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The Role of Logic in Teaching Proof

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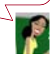
Proof?

Is the square of any odd integer odd?

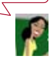
Yes: $1^2 = 1$, $3^2 = 9$, $5^2 = 25$, $(-1)^2 = 1$, $(-3)^2 = 9$

How can you be 100% sure that the square of *any* odd integer odd?



That's what my teacher said.

Is the square root of 3 irrational?



Sure. I can tell from my calculator.

Why Teach Proof?

Skeptical Student: I've known certain mathematical facts since I was a child. Others become obvious from software like Geometer's Sketchpad or GeoGebra. Why justify them with a mathematical proof?

Michael de Villiers: Proofs are sometimes needed to convince us that a mathematical property is always true. But we generally don't work very hard to prove something if we don't believe it to be true. **A more important function of proof is that it explains *why* the thing is true.**

Michael de Villiers

http://mzone.mweb.co.za/residents/profmd/

The Standards and Proof

What Learning Proof Teaches

- How to understand and build a logically connected chain of statements - think in a tightly disciplined way
- The power of logical principles
- How to use symbols in thinking
- How to move between the abstract and the particular
- The need to give a valid reason for each step in an argument
- Respect for careful use of language

Mathematical Practice Standards

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

A Simple Proof

Statement: The square of any odd integer is odd.

Proof 1: $(2k+1)^2 = 4k^2 + 4k + 1$
$= 2(2k^2 + 2k) + 1$

Is anything missing? Would students be able to understand why this proves the statement?

Proof 2: Suppose n is any odd integer. Then $n = 2k+1$ where k is an integer, and so

$$n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\cdot(2k^2 + 2k) + 1.$$

Would anything cause a student difficulty in following and/or accepting this proof?

Now $2k^2 + 2k$ is an integer, and thus n^2 is odd

Quantifiers and Variables

- Words like "all," "some," "any," and "every," are called "quantifiers."
- "All" important mathematical statements use quantifiers.
  Ex: The square of any odd integer is odd.
- To work with quantified statements in mathematics, we need letters (variables). (Ex: For every odd integer n, n^2 is odd.)

A. N. Whitehead: The ideas of 'any' and 'some' are introduced to algebra by the use of letters….it was not till within the last few years that it has been realized how fundamental *any* and *some* are to the very nature of mathematics…

Felix Klein (On the transition to operations with letters): This represents such a long step in abstraction that one may well declare that real mathematics begins with operations with letters.

The Basic Logical Principle for Mathematical Proof: Generalizing from the Generic Particular

If we can prove that a property is true for one particular, but arbitrarily chosen, element of a set, then we can conclude that the property is true for *every* element of the set.

i.e., a generic element of the set

Prove for one; know for all!

Thus: To prove "For every odd integer n, n^2 is odd," suppose n is *any* *[particular but arbitrarily chosen]* odd integer and deduce that n^2 is odd.

Mathematics, as a science, commenced when first someone, probably a Greek, proved propositions about 'any' things or about 'some' things without specification of definite particular things.

Alfred North Whitehead:
An Introduction to Mathematics 1911

Another Logical Principle: Universal Instantiation

If a property is true for all elements of a set, then it is true for each individual element of the set.

Duh!

Two main uses:

1. Every time we do algebra

Ex. We know that for **all** real numbers A, B and C,

$$AB + AC = A(B + C)$$

$$\text{So } 2\cdot 2k^2 + 2\cdot 2k + 1 = 2(2k^2 + 2k) + 1.$$

Universal instantiation was repeatedly applied for all the other computations in the proof, but its use can become automatic.

2. Extensively in explanation/justification/proof

Ex. Because a sum of products of *any* two integers is an integer and because 2 and k are integers, then $2k^2 + 2k$ is an integer.

The Logic of Definitions

if and only if

An integer is **even** ⇔ it can be expressed as 2 times some integer.

An integer is **odd** ⇔ it can be expressed as 2 times some integer plus 1.

What it means for a definition to be ⇔:

- If an integer is even then it can be expressed as 2 times some integer. *AND*
- If an integer can be expressed as times some integer, then it is even.

Use a definition as a **test**.
***Ex:* Is 0 even?**
Well --- it passes the test!

