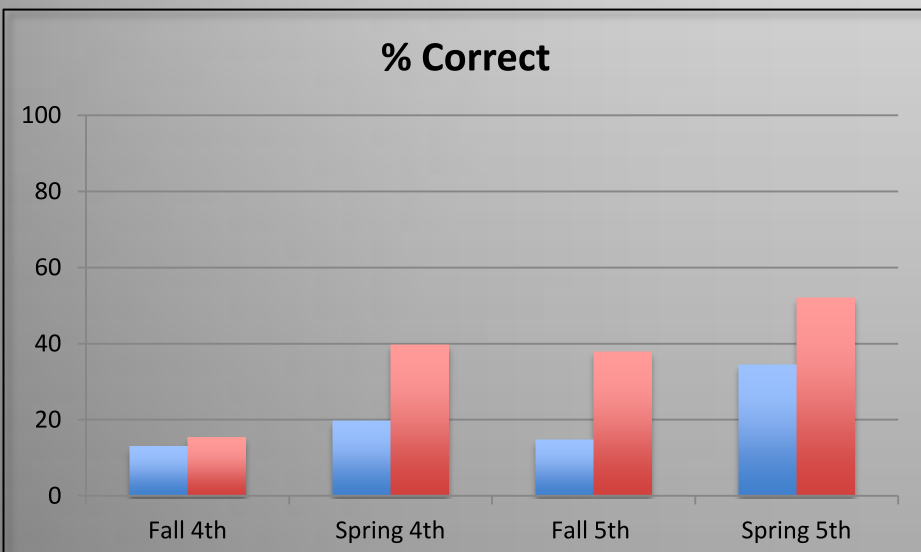
In which of the following are the three fractions arranged from least to greatest?

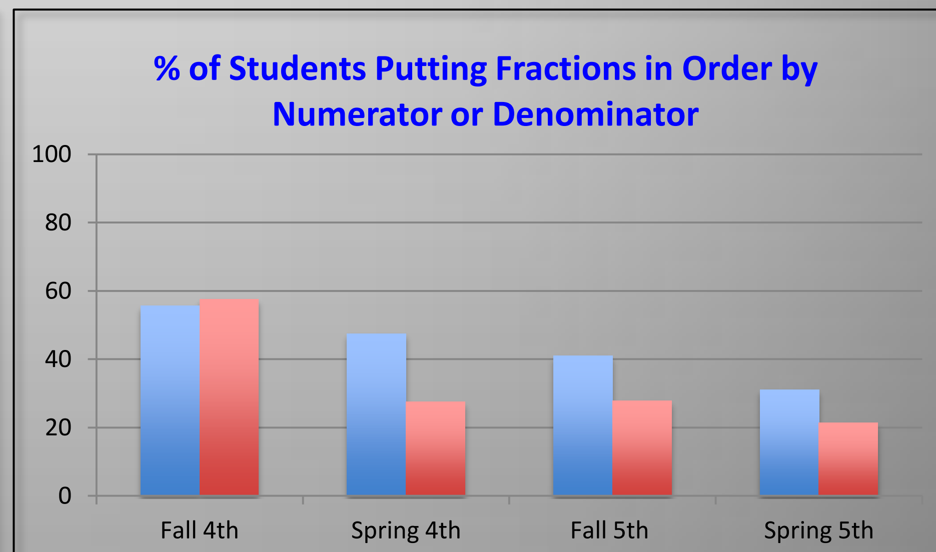
- (A.) $\frac{2}{7}, \frac{1}{2}, \frac{5}{9}$
- B. $\frac{1}{2}, \frac{2}{7}, \frac{5}{9}$
- C. $\frac{5}{9}, \frac{1}{2}, \frac{2}{7}$
- D. $\frac{5}{9}, \frac{2}{7}, \frac{1}{2}$

Incorrect Response:

In which of the following are the three fractions arranged from least to greatest?

