Teaching Logarithms for Understanding

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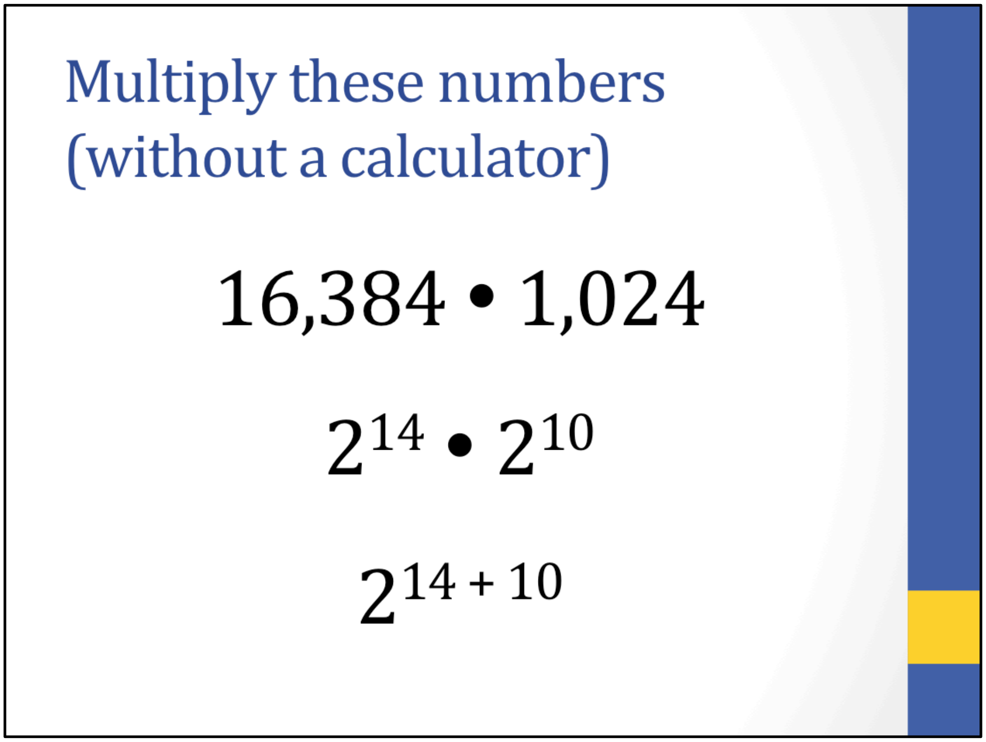
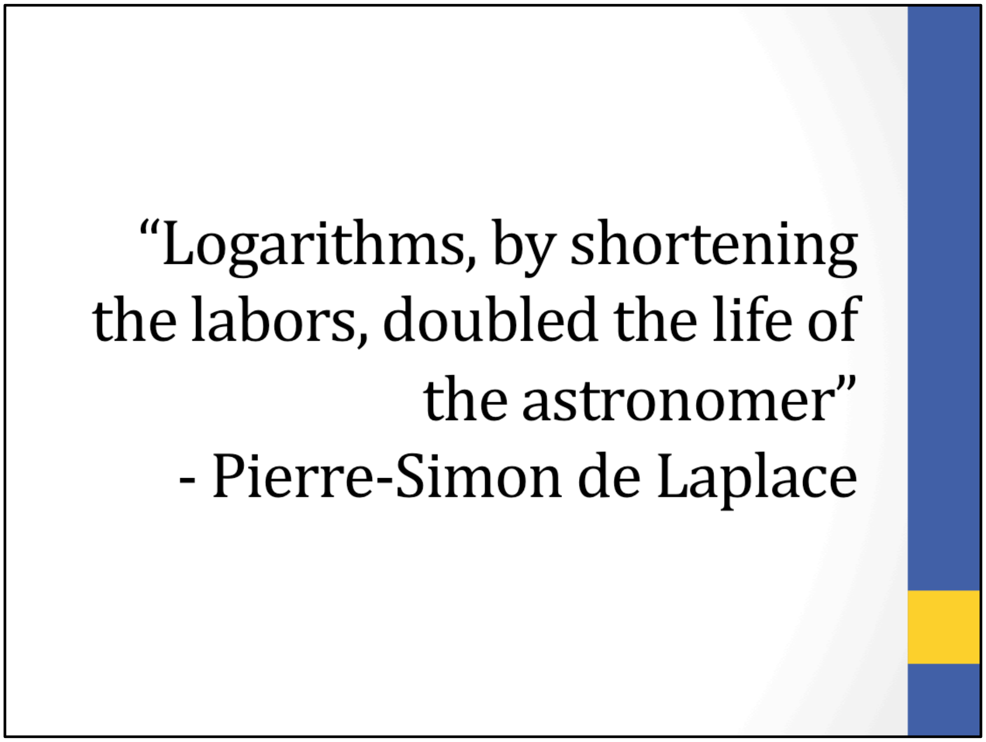
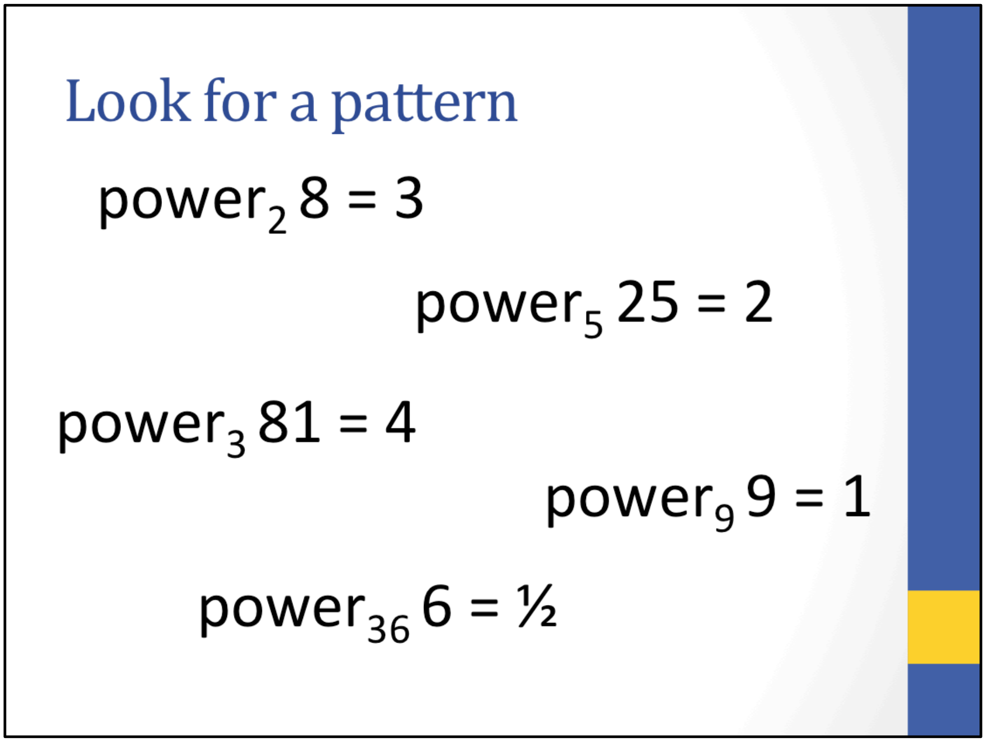
I never really liked how I taught logarithms. It always seemed contrived. And, my calculus students complained about logarithms because they never remembered the properties AND they didn't really know why we used them. We decided to try something different.