Another Logical Principle: Existential Instantiation

If we know or suspect that an object exists, then we may give it a name as long as we are not using the name for another object in our current discussion.

Example: Given an odd integer, we know that it equals twice some integer plus 1. It's existential instantiation that entitles us to give that integer a name. For instance, we can call it k (or a or r or whatever) and write the odd integer as $2k + 1$, where we identify k as being an integer.

Thanks to Dan Velleman (U Mass Amherst) for making me really aware of how often we use this principle.

A Simple Proof - Again

Generalizing from the generic particular governs the shape of the proof.

Prove: The square of any odd integer is odd.

For every odd integer n, n^2 is odd.

Proof: Suppose n is any *[particular but arbitrarily chosen]* odd integer. *[We must show that n^2 is odd.]*

By definition of odd, there is an integer *[which we may call]* k with $n = 2k + 1$. Then

by existential instantiation

$n^2 = (2k + 1)^2$ by substitution

$= 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1.$

Both the "if" and the "only if" aspects of the definition are used.

Universal instantiation: It is a sum of products of integers, ∴ an integer

But $2k^2 + 2k$ is an integer. So by definition of odd, n^2 is odd *[as was to be shown].*

Universal instantiation is used repeatedly in the algebra steps.

Why Do Students Have Difficulty with Proof?

- Little experience with logical flow of an argument/deductive reasoning
- Words like "all," "some," "if-then," "and," and "or" have multiple meanings in ordinary language but just one meaning in math
- Some experience using variables as unknowns but very little as generic elements of sets
- Very little experience expressing mathematical statements in a variety of different ways, especially with variables

What can teachers do?

Help students develop a sense for the logical flow of an argument and the use of deduction in problem solving

Idea. Use logical puzzles

- They're fun and they help students learn that deductive reasoning consists of chains of inferences.
- Many introduce reasoning by contradiction (certain possibilities lead to contradictions and can therefore be eliminated).
- They can be created for every K-12 grade and formulated to advance the content goals for that grade.

Logical puzzles help students reach many of the Mathematical Practice goals: • Make sense of problems and persevere in solving them • Reason abstractly and quantitatively • Construct viable arguments and critique the reasoning of others • Use appropriate tools strategically • Attend to precision.

Exercise

CCSS: Number and Operations in Base Ten

Meg's number is a multiple of 7.
Meg's number has two digits.
The digit in the tens place of Meg's number is either 9 or an even number.
If the digits of Meg's number are reversed, a smaller number is obtained.
The sum of the digits of Meg's number is greater than 9.
The product of the digits of Meg's number is less than 40.
What is Meg's number?

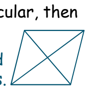
An exercise like this can be used as a group activity. Each student in the group is given one of the clues, and the group as a whole is to find the answer. Group members are supposed to tell their clue to the other members of the group without actually showing it to them.

Help students understand the special ways language is used in mathematics

Ex. **Parent:** If you don't eat your dinner, then you won't get dessert.
Child: Eats dinner and asks for dessert. Not a real-world parent!
Parent: No!. . . I didn't tell you what would happen if you ate your dinner, only what would happen if you didn't eat your dinner.

Ex. If the diagonals of a parallelogram are not perpendicular, then it is not a square. True

Okay, so if the diagonals of a parallelogram are perpendicular, then it is a square. Right?



Well no. Although the first statement is true, the second is false; the parallelogram could be a non-square rhombus.

In ordinary speech, "If *A* then *B*" is often interpreted to mean **both** "If *A* then *B*" **and** "If not *A* then not *B*." … But not in mathematics! Yet to prove or disprove a given statement, we have to know what it means.

An Equity Issue

Students who are exposed to sophisticated use of language at home may come to understand subtle linguistic distinctions (like the difference between an if-then statement and its inverse) without explicit instruction at school.

Example: An interchange between a five-year-old boy, Jacob, and his mother when Jacob was supposed to go with his father to get his hair cut. Both Jacob's mother and father are attorneys.

Jacob: What will you give me if I get my hair cut?

Jacob's mother: Jacob, if you get your hair cut, I'll let you live.

Jacob (wide-eyed): Does that mean that if I don't get my hair cut you won't let me live?