- A. $\frac{2}{7}, \frac{1}{2}, \frac{5}{9}$
- (B.) $\frac{1}{2}, \frac{2}{7}, \frac{5}{9}$
- C. $\frac{5}{9}, \frac{1}{2}, \frac{2}{7}$
- D. $\frac{5}{9}, \frac{2}{7}, \frac{1}{2}$





> 36th percentile in math achievement (red)

≤ 35th percentile in math achievement (blue)

Instructional Research Group

Even by the end of fifth grade, less than half of students in both groups can correctly identify fractions on the number line.

This points towards the need to further develop the understanding of a fraction as a location on a number line.

Instructional Research Group

MAGNITUDE OF FRACTIONS

It develops slowly and precludes accurate magnitude comparison.

STRAND 3 – INTERVENTION FOR AT RISK 4TH GRADERS: RANDOMIZED CONTROLLED TRIAL (RCT) RESEARCH

CENTER FOR IMPROVING THE LEARNING OF FRACTIONS

1. **Strand 1 – Longitudinal Research: Students With Math Difficulties**: Improving At-Risk Learners' Understanding of Fractions (Nancy Jordan)

2. **Strand 2** – The **Centrality of Fractions to Mathematics Learning** and the **Centrality of Magnitude Comparison** and the **Number Line** to Mathematics Proficiency (Bob Siegler)

3. **Strand 3 – Interventions for Students At-Risk for Failure**: randomized controlled trials (Lynn Fuchs and Robin Schumacher)

4. **Strand 4 – Dissemination and Leadership**

(Note: slides adapted from those developed by Siegler, Jordan and Fuchs and Schumacher)

EMPHASIS ON FRACTION MAGNITUDE

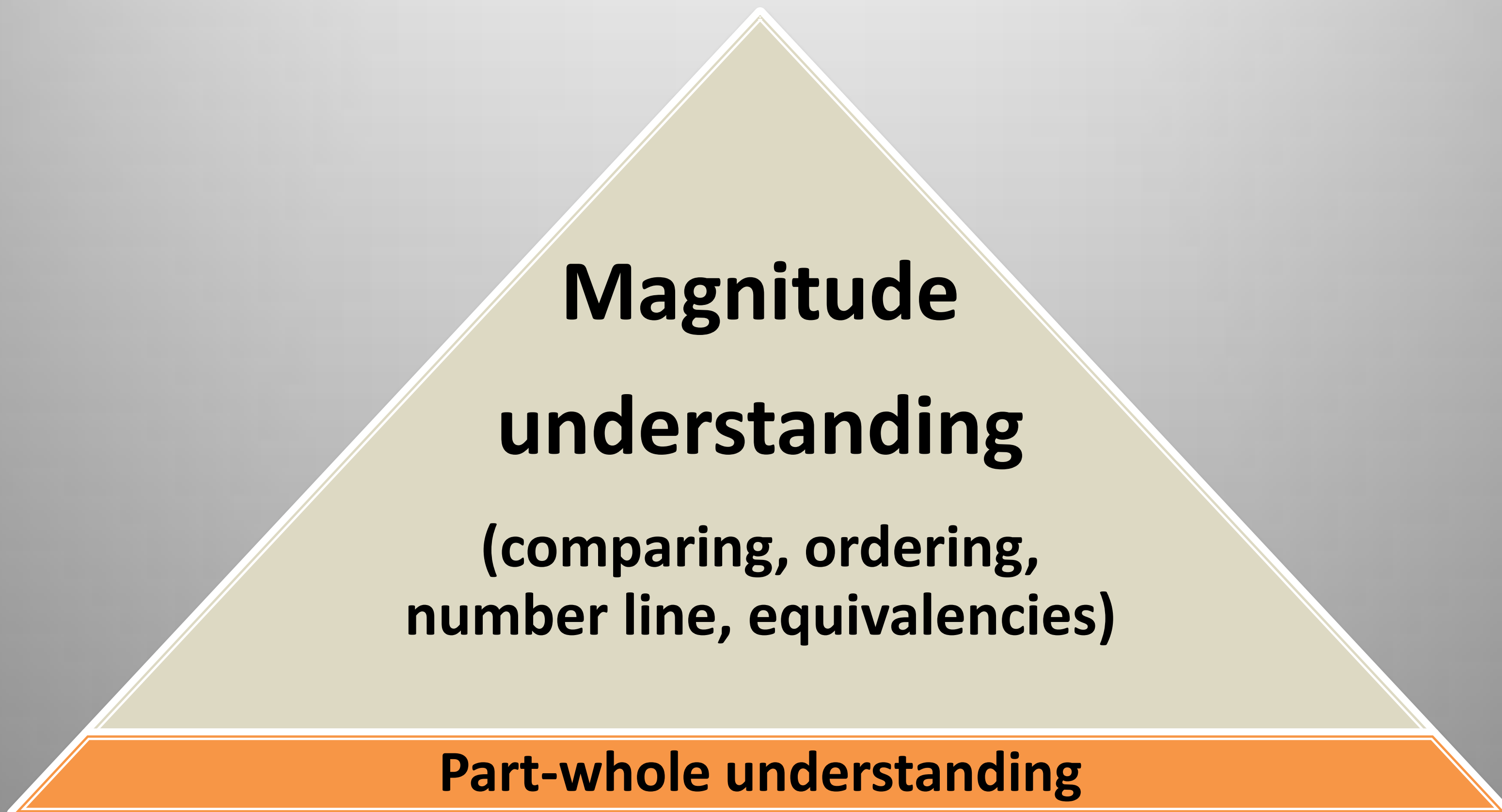
- Understanding that fractions have magnitude and can be ordered, compared, and placed on the number line is essential for future success in higher-level mathematics courses (e.g., Bailey, Hoard, Nugent, & Geary, 2012; Booth & Newton, 2012; Brown & Quinn, 2007; Siegler et al., 2012)