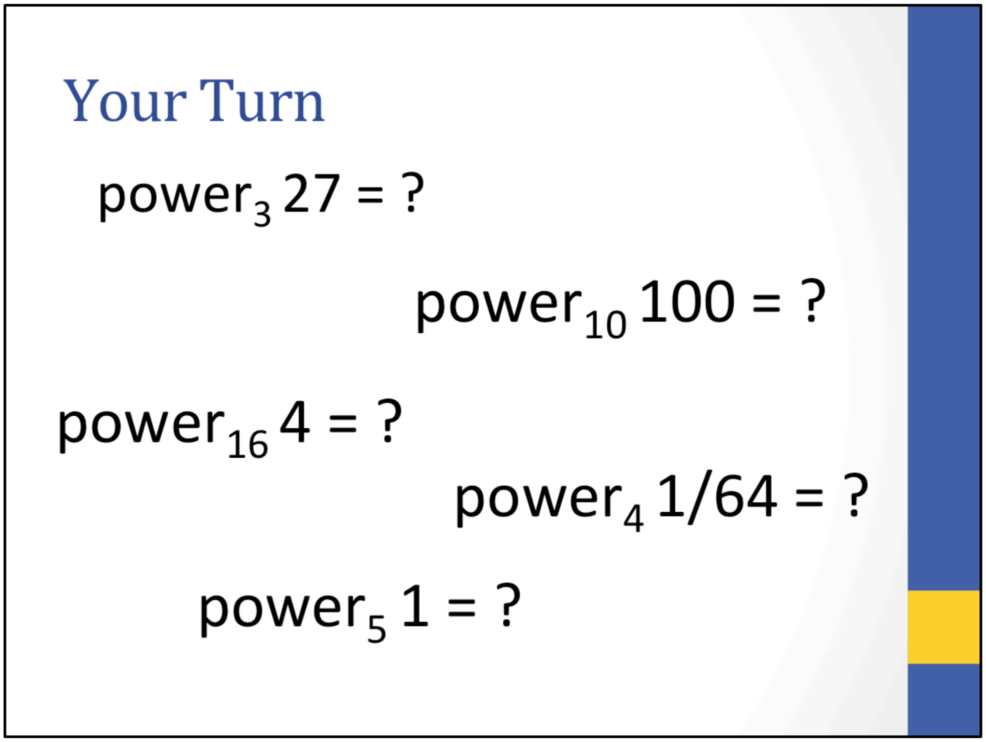
This is the hook. Why were logarithms invented in the first place? Because Napier was trying to help out by inventing a way to turn big numbers into small numbers and turn complicated multiplication into easier addition. It was about a hundred years later when someone realized that what Napier done amounted to basically the inverse of exponentiation.



The whole purpose of logarithms was to make computation easier. Even if students don’t care about the actual computations, they can at least empathize with wanting to to make your life easier.



We shamelessly cribbed this activity from a James Tanton curriculum essay. We liked it a lot, but had to adapt some of the numbers for our group. We had a class that was 75% students with IEPs, a mixture of Learning Disabilities and Emotional Disturbances. We knew they could do it, but we needed to meet them where they could be successful and then ramp up the difficulty.



Take a second and collaborate with a neighbor.