Jacob's mother: Of course not!

Math classes can also offer examples in ordinary language:
If it's raining, then the ground is wet.
So if the ground is wet, then it's raining. Right?

Math exercises for "if-then," "for all," and "for some"

a. It's true that if an number is irrational then it has a nonterminating decimal expansion. So is it also true that if a number has a nonterminating decimal expansion, then it is irrational? Why or why not? No!

b. It's false that "All rational numbers are integers." So is it true that "No rational numbers are integers"? Why or why not? No!

c. It's true that "For all positive numbers x, if $x < 1$ then $x^2 < x$." So is this also true: "For all positive numbers x, if $x^2 \nless x$ then $x \nless 1$"? Why or why not? Yes!

In grades 7-12, exercises to increase students' ability to work with these kinds of statements can help them reach CCSS goals for The Real Number System while contributing to all the Mathematical Practice goals.

Exercises: Evaluating truth and falsity of statements that involve words like "all," "some," "if-then," "and," and "or."

Ex 1. 

True or false?

a. All the gray objects are small.
b. All the small objects are gray.
c. No square objects are white.
d. Some white object is larger than every gray object.
e. If an object is gray, then there is a black object next to it.
f. There is a black object that has all the gray objects next to it.
g. If an object is small, then it is gray or white.

An exercise like this can help even very young students reach both CCSS Geometry Content goals and Mathematical Practice goals.

Ex 2. Sam claims the equation $(a + b)^2 = a^2 + b^2$ is true for all values of a and b. Ann disagrees. Who is right? Justify your answer.

Students can use a counterexample to disprove an "all" statement long before they can prove a general "all" statement.

An exercise like this can help students reach both CCSS Expressions and Equations goals and Mathematical Practice goals.

Help students understand the idea of generalizing from the generic particular – the basic idea of proof

Ex. Will the answer always be 5 regardless of what number is put into the empty box? Why or why not?

Verbal (Mental Mathematics)	Numerical (Example)	Visual (Manipulative/ Pictorial)	Algebraic (Abstract)
Think of a number from 1 to 10.	7	□	n
Add 7.	14	□ ● ● ● ● ● ● ●	$n + 7$
Multiply by 2 (double it).	28	□ ● ● ● ● ● ● ● □ ● ● ● ● ● ● ●	$2n + 14$
Subtract 4.	24	□ ● ● ● ● ● □ ● ● ● ● ●	$2n + 10$
Find 1/2 of the result.	12	□ ● ● ● ● ●	$n + 5$
Subtract the original number.	5	● ● ● ● ●	$n + 5 - n$
Write your answer.	5	● ● ● ● ●	5

Based on: Davis, S. and Thompson, D. R. To encourage "algebra for all," start an algebra network. The Mathematics Teacher. Apr 1998. Vol. 91 (#4), p. 282

Exercise: Express a mathematical statement in a variety of different ways

Ex. Given: **The square of any odd integer is odd.**

Two of the statements below do not mean the same as the given statement. Which are they?

a. All odd integers are squares of odd integers.

b. Any square of an odd integer is odd.

c. If x is an odd integer, then x^2 is odd.

d. Every number that is a square is nonzero.

e. For every odd integer x, x^2 is odd.

An exercise like this helps develop mental flexibility and contributes to several Mathematical Practices: • Model with mathematics • Use appropriate tools strategically • Look for and make use of structure • Attend to precision. The subject matter can be any mathematical content.

What's Going On Here??

Def: For all integers n, n is even if, and only if, $n = 2k$ for some integer k.

For all integers n, n is odd if, and only if, $n = 2k + 1$ for some integer k.

Theorem: The sum of any even integer and any odd integer is odd.

Frequently proposed "proof":

Suppose m is any even integer and n is any odd integer. By definition of even and odd, $m = 2k$ and $n = 2k + 1$, and so

$$m + n = 2k + (2k + 1) = 4k + 1 \quad \ldots \text{etc.}$$

Fact: In each definition, the variable k is just a placeholder!