- Many students do not understand how to assess fraction magnitude (e.g., NAEP, 2013), and often operate with **whole number bias,** i.e., the overgeneralization of whole number properties to fractions (Ni & Zhou, 2005).

RATIONALE FOR THE INTERVENTION

1. Core of intervention is aligned with grade level mathematics instruction.
2. Content linked to contemporary state standards and current thinking about best practice in mathematics education.
3. Small group instruction targets an area that longitudinal research shows is particularly difficult for this group of students.
4. Intervention attacks an area that some elementary teachers are unsure how to teach.
5. Adequate practice provided.

FOCUS OF INSTRUCTION IN FRACTIONS FACE-OFF!



TOPICS COVERED IN FRACTIONS FACE-OFF!

1. Naming fractions from regions (part-whole foundation).
   - ✓ E.g., $\frac{1}{4}$ = 1 of 4 equal parts (supported by pictures, tiles, and circles).
2. Define numerator and denominator
   - ✓ Differentiate between number of parts (i.e., numerator) and the size of the parts (i.e., denominator).

TOPICS COVERED IN FRACTIONS FACE-OFF!

3. Comparing two fractions

✓ Conceptual comparing strategies

- Same numerator
- Same denominator
- $\frac{1}{2}$ as the benchmark

TOPICS COVERED IN FRACTIONS FACE-OFF!

1. Using $\frac{1}{2}$ as a benchmark:
   - ✓ Teach fractions equivalent to $\frac{1}{2}$ with multiplication, supported by fraction tiles, circles, and number lines
   - ✓ For fluency – teach the **Doubling Rule**
     - If the numerator is double the denominator, the fraction is equal to $\frac{1}{2}$. Quick strategy for rewriting $\frac{1}{2}$.
     - Students can then quickly write a fraction equivalent to $\frac{1}{2}$ with a different denominator

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} = \frac{5}{10} = \frac{6}{12}$$