Your Turn

$\text{power}_3\, 27 = 3$

$\text{power}_{10}\, 100 = 2$

$\text{power}_{16}\, 4 = ½$

$\text{power}_4\, 1/64 = -3$

$\text{power}_5\, 1 = 0$

Fix these with the blanks we had from last year

Terminology

$\text{power}_2\, 8 = 3$

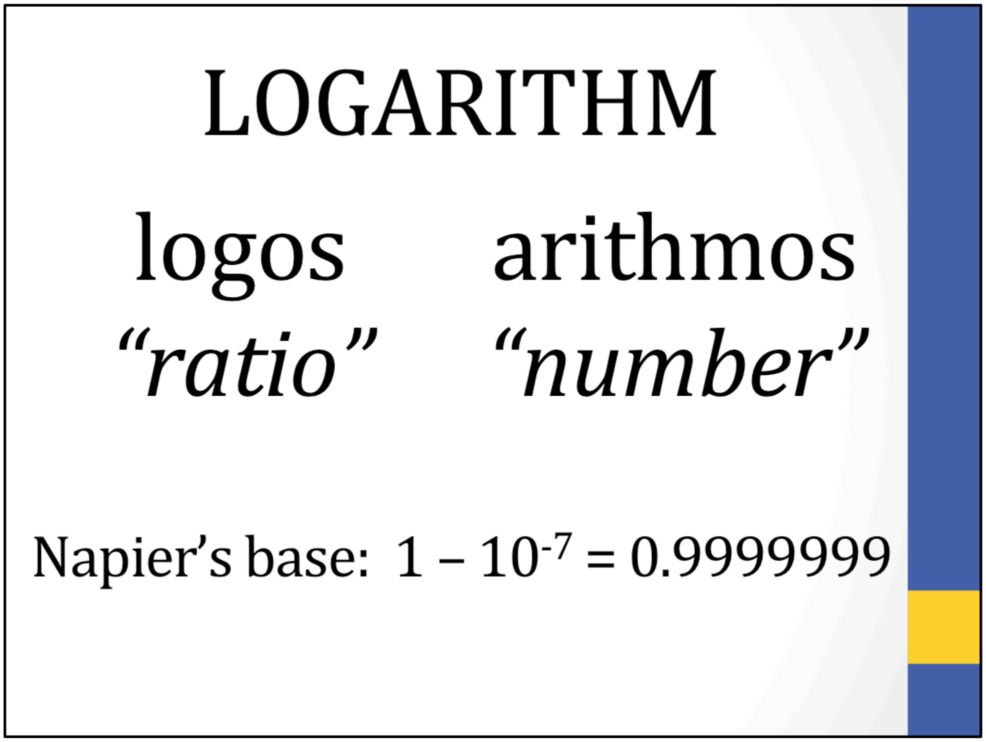
$\text{power}_5\, 25 = 2$

$\text{power}_3\, 81 = 4$

$\text{power}_9\, 9 = 1$

$\text{power}_{36}\, 6 = ½$

For historical reasons, we don't use the word "power" for the relationship shown (although it would be nice if we did). So, the usual way to write these is with the abbreviation "log".



Napier made up the word “Logarithm” from the Greek words for “ratio” and “number”. He was using a proportional relationship to compute his original logarithm table which used the base 1-10^-7. Which seems weird to us, but obviously made sense to him somehow.

