Intellectually Engaging Problems: The Heart of a Good Lesson

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Intellectually Engaging

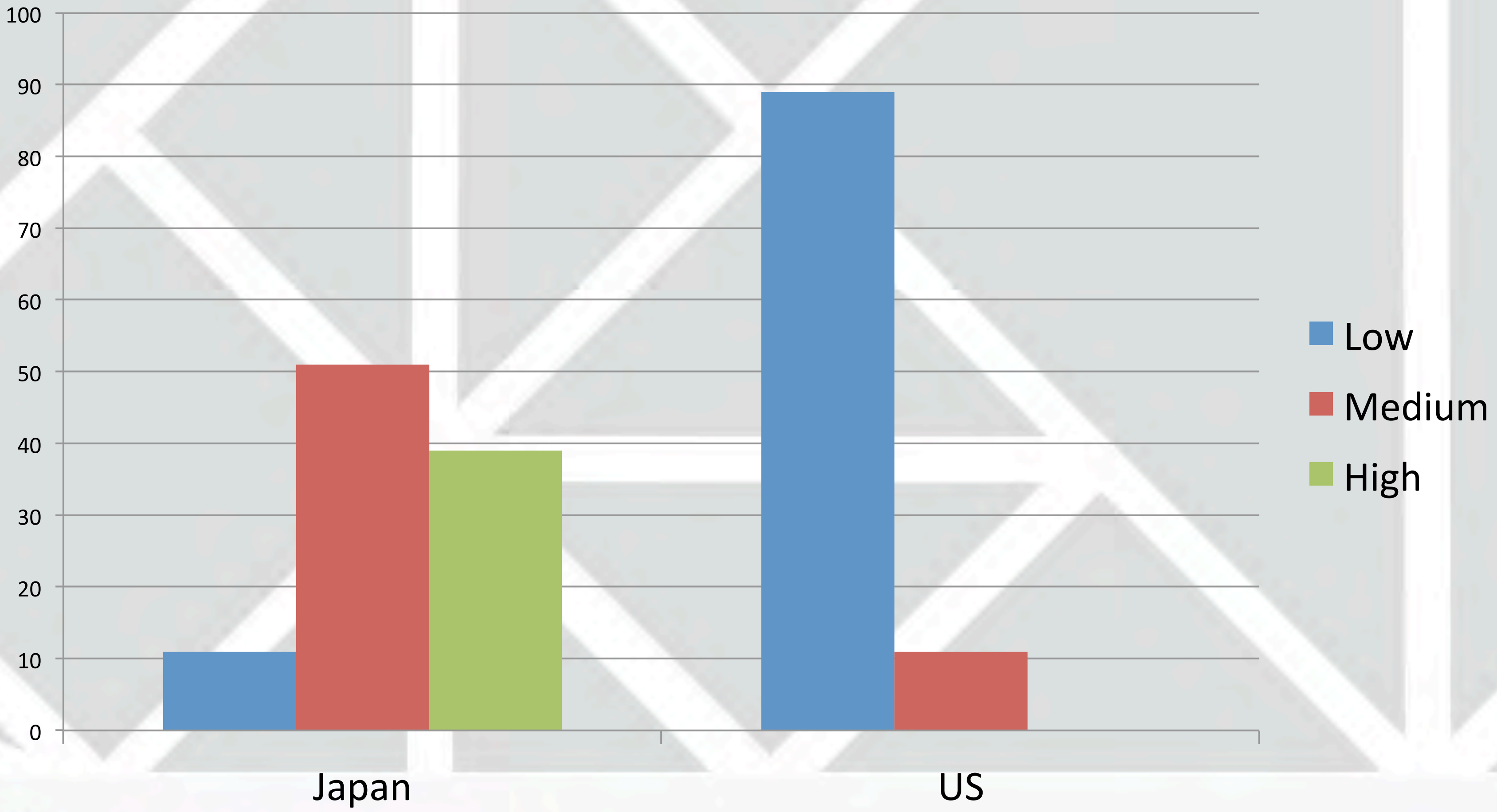
In comparing 8th grade mathematics instruction in the U.S. with high achieving countries, Hiebert and Stigler found "the *engagement of students in active struggle with core mathematics concepts and procedures*" was common among high achieving countries and "missing in the United States"

(Hiebert and Stigler, 2004, p. 12)

What Does This Intellectually Engaging Instruction Look Like in Japan

- The Teaching Gap
  - Video study of 8th grade classrooms in Japan, U.S., and Germany.
- Japanese Student Teaching

Teaching Gap Results
Overall Mathematical Quality



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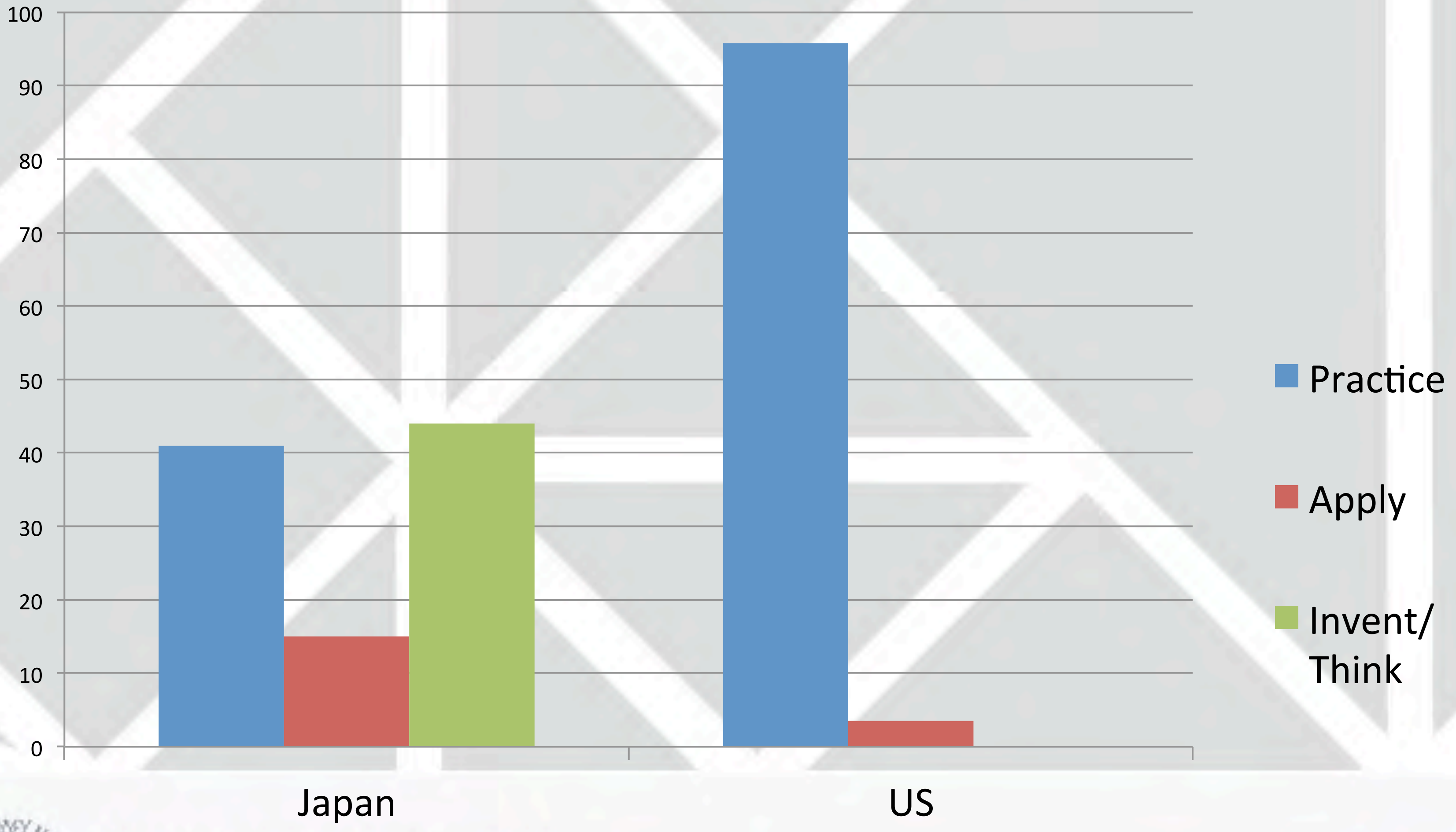
Teaching Gap Results
Work Done by Students



A Look at Japanese Student Teaching

- Japanese junior high school
- Three student teachers (ST)
- Three cooperating teachers (CT)
- Each ST taught three lessons once with each CT
- The ST and CT met for about 60 minutes across ~3 sessions prior to the teaching of each lesson
- These sessions were analyzed to see what the CT focused on as important to crafting a high-quality lesson

What do Cooperating Teachers Emphasize?