COMPARING FRACTIONS USING $\frac{1}{2}$ AS A BENCHMARK

$$\frac{3}{6}\ \cancel{\frac{1}{2}} < \frac{5}{6}$$

Same Denominator

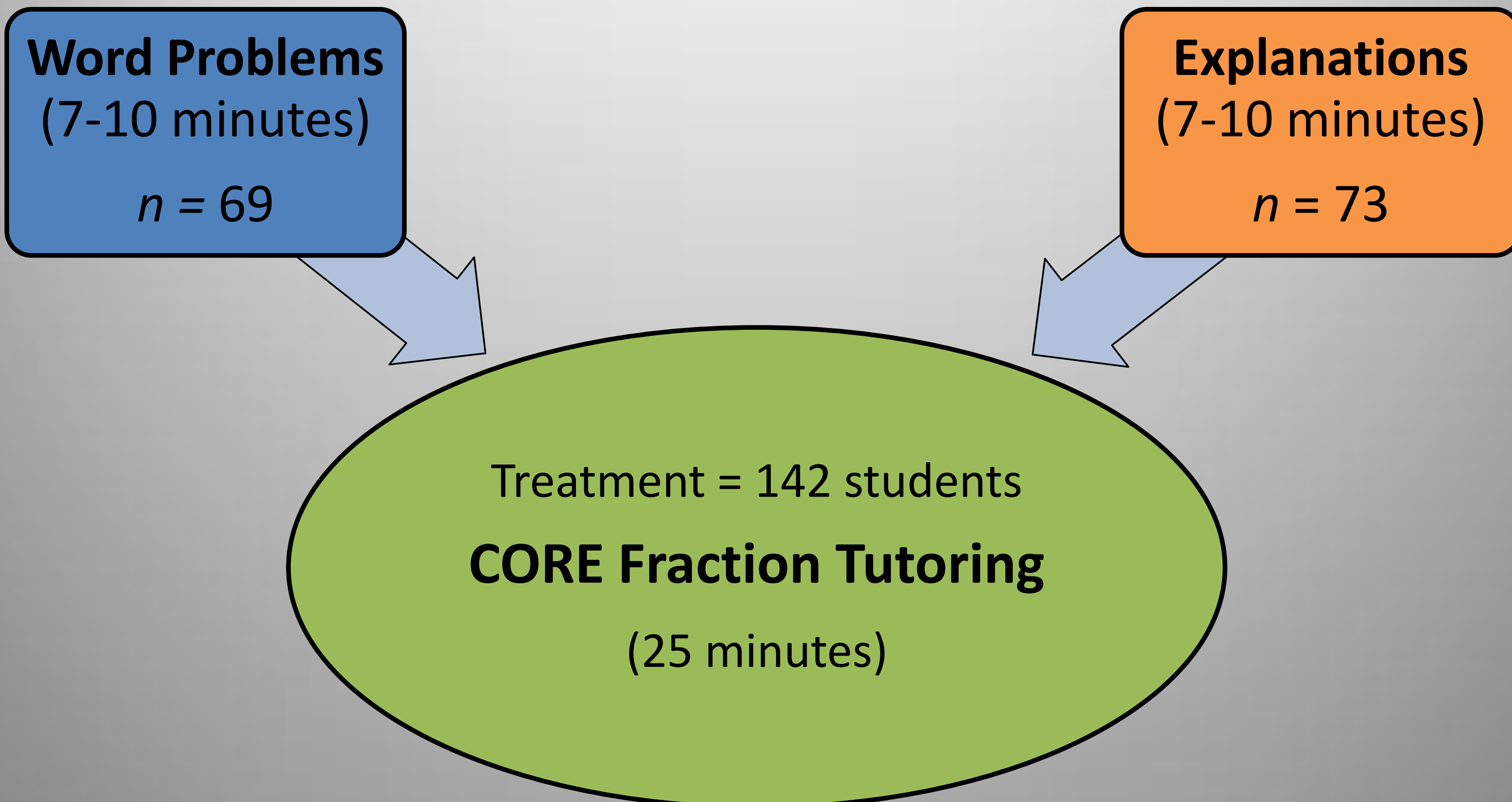
Bigger numerator,
Bigger Fraction

5 RANDOMIZED CONTROLLED TRIALS

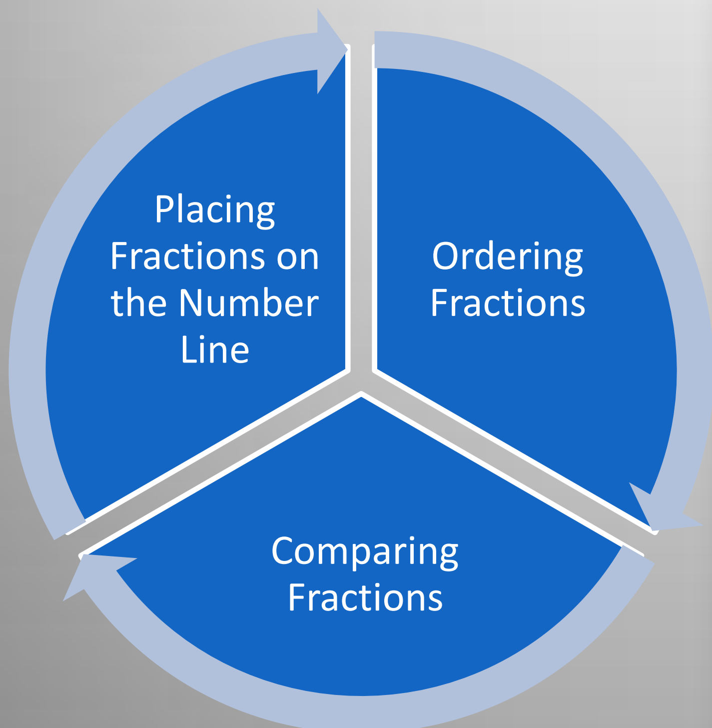
Lynn Fuchs, Robin Schumacher and colleagues, Vanderbilt University

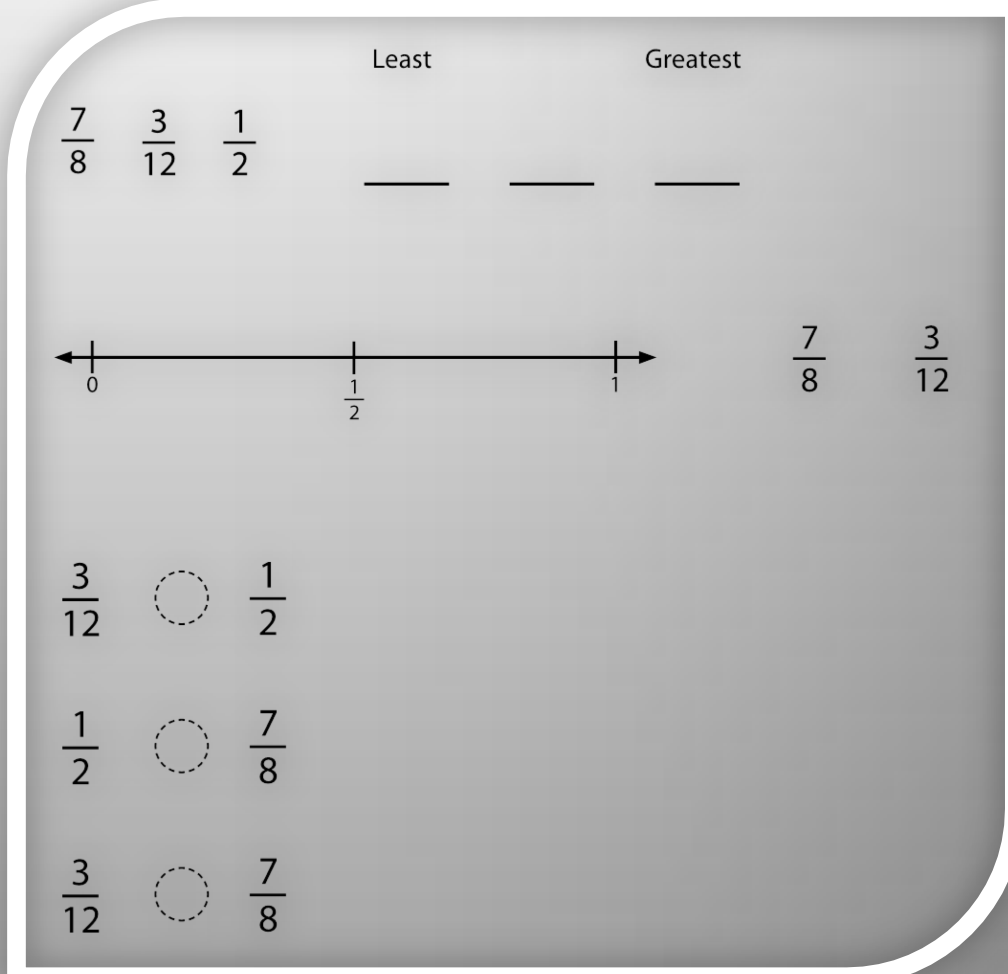
1. In Year 1, contrasted core intervention program vs. regular instruction in 4th grade classroom (Business as Usual)
2. In Years 2-5,
   - ✓ Replicating this effect for refined versions of core intervention program
   - ✓ Isolating contribution of a series of additional program components (3 study conditions: 2 versions of intervention and Business As Usual).