Terminology

$\log_2 8 = 3$

$\log_5 25 = 2$

$\log_3 81 = 4$

$\log_9 9 = 1$

$\log_{36} 6 = ½$

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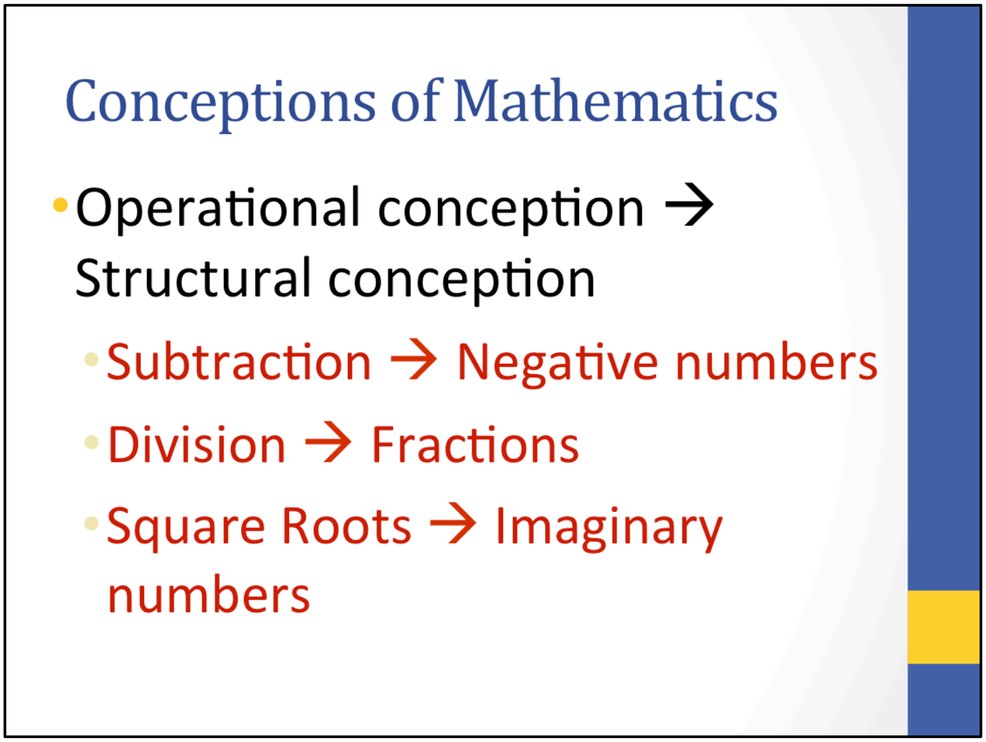
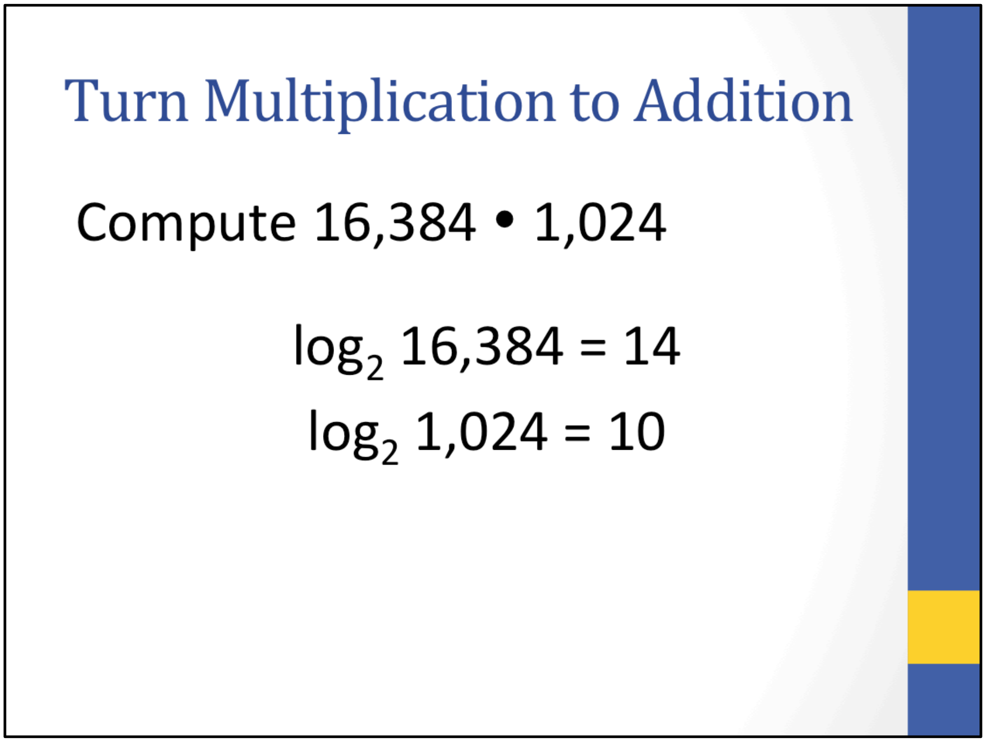
Why this approach?

- Structural and Operational conceptions of mathematical concepts (Sfard, 1991)
- Operational: dynamic, sequential, detailed
- Structural: static, instantaneous, integrative

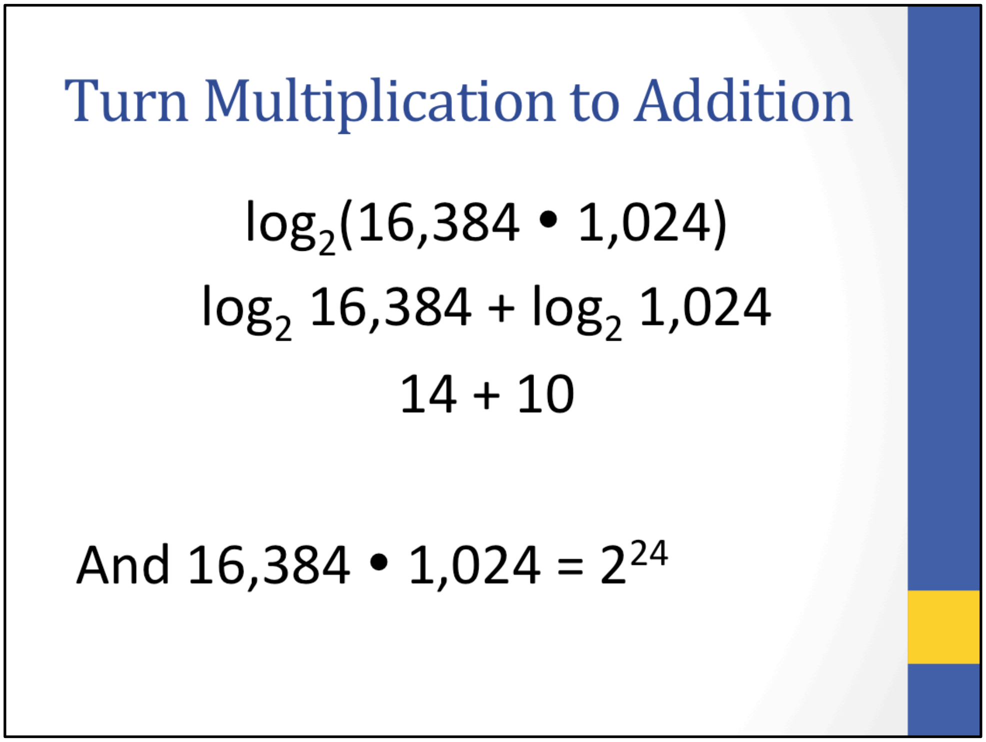
Why did we try this approach? One of the fascinating and frustrating parts of mathematics is that most concepts can be thought of in both structural and operational terms. This is similar to the procedural / conceptual dichotomy that other people have noticed and wrote about. Sfard claims that these two approaches are complementary: you can't have one without the other! Operational conceptions are action oriented – something to do, a list of instructions. Structural conceptions are object oriented – something to observe, a connected whole.