- Intellectual Engagement Principle
- Goal Principle
- Flow Principle
- Unit Principle
- Adaptive Instruction Principle
- Preparation Principle

Peterson, Blake E., Corey, Doug, Lewis, Ben M., & Bukarau, Jared (2013). Intellectual Engagement and Other Principles of Mathematics Instruction. *Mathematics Teacher* 106(6), 446-450

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Intellectual Engagement Principle

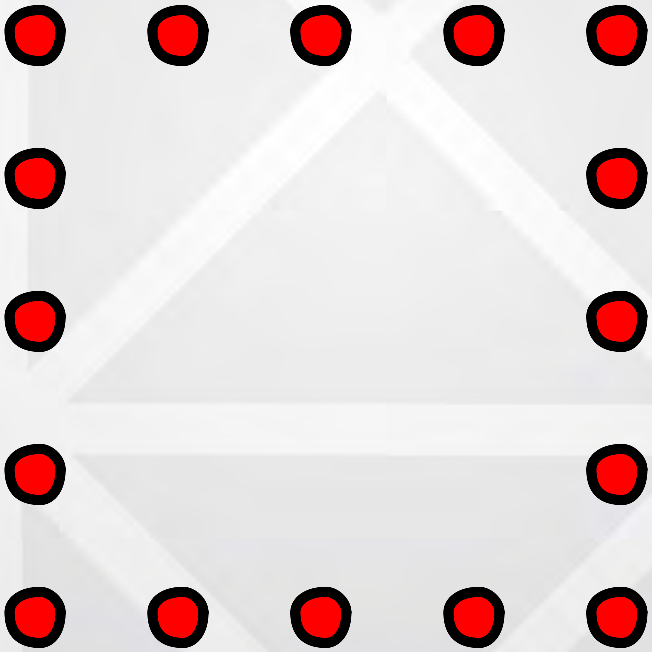
High-quality mathematics instruction intellectually engages students with important mathematics.

The emphasis by these 3 Japanese teachers seems to focus on *intellectual* engagement as opposed to just physical engagement or on-task behavior.

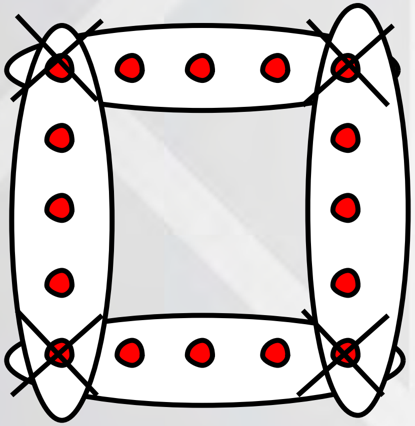
Questions

What kind of problems or tasks generate good intellectual engagement?

How many dots are in the figure below?

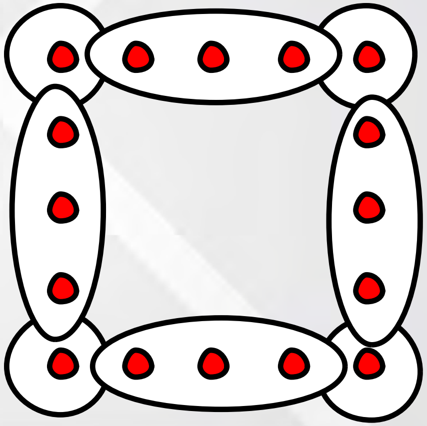


What arithmetic expression corresponds to how you counted or saw the dots?



4 x 5 - 4

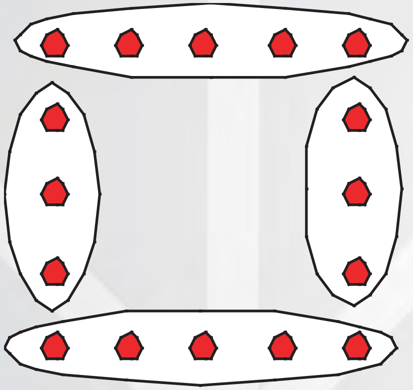
There are 5 dots on each side and there are 4 sides so that is 4 x 5. Since the corners are double counted, we subtract 4.



4 x 3 + 4

If we just count the dots on the interior of each side there are 3 dots on each of the 4 sides so that is 4 x 3. Now we add in the 4 corners.

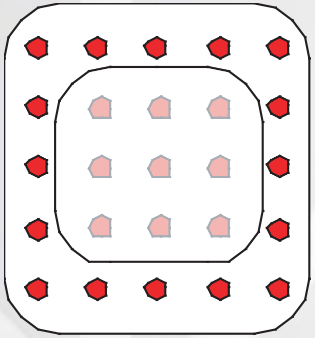
What arithmetic expression corresponds to how you counted or saw the dots?



$$\Rightarrow \left\{\begin{array}{l} 5+5+3+3 \\ 2\times 5+2\times 3 \\ 2\times 5+2\times (5-2) \end{array}\right. \qquad =16$$

You can see two sides with all of the dots (5) and the other two side would only include the interior dots (3). Thus we have 2x5 + 2x3 = 16

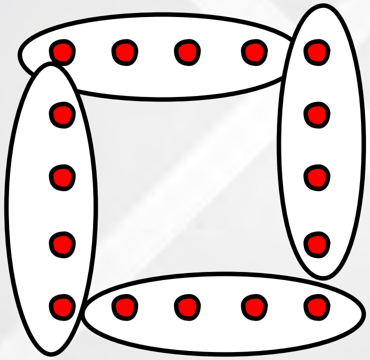
What arithmetic expression corresponds to how you counted or saw the dots?



$$\Rightarrow \begin{cases} 5\times 5-3\times 3 \\ 5^2-3^2 \\ 5^2-(5-2)^2 \end{cases} =16$$

You can fill in the entire square array of dots to have 5x5 dots but then need to subtract out the dots in the interior dots (3x3). Thus the total is 5x5 – 3x3 = 16.

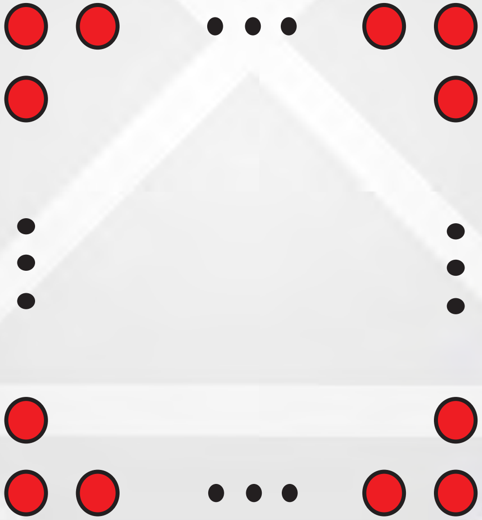
What arithmetic expression corresponds to how you counted or saw the dots?



4 x 4

In order to deal with the corners, we will just count 1 corner on each side or 4 dots on each side. With 4 sides, we have 4 x 4 dots.

How many dots would there be if there were *n* dots on a side?