STRUCTURE OF FRACTIONS FACE-OFF!



BUILDING MAGNITUDE UNDERSTANDING



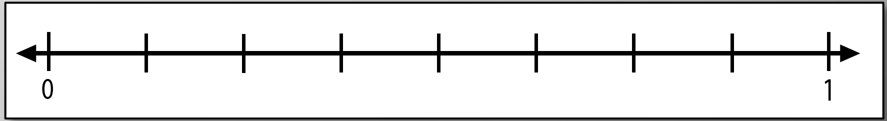


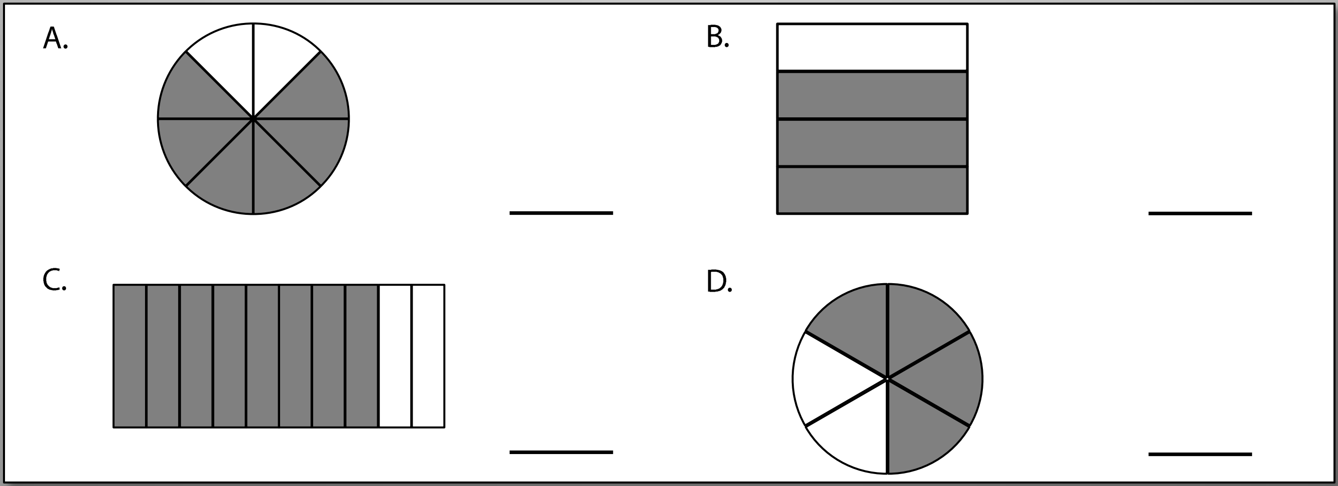
Instructional Research Group

EARLY SKILLS:
UNIT FRACTIONS AND NAMING FRACTIONS

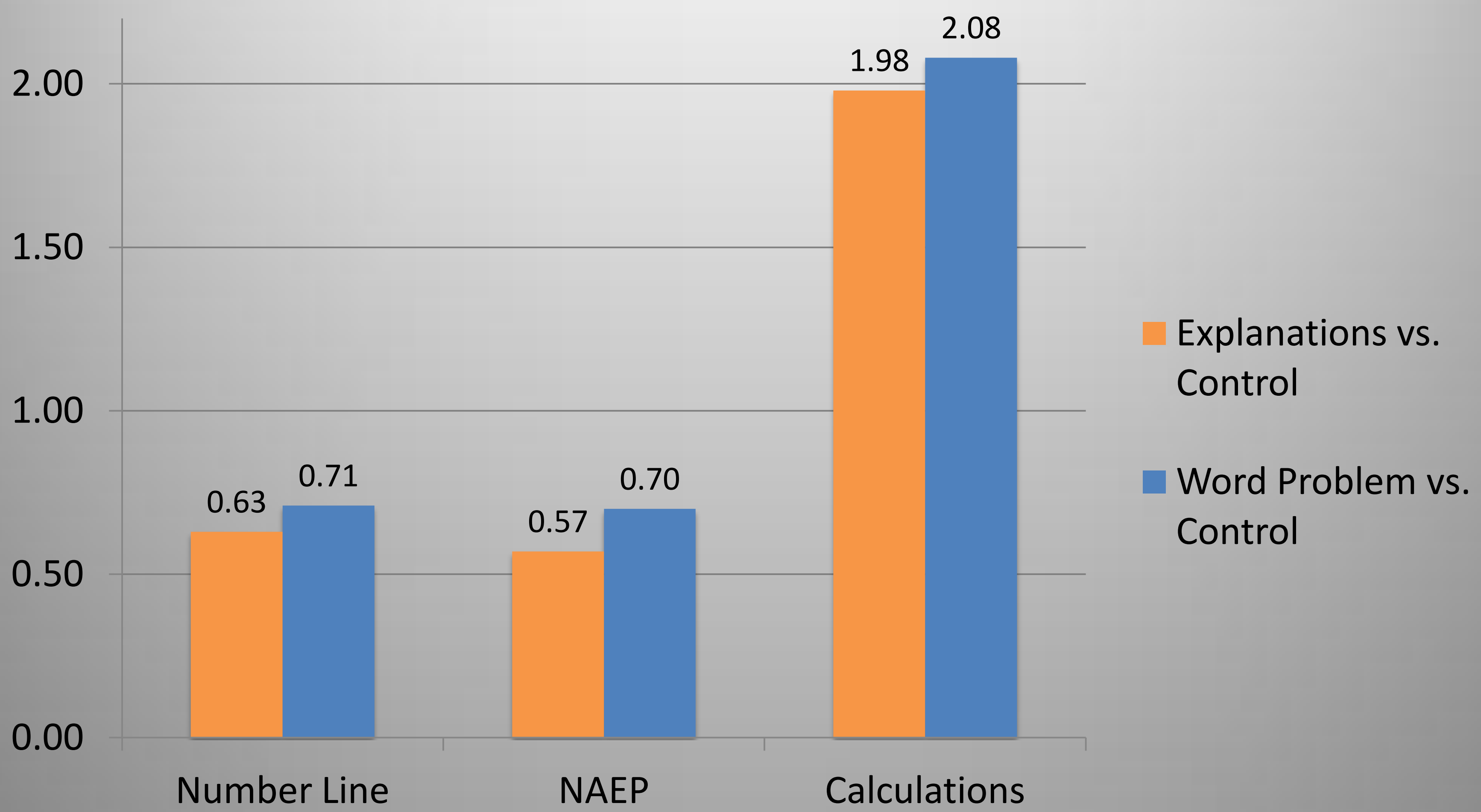
- ✓ Introduce unit fractions with Circles and Tiles
- ✓ Show fractions with shaded regions to show unit fractions
- ✓ Show how *unit fractions* make larger fractions with manipulatives, number lines, and numbers
- ✓ Name fractions from shaded representational regions (see example below)

$$\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{5}{8}$$





RESULTS:
FRACTION OUTCOMES



Instructional Research Group

FINDINGS TO DATE

Results on 3 core outcome measures, across 4 years for intervention vs. Business As Usual

1. *Fractions Number Line m*ean effect size for intervention of 0.99
2. *Adding/Subtracting Fractions* mean effect size of 1.57, supporting idea of reciprocal relationship

QUESTIONS?

To receive intervention materials, email:
lynn.fuchs@vanderbilt.edu

Instructional Research Group