Historically, operational conceptions almost always precede the structural ones. For example, the action of subtracting a larger quantity from a smaller one led to negative numbers. The function concept was operational for a long time before mathematicians settled on the structural definition. The modern definitions of nearly every concept are structural in nature. We treat logarithms like an object. At my first job, the mantra among the Alg2 teachers was, “a log is an exponent”. But, Napier clearly had an operational conception of his invention. He conceived logarithms as a way to convert large numbers into small values that would be easier to compute.



We can simplify our computations by choosing a convenient base, converting the large numbers to their exponents.



This is the fundamental property of logarithms that Napier wanted to create. We can justify this by saying that if we're thinking about the exponents, then the multiplication of the values can be performed by adding the exponents. Then anti-logarithming back to the base with the new exponent. Sure, it's a lot of steps and there are a decent number of moving parts but it's still way easier than actually doing the multiplication.

Your Turn

- Compute 1024 • 4096

- Compute 3125 • 125

Your Turn

- Compute 1024 • 4096

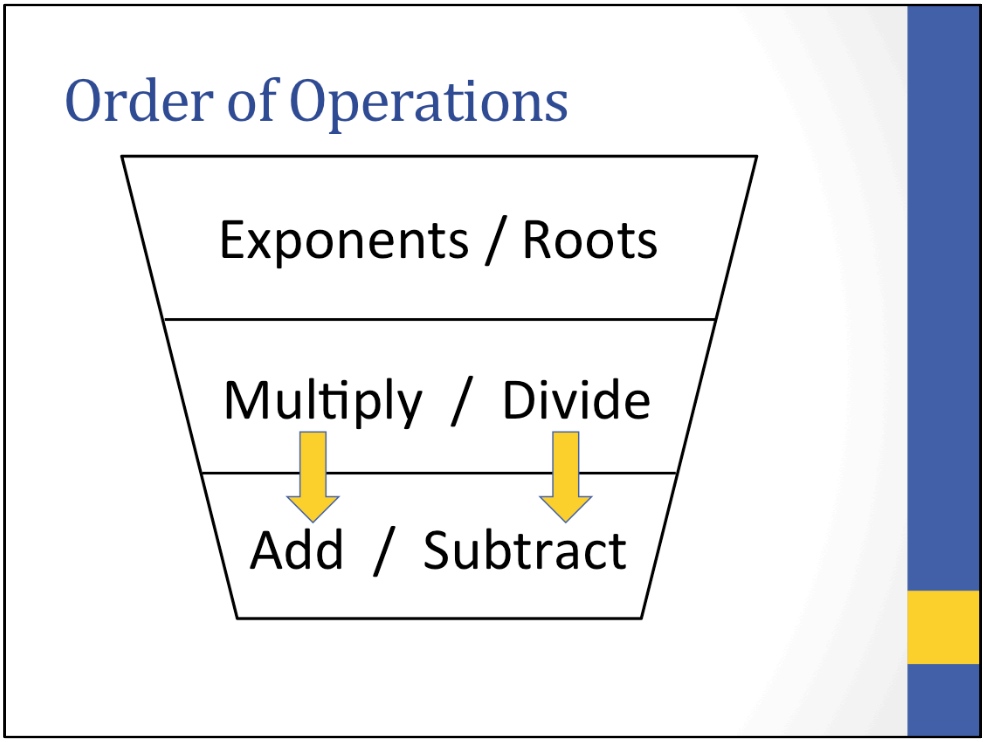
  $\log_4 1024 + \log_4 4096 = 11$

  $1024 \cdot 4096 = 4^{11}$

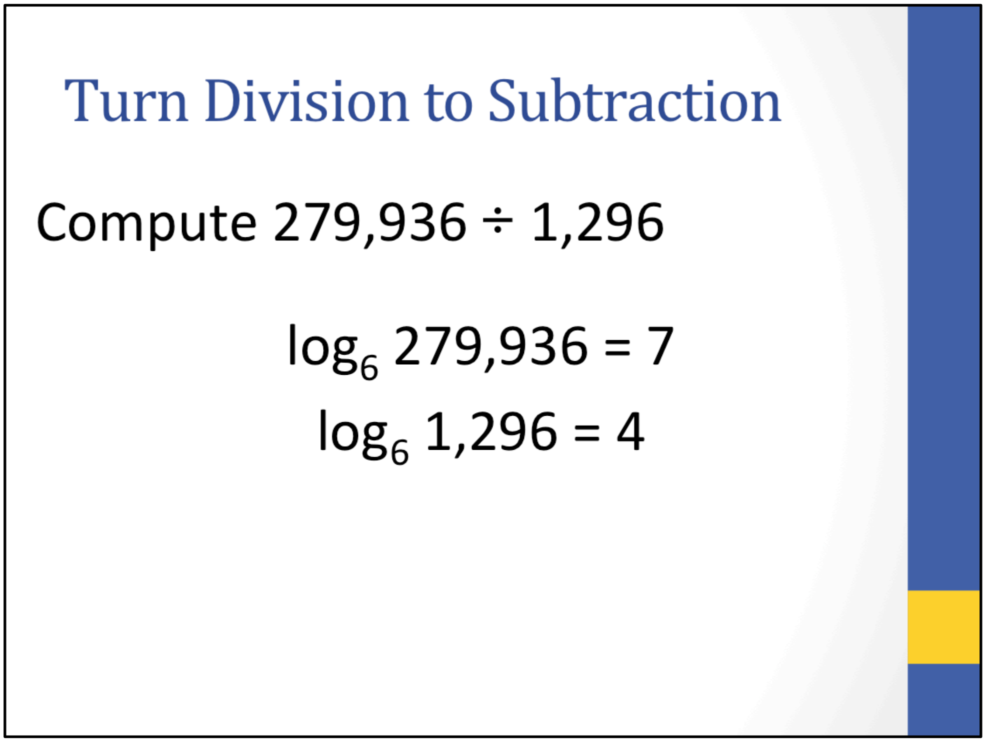
- Compute 3125 • 125

  $\log_5 3125 + \log_5 125 = 8$

  $3125 \cdot 125 = 5^{8}$



We try to encourage understanding throughout the course. We use this graphic organizer for order of operations because is helps students see inverse relationships between the operations and it helps to evoke some thoughts about a hierarchical relationship among the operations. It's good for visualizing when a distributive (or distributive-like) property holds. Once students start thinking about how a major goal of logarithms is to turn multiplication into addition, then we can ask, "what do you think a log will turn division into?" Most students can guess subtraction.