Counting total dots on a square with *n* dots on each side.

$$2 \times 5 + 2 \times 3 \quad \Rightarrow \quad 2 \times n + 2 \times (n - 2)$$

$$\left.\begin{array}{l} 4 \times 3 + 4 \\ 4 \times (5 - 2) + 4 \end{array}\right\} \quad \Rightarrow \quad 4 \times (n - 2) + 4$$

$$4 \times 5 - 4 \quad \Rightarrow \quad 4 \times n - 4$$

Counting total dots on a square with *n* dots on each side.

$$\left.\begin{array}{l} 4\times 4 \\ 4\times(5-1) \end{array}\right\} \Rightarrow \quad 4(n-1)$$

$$\left.\begin{array}{l} 5\times 5-3\times 3 \\ 5^2-3^2 \end{array}\right\} \Rightarrow \quad n^2-(n-2)^2$$

What Mathematics Could Students Learn by Engaging with this Problem?

- Arithmetic expressions can/should have meaning – connecting the geometric situation to the arithmetic.
- Generalizing patterns – By identifying which numbers in the expressions are connected to the number of sides and which are connected to the number of dots on a side, they can see which numbers need to be replaced by an expression involving *n*.

What Mathematics Could Students Learn by Engaging with this Problem?

- Equivalent expressions – since all of the expressions represent the same value, they must be equivalent. Geometrically that equivalence becomes more obvious.
- Simplifying expressions – since the expressions $4(n-1)$ and $n^2-(n-2)^2$ are equivalent, I should be able to simplify one to get the other.

What makes this problem "Intellectually Engaging"?

- Multiple entry points
  - – Starts simple by counting dots but gets more complex when asked to generalize.
- Multiple solutions
  - – Students can be encouraged to find more than one solution.
  - – Students have opportunities to determine if the expressions are equivalent

How do the potential engagement of the two tasks below compare?

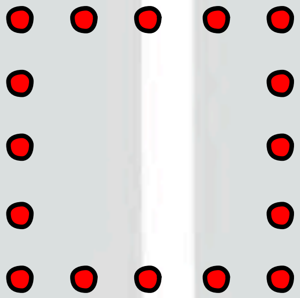
TASK 1

For the table find an equation for the *n*th term.

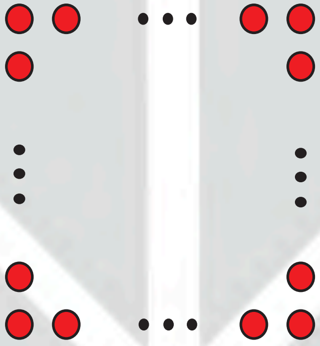
x	y
1	0
2	4
3	8
4	12
5	16
n	

TASK 2

A. Write an arithmetic expression the corresponds to the way you counted the dots in the figure at the right.



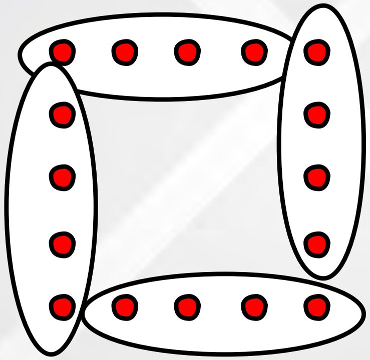
B. Describe the number of dots in the square figure shown below with *n* dots on a side.



Detail Matters!

- Why does the initial prompt have 5 dots on a side?
- Does geometric shape have to be a square?

A Surprising Result

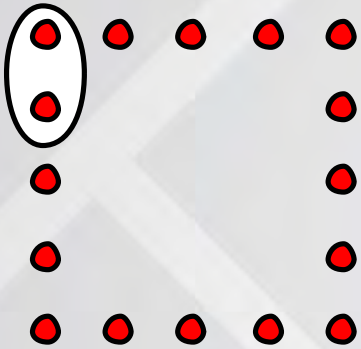


4 x 4

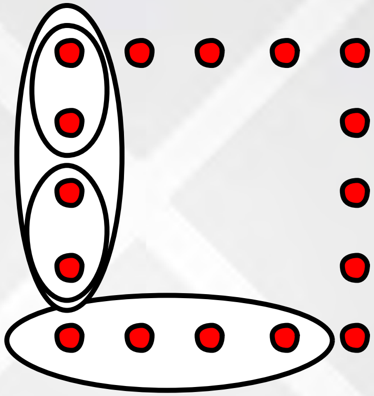
In order to deal with the corners, we will just count 1 corner on each side or 4 dots on each side. With 4 sides, we have 4 x 4 dots.

After several solutions were presented, a student said that she counted the dots using the arithmetic expression of 2^4

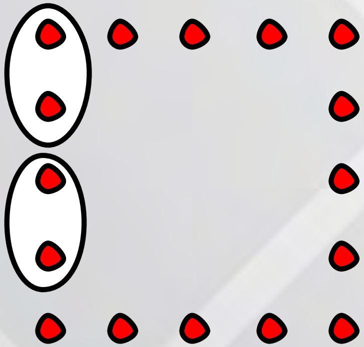
Can you see 2^4?