Variables Are Just Placeholders: Two Main Uses

Variables are the "John Doe"s of mathematics. We use a variable as a **placeholder** when we want to talk about a quantity that is

A. Fixed but unknown: We know or hypothesize that a quantity has certain values but we don't know what they are.

Ex. Solve $4x + 20 = 0$. In other words, find a number that when put in place of x makes $4x + 20 = 0$.

Ex. When you know that a number is even, you know that it equals twice *some* integer. So you can use a variable like k (or r, or c, etc.) to hold a place for that integer even though you don't know what it is.

These examples are related to CCSS goals for • Expressions and Equations • Reason abstractly and quantitatively. • Model with mathematics. • Use appropriate tools strategically. • Construct viable arguments • Attend to precision.

A variable is like a "John Doe." We use it as a placeholder when we want to talk about a quantity that is

B. Individual but generic: We want whatever we say about it to be equally true for all elements in a given set.

Ex. If we write the commutative law as $a + b = b + a$, we mean that this equation is true no matter what real numbers are put in place of a and b.

Ex. We can define the squaring function by writing $x \rightarrow x^2$. This means that when any real number is put in place of x, the function will associate the number x^2.

Ex. To prove that the square of any odd integer is odd, we can let n hold a place for a "generic" odd integer and then deduce that n^2 is odd.

These examples are related to CCSS goals for • Expressions and Equations • Functions • Reason abstractly and quantitatively. • Model with mathematics. • Use appropriate tools strategically. • Construct viable arguments • Attend to precision.

Variables: What We Say vs. What We Mean

What we say	What we mean
the value of x	the quantity that is put in place of x
as the value of x increases	as larger and larger numbers are put in place of x
as the value of x increases, the value of y increases	If larger and larger numbers are put in place of x, the corresponding numbers that are put in place of y become larger and larger
where x is any real number	for all possible substitutions of real numbers in place of x
Let n be any even integer.	Imagine substituting an integer in place of n but do not assume anything about its value except that it is even.
By definition of even, $n = 2k$ for some integer k.	By definition of even, there is an integer we can substitute in place of k so that the equation $n = 2k$ will be true. (Actually there is exactly one such integer; its value is $n/2$.)
the function x^2	the function that relates each real number to the square of that number. In other words, for each possible substitution of a real number in place of x, the function corresponds the square of that number.
where x is some real number that satisfies the given property	There is a real number that will make the given property true if we substitute it in place of x.
A general linear function is a function of the form $f(x) = ax + b$ where a is any real number and b is any real number.	A general linear function is a function defined as follows: for all substitutions of real numbers in place of a and b, the function relates each real number to a times that number plus b.

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For Completeness: Other Needed Logic

- Meaning of and, or, not, if-then, if-and-only-if, for-all, there-exists
- Ability to form negations *[To evaluate the truth of a (mathematical) statement, one needs to know what it means for the statement to be true and what it means for the statement to be false.]*
- Modus ponens (aka Law of Detachment): if p then q, p, ∴q
- Transitivity: [if p then q] and [if q then r], ∴ [if p then r] (aka chaining, hypothetical syllogism)
- Non-equivalence between a conditional statement and its converse/inverse
- Equivalence between a conditional statement and its contrapositive
- Conditional proof (Assume p, deduce q. Conclude: if p then q)
- Generalizing from the generic particular, Universal instantiation, Existential instantiation, Disproof by counterexample
- Proof by contradiction

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Quote

The mathematics profession as a whole has seriously underestimated the difficulty of teaching mathematics.

Ramesh Gangolli
MER Workshop
May 31, 1991

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Some References

1. The Role of Logic in Teaching Proof. *The American Mathematical Monthly* (110)10, 2003.
2. The Language of Quantification in Mathematics Instruction. In Developing Mathematical Reasoning in Grades K-12. Frances R. Curcio and Lee V. Stiff, Eds. Reston, VA: National Council of Teachers of Mathematics, 1999, 188-197.
3. Variables in Mathematics Education. Reprinted in The Best Writing on Mathematics 2012, M. Pitici, Ed., Princeton Univ. Press. (Originally published in Tools for Teaching Logic. Blackburn, P. et al., Eds. Springer Publishing, 2011.)
4. A Unified Framework for Proof and Disproof. The Mathematics Teacher 91 (8) November 1998, 708-713.

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Thank you!

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