Division to subtraction is basically the same procedure as multiplication to addition. Choose a convenient base, convert the large numbers to their exponents, perform the subtraction, do the anti-logarithm back to the base. Of course, for Napier, there was only one base and it was that crazy 0.999999. And you had to look up the values in a logarithm table. And the process was considerably more complicated but still less challenging than trying to brute force your way through a difficult computation.

Turn Division to Subtraction

$$\log_6 (279{,}936 \div 1{,}296)$$

$$\log_6 279{,}936 - \log_6 1{,}296$$

$$7 - 4$$

And $279{,}936 \div 1{,}296 = 6^3$

Division to subtraction is basically the same procedure as multiplication to addition. Choose a convenient base, convert the large numbers to their exponents, perform the subtraction, do the anti-logarithm back to the base. Of course, for Napier, there was only one base and it was that crazy 0.999999. And you had to look up the values in a logarithm table. And the process was considerably more complicated but still less challenging than trying to brute force your way through a difficult computation.

Your Turn

- Compute 2187 ÷ 729

- Compute 49 ÷ 16807

Your Turn

- Compute 2187 ÷ 729

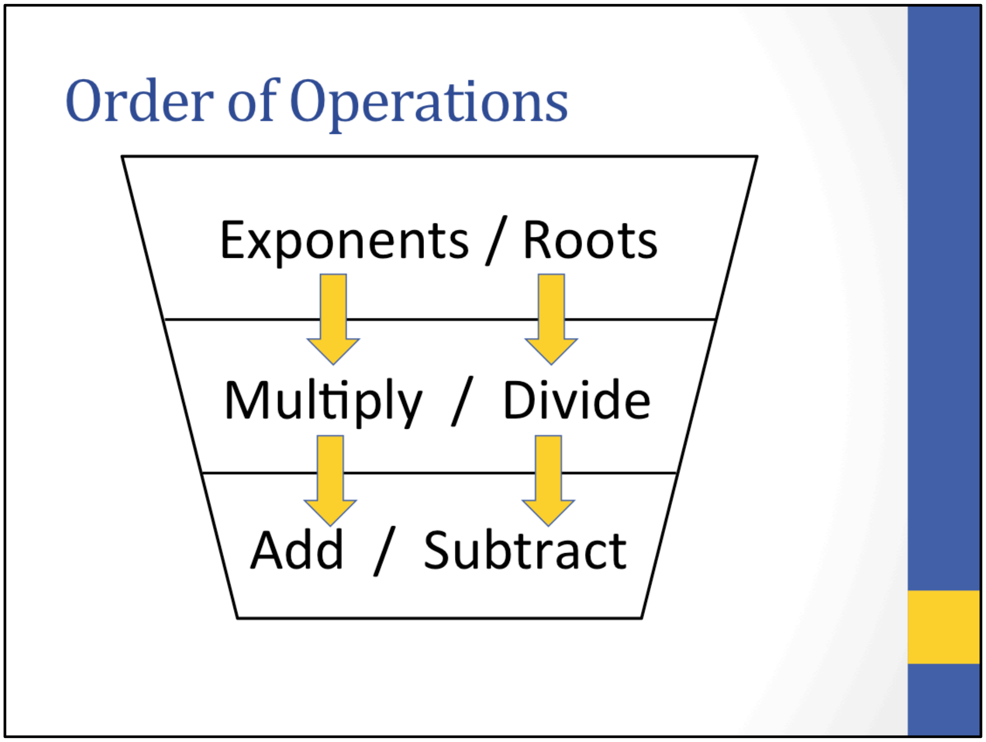
  $\log_3 2187 - \log_3 729 = 1$

  2187 ÷ 729 = 3

- Compute 49 ÷ 16807

  $\log_7 49 - \log_7 16807 = -3$

  $49 \div 16807 = 7^{-3} = 1/7^3$



Back to the graphic organizer: We can also ask students what they think a logarithm might do to an expression that involves an exponent. Turn exponents into multiplication? Yes indeed! In fact, it would not be at all strange if a student asked that question while looking at the diagram. We've realized that students will spontaneously start making conjectures and connections when the story has a tendency to unfold itself.