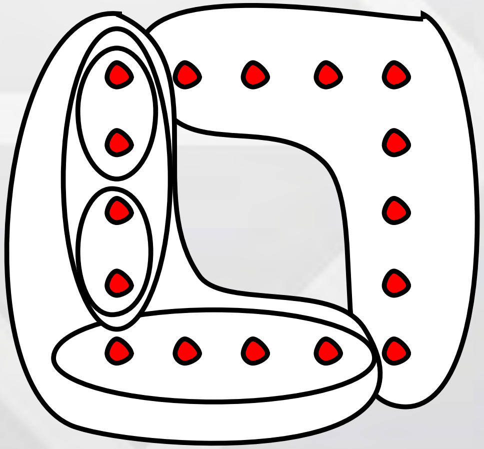
2 dots



2 groups of 2 groups of 2 dots



2 groups of 2 dots

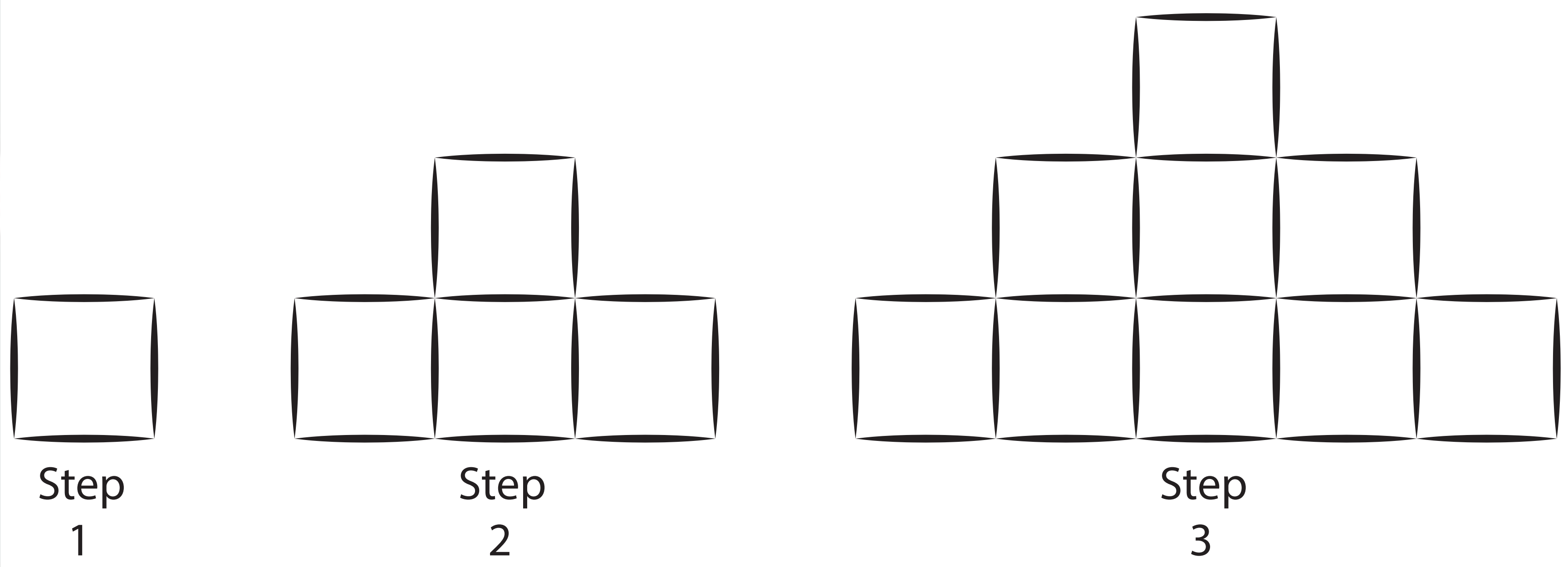


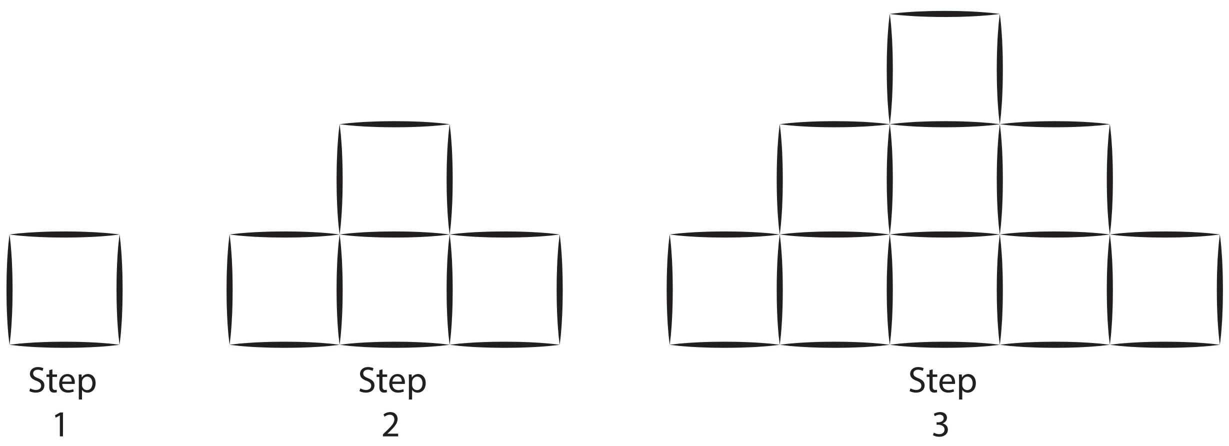
2 groups of 2 groups of 2 groups of 2 dots

For More Information

Peterson, Blake E. (2006) Counting Dots and Measuring Area: Rich Problems from Japan. *Mathematics Teaching in the Middle School, 12(4)*, 214-219

In the figures below, as the step changes, _________ also changes.

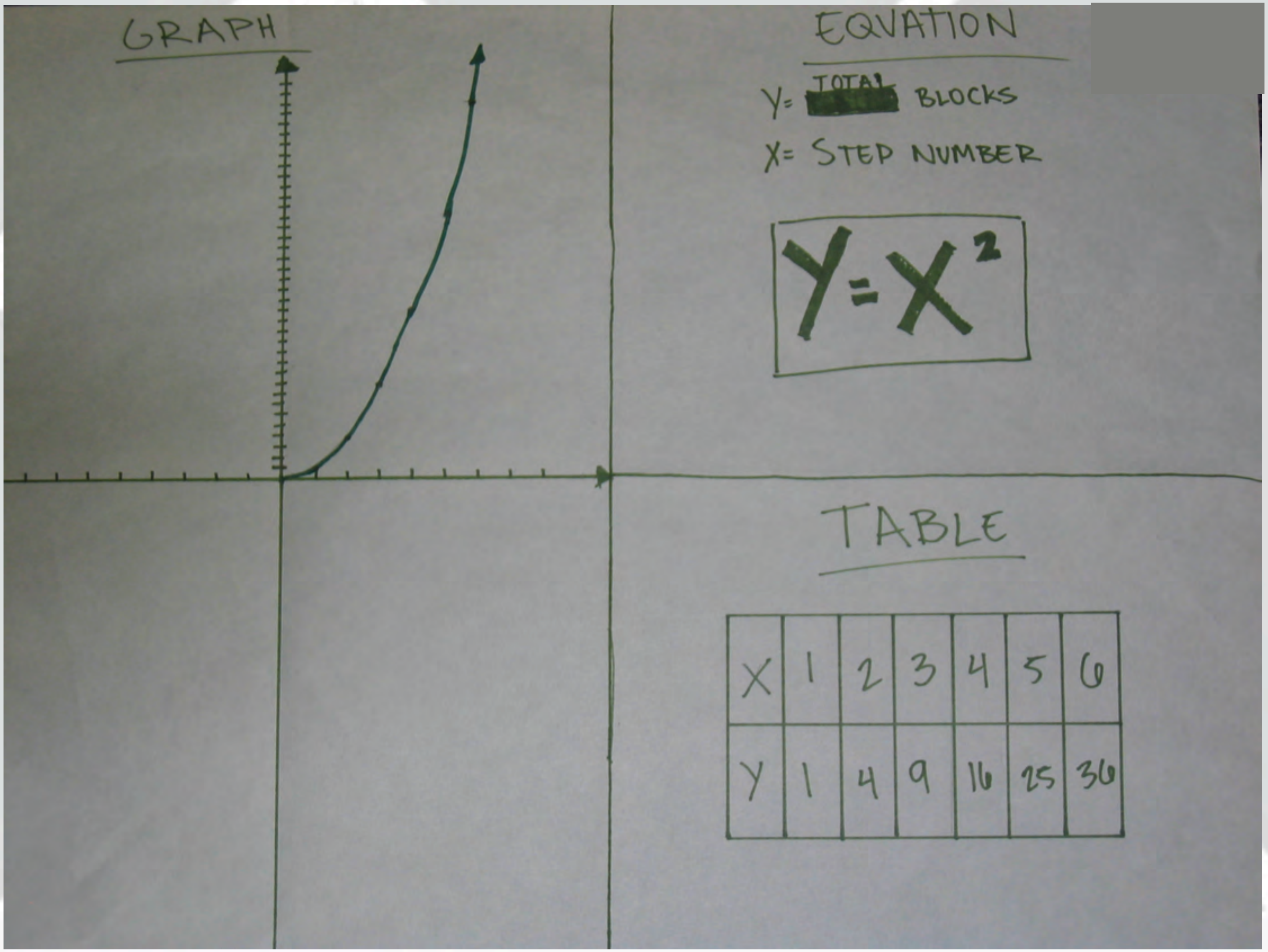




- Perimeter
- Height
- Width
- Size of enclosing rectangle
- # of “toothpicks”
- # of interior toothpicks
- # of intersections
- # of corners
- # of convex corners
- # of squares
- # of diagonals
- leftover space
- # of segments
- # of parallel lines
- Length of longest line
- # of rectangles

Using a table, graph and equation, describe the change.