Turn Exponents to Multiplication

$$\log_3(16)$$

$$\log_3 (2^4)$$

$$\log_3 (2 \cdot 2 \cdot 2 \cdot 2)$$

$$\log_3 2 + \log_3 2 + \log_3 2 + \log_3 2$$

$$4 \log_3 2$$

Therefore, $\mathbf{\log_3 2^4 = 4 \log_3 2}$

Your Turn

- Rewrite: $\log_4 625$
- Rewrite: $\log_{10} 1024$

Your Turn

- Rewrite: $\log_4 625$

  $4 \log_4 5$

- Rewrite: $\log_9 1024$

  $10 \log_9 2$

That's not right…

$$817\sqrt{29}$$

$$\log\left(817\sqrt{29}\right)$$

$$\log 817+\frac{1}{2}\log 29$$

But wait, you're saying. That's not really how logarithms were used. After all, not all numbers are nice integer powers of another number. What if you have something to compute like 817 times the square root of 29. If it was before the 1970s when hand held calculators started to become affordable, you would have used the log properties to make your calculation something like this. Then consult a log table.

Log Tables

N	0	1	2	3	4	5	6	7	8	9
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289

Because 100 < 817 < 1000,

log 817 = 2.****

log 817 = 2.9122

Look up 81 then move to the 7 column for 817 for the decimal part (the mantissa)

Log Tables

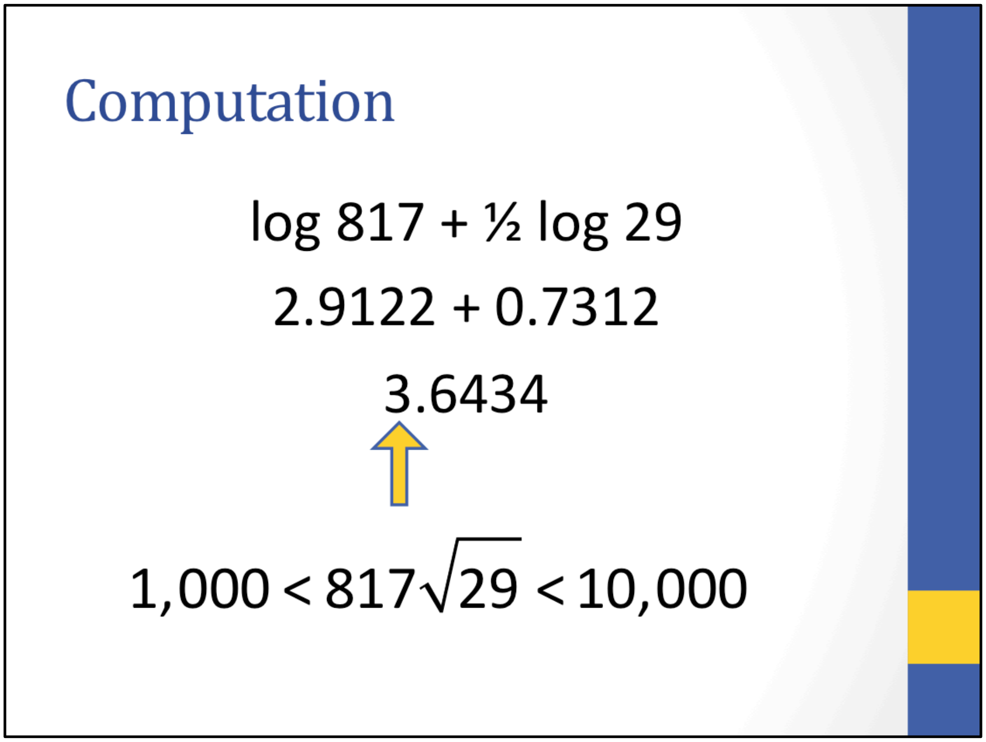
N	0	1	2	3	4	5	6	7	8	9
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757

Because $10 < 29 < 100$,

log 29 = 1.****

log 29 = 1.4624   ½ log 29 = 0.7312

Look up 29 then move to the 0 column for the decimal part of 29, then half it



Add the two values we obtained from the table. Now we have to go backwards. The 3 tells us that the answer is between 1,000 and 10,000. Then we look in the table for the 6434.

Anti-Logarithm

N	0	1	2	3	4	5	6	7	8	9
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522

$$817\sqrt{29} \approx 4400$$

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817√29
      4399.679647
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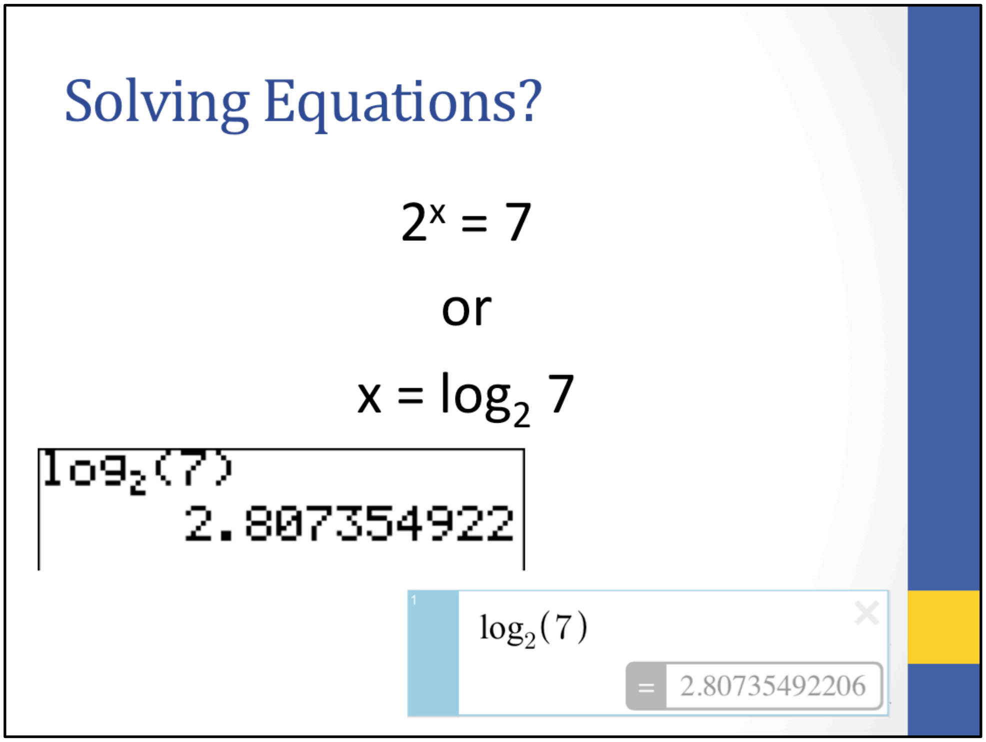
6435 is close enough for us. Then back track to see the values 44 and 0. This tells us that 817 sqrt 29 is approximately 4400 (because 4400 is between 1,000 and 10,000).



But with the advent of graphing calculators and web-based apps that will calculate roots and logs of all bases, why do we even need logarithms anymore? We used to have to use the “change of base” formula for this.

Usefulness of Logarithms

- Change big numbers into small numbers
- Change multiplication into addition
- Change division into subtraction
- Change exponentiation into multiplication

Solving Equations!

$$2^x = 7$$

$$\log 2^x = \log 7$$

$$x \log 2 = \log 7$$

$$x = \log 7 / \log 2$$

We can use the log properties to solve this equation because we want to use what logs can do – change exponentiation to multiplication – to get the variable out of the exponent spot

Solving Equations!

$$2^{x-4} = 3^{3x+1}$$

$$\log 2^{x-4} = \log 3^{3x+1}$$

$$(x - 4) \log 2 = (3x + 1) \log 3$$

$$x \log 2 - 4 \log 2 = 3x \log 3 + \log 3$$

$$x \log 2 - 3x \log 3 = \log 3 + 4 \log 2$$

$$x (\log 2 - 3 \log 3) = \log 3 + 4 \log 2$$

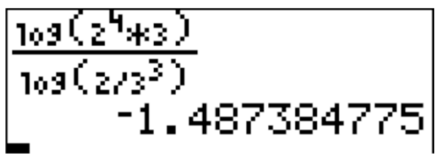
Maybe with a more complicated equation?

Solving Equations!

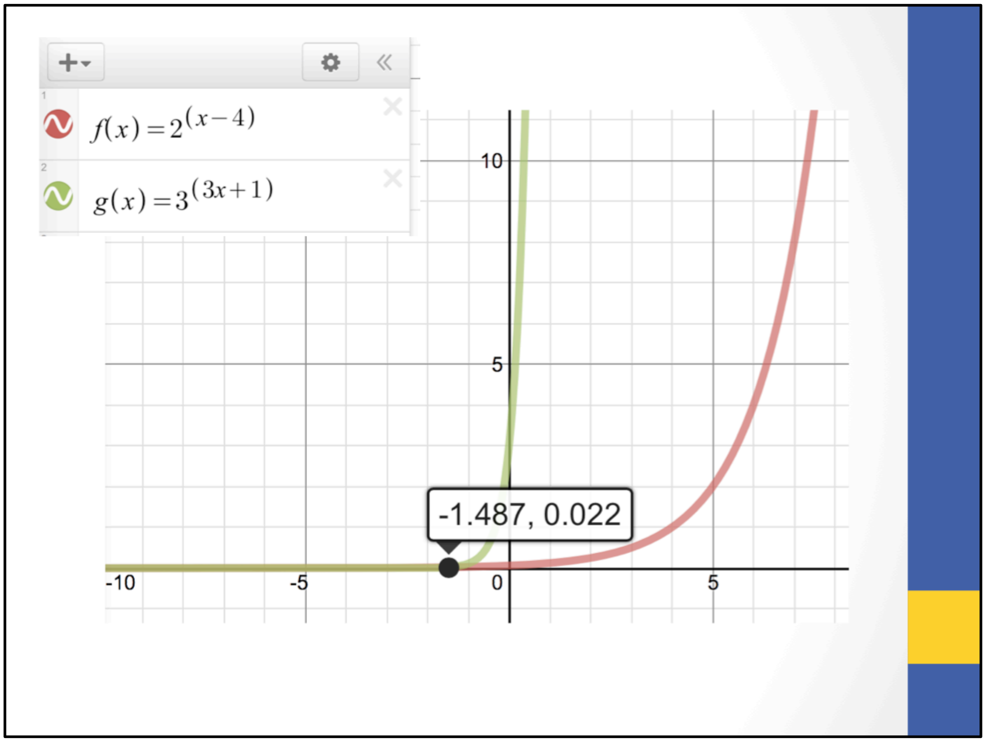
$$x(\log 2 - 3\log 3) = \log 3 + 4\log 2$$

$$x = \frac{\log 3 + 4\log 2}{\log 2 - 3\log 3}$$

$$x = \frac{\log 2^4 \bullet 3}{\log 2/3^3}$$



A little bit of log properties to condense the computations



But, you can get a decent decimal approximation using a graphing utility

Usefulness of Logarithms

- Statistics: smoothing out data
- Fast growing magnitudes
- Solving Differential Equations

Questions?

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