Total Blocks (Area)



GRAPH

EQUATION

Y= TOTAL [redacted] BLOCKS

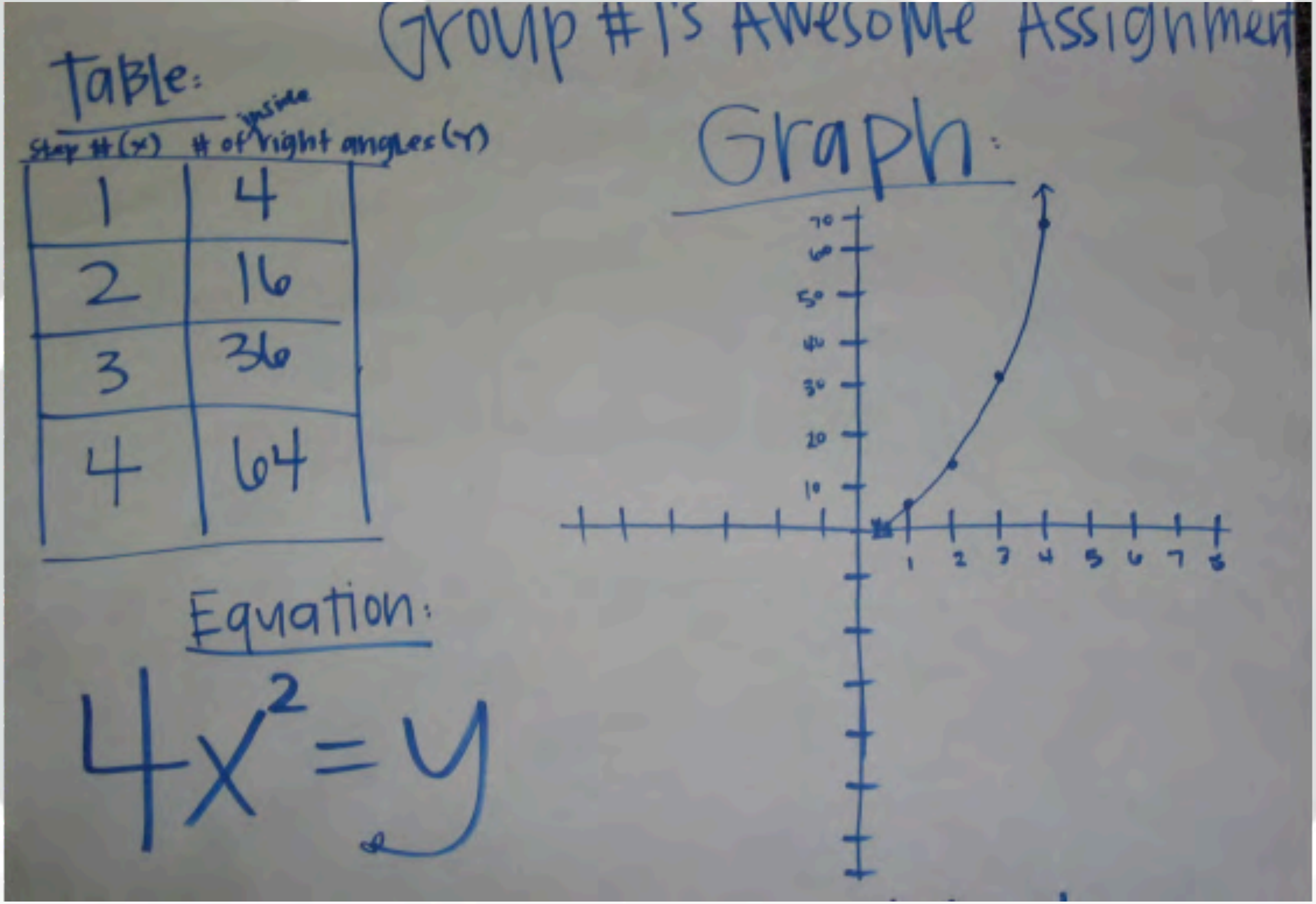
X= STEP NUMBER

$$\boxed{Y=X^2}$$

TABLE

X	1	2	3	4	5	6
Y	1	4	9	16	25	36

Number of inside right angles.



Group #1's Awesome Assignment

Table:

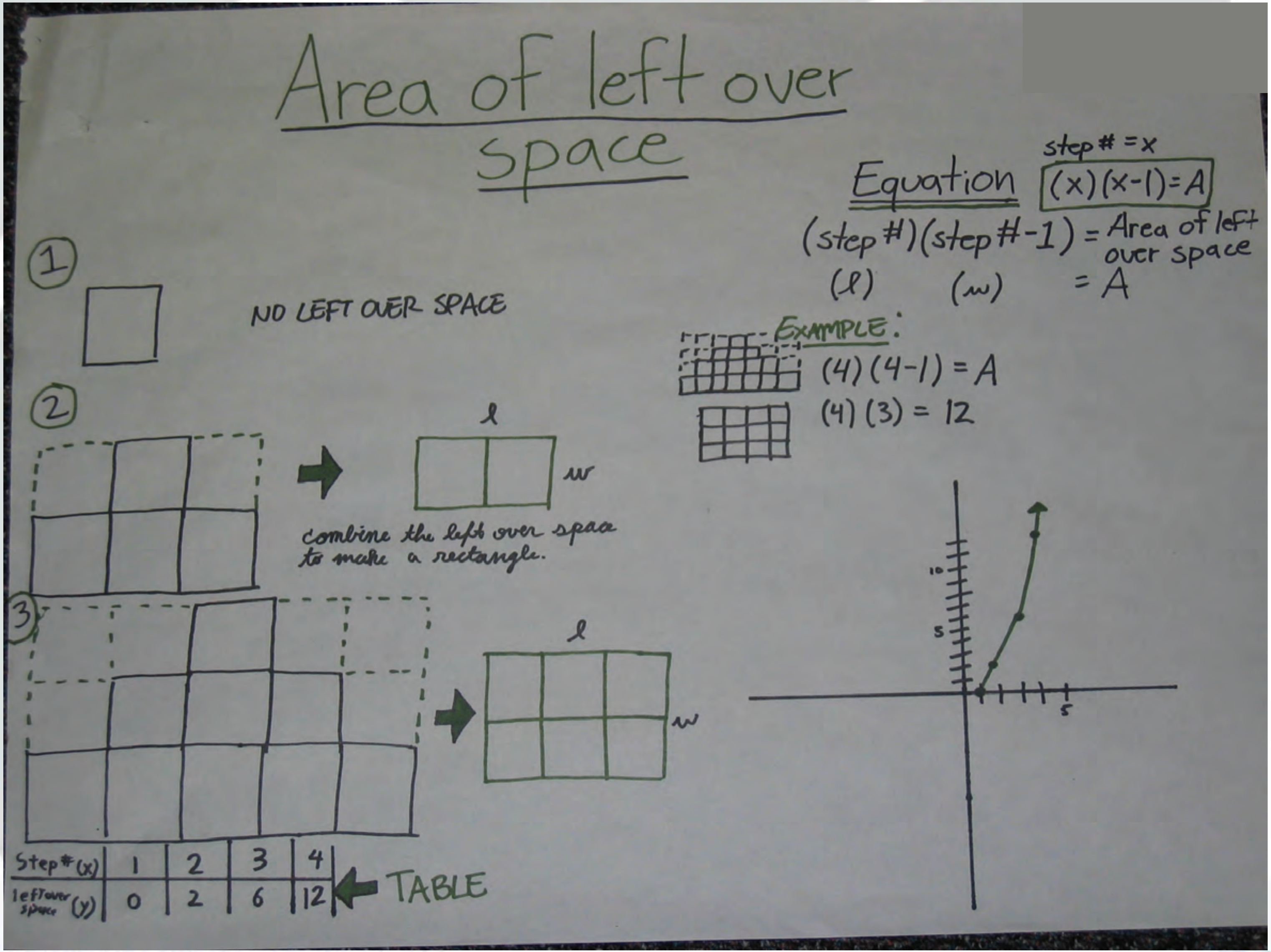
Step #(x)	# of inside right angles (y)
1	4
2	16
3	36
4	64

Equation:

$$4x^2=y$$

Graph:

Left-over Space



Area of left over space

Equation step # = x $\boxed{(x)(x-1)=A}$

(step #)(step # -1) = Area of left over space

(l) (w) = A

EXAMPLE:

$(4)(4-1) = A$

$(4)(3) = 12$

① NO LEFT OVER SPACE

②

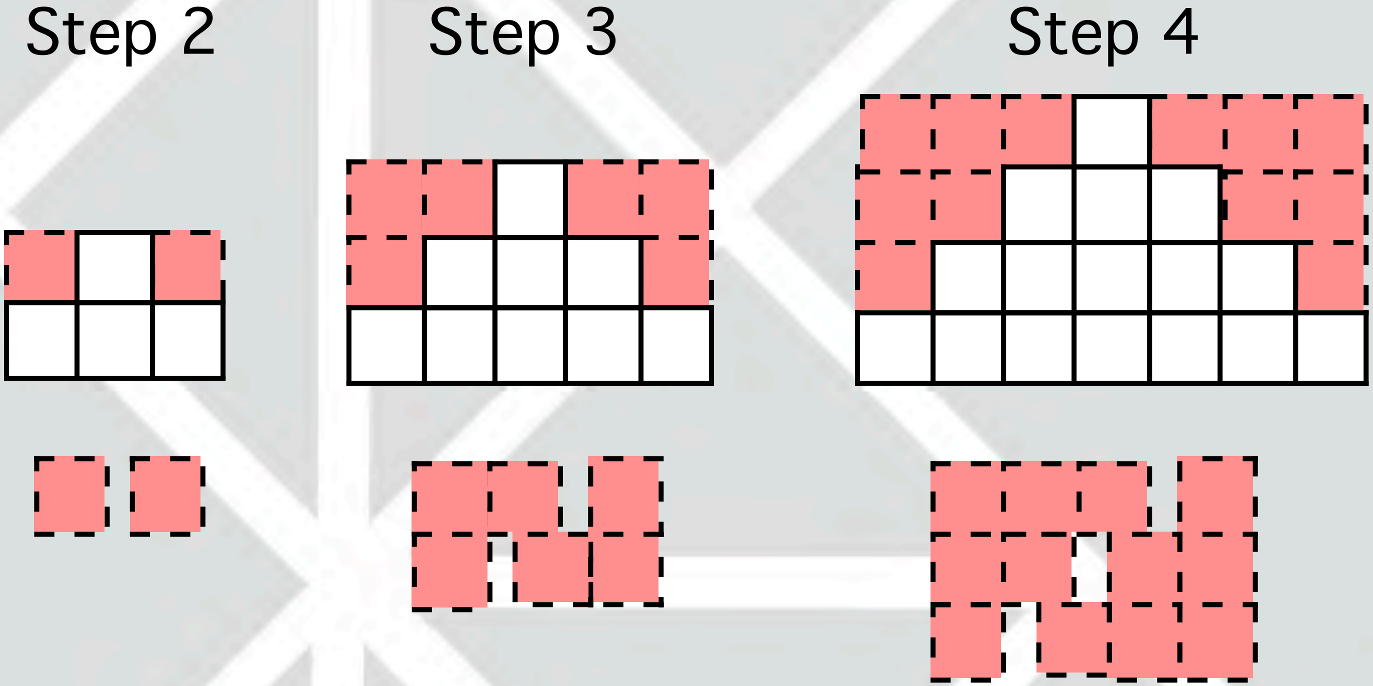
combine the left over space to make a rectangle.

③

Step # (x)	1	2	3	4
left over space (y)	0	2	6	12

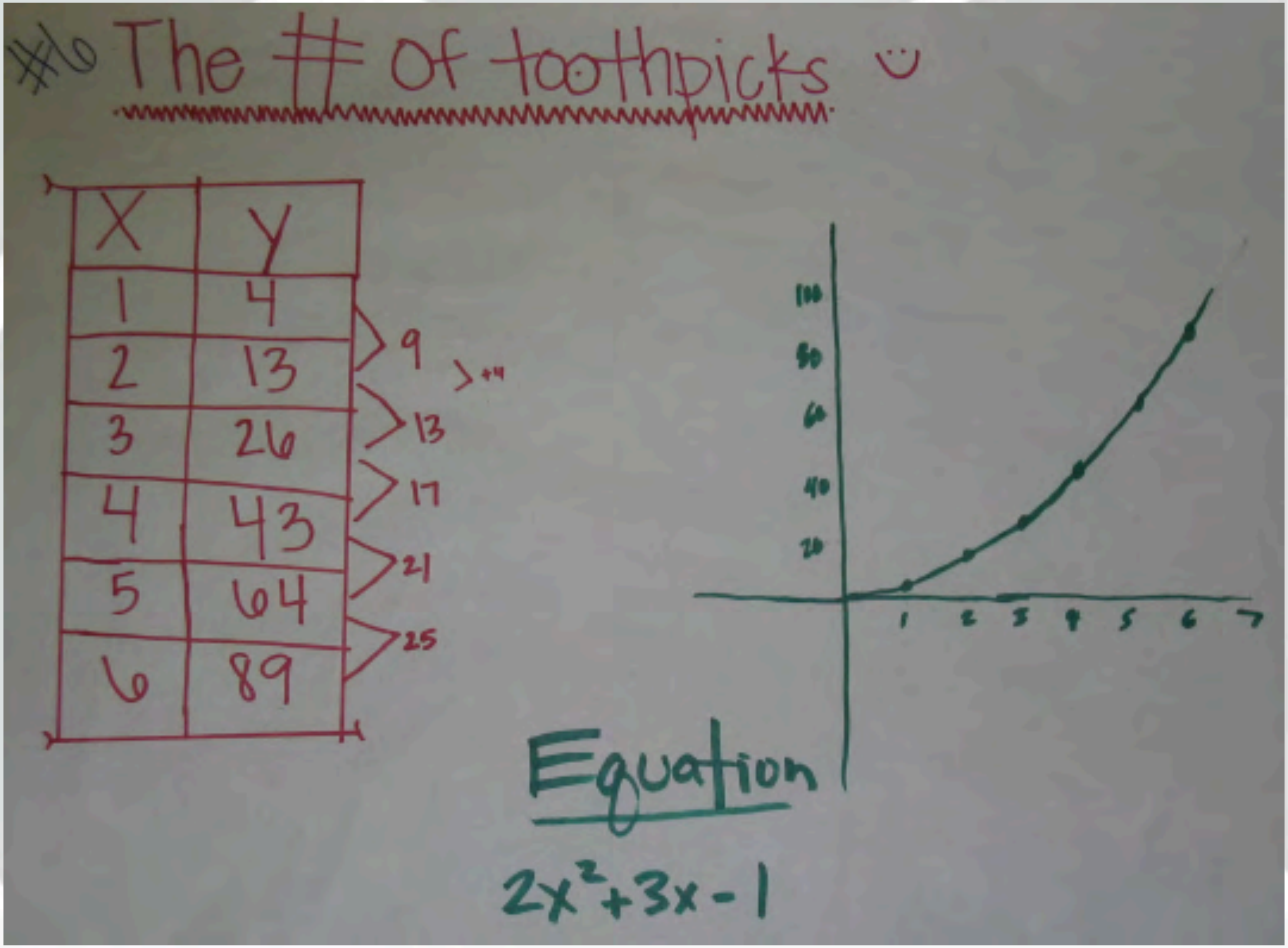
← TABLE

Left-over Space (cont.)



$$A(x) = x(x-1)$$

Number of toothpicks



#6 The # of toothpicks ☺

X	y
1	4
2	13
3	26
4	43
5	64
6	89

9

13

17

21

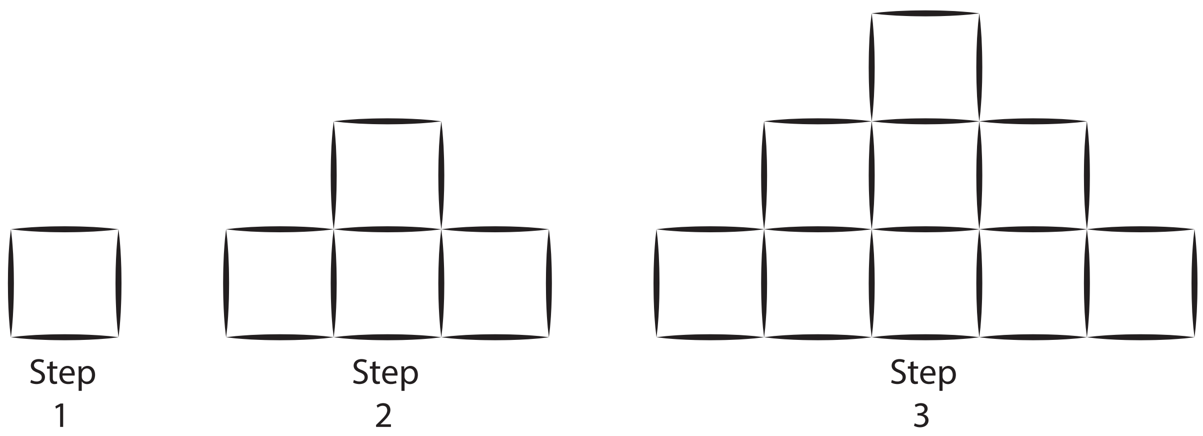
25

> +4

Equation

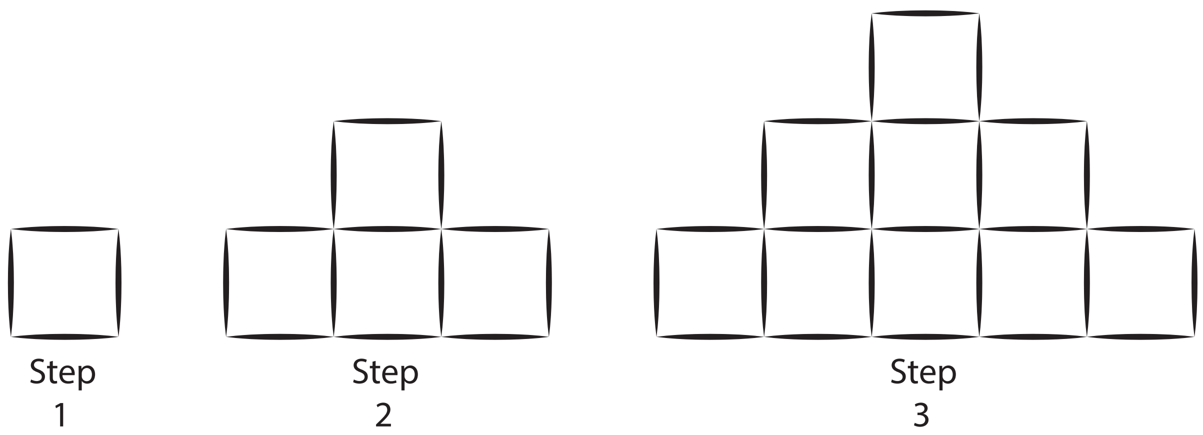
$2x^2+3x-1$

Number of Toothpicks (cont.)



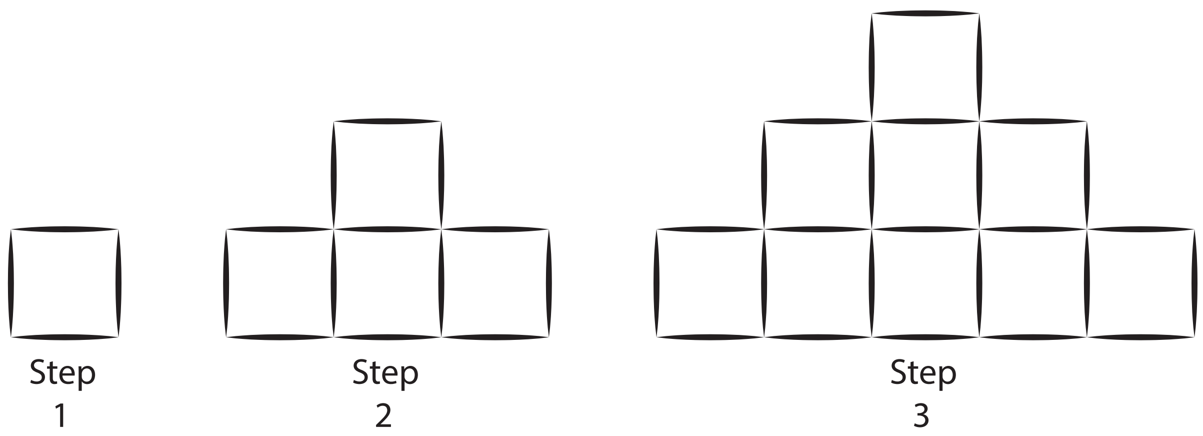
Step	Vertical Toothpicks	Horizontal Toothpicks	Total
1	1+1	1+1	4
2			
3			
4			
n			

Number of Toothpicks (cont.)



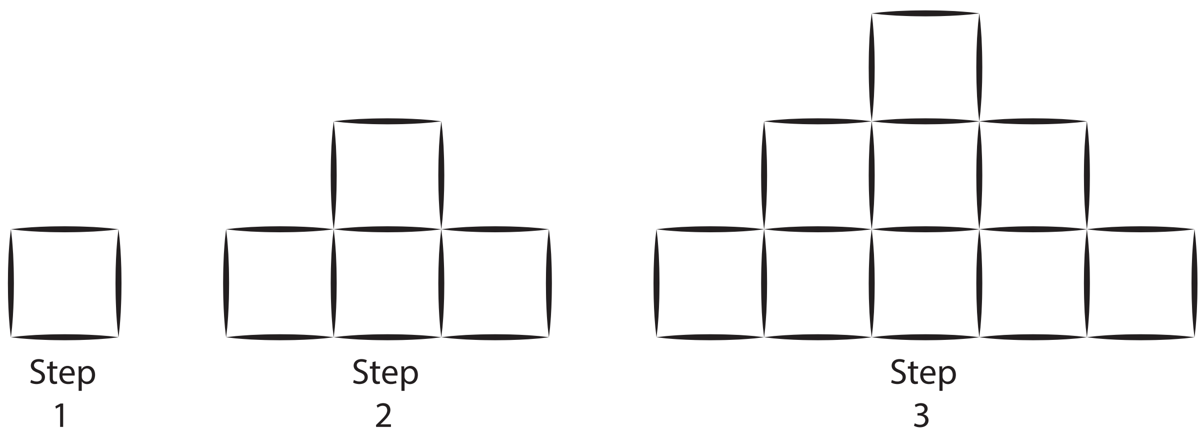
Step	Vertical Toothpicks	Horizontal Toothpicks	Total
1	1+1	1+1	4
2	1+2+2+1	1+3+3	13
3			
4			
n			

Number of Toothpicks (cont.)



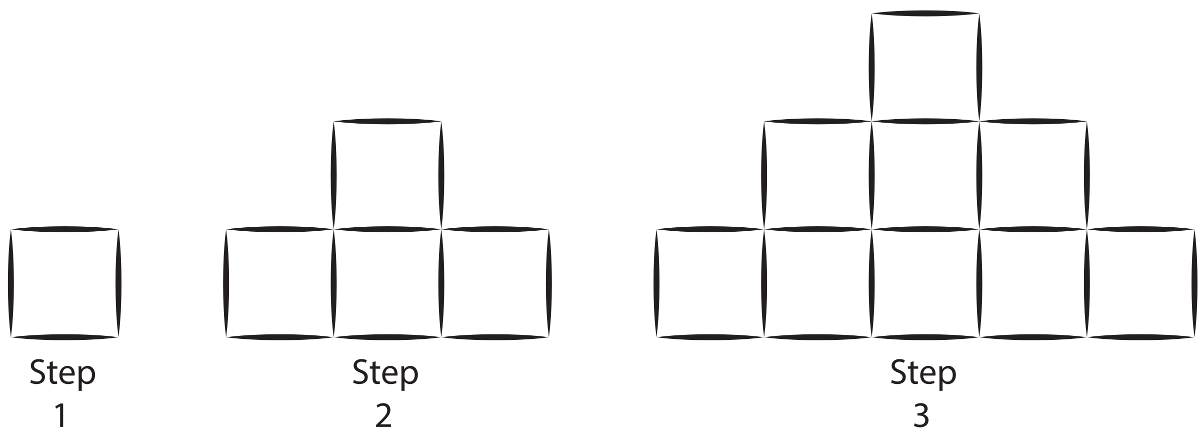
Step	Vertical Toothpicks	Horizontal Toothpicks	Total
1	1+1	1+1	4
2	1+2+2+1	1+3+3	13
3	1+2+3+3+2+1	1+3+5+5	26
4			
n			

Number of Toothpicks (cont.)



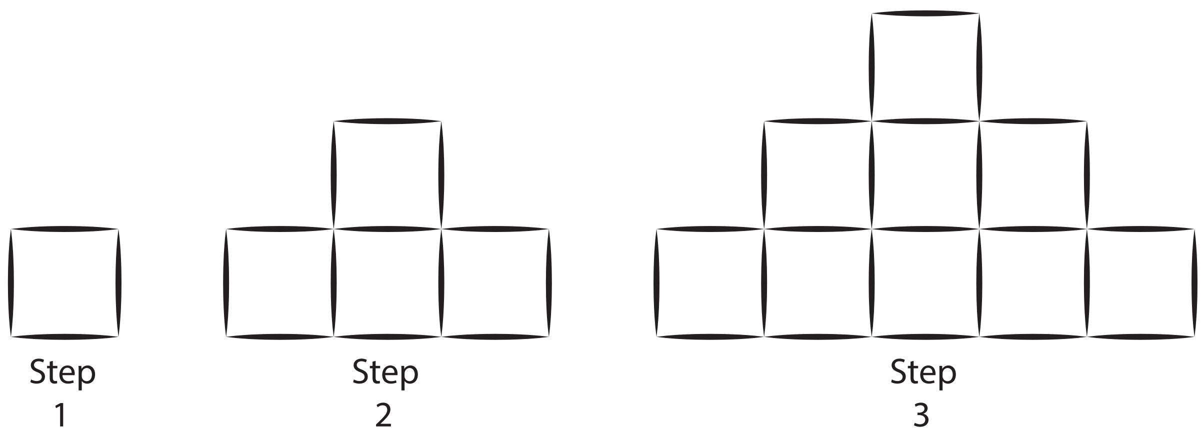
Step	Vertical Toothpicks	Horizontal Toothpicks	Total
1	1+1	1+1	4
2	1+2+2+1	1+3+3	13
3	1+2+3+3+2+1	1+3+5+5	26
4	1+2+3+4+4+3+2+1	1+3+5+7+7	
n			

Number of Toothpicks (cont.)



Step	Vertical Toothpicks	Horizontal Toothpicks	Total
1	1+1	1+1	4
2	1+2+2+1	1+3+3	13
3	1+2+3+3+2+1	1+3+5+5	26
4	1+2+3+4+4+3+2+1	1+3+5+7+7	
	2(1+2+3+4)	(1+3+5+7)+7	43
n			

Number of Toothpicks (cont.)

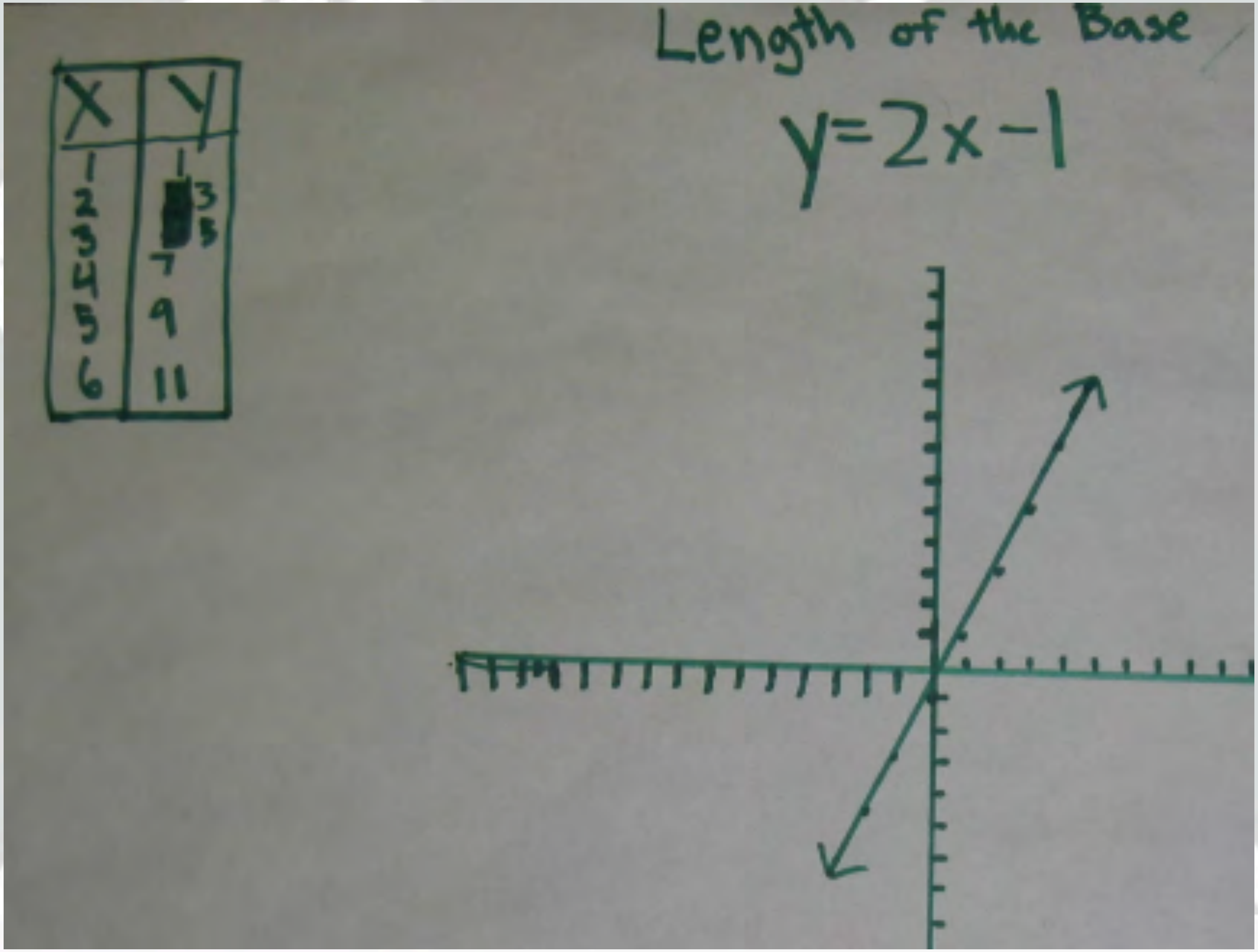


Step	Vertical Toothpicks	Horizontal Toothpicks	Total
1	1+1	1+1	4
2	1+2+2+1	1+3+3	13
3	1+2+3+3+2+1	1+3+5+5	26
4	1+2+3+4+4+3+2+1	1+3+5+7+7	
	2(1+2+3+4)	(1+3+5+7)+7	43
n	2(1+2+3+ … + *n*)	(1+3+5+ … + (2*n*-1))+(2*n*-1)	

Number of Toothpicks (cont.)

$$\begin{tikzcd}
2(1+2+3+\ldots+n) \arrow[d] & +(1+3+5+\ldots+(2n-1)) \arrow[d] & +(2n-1) \arrow[d] \\
2\left(\frac{n(n+1)}{2}\right) \arrow[dr] & n^2 \arrow[d] & 2n-1 \arrow[dl] \\
 & (n^2+n)+n^2+(2n-1) \arrow[d] & \\
 & 2n^2+3n-1 &
\end{tikzcd}$$

Length of the Base



Length of the Base

$y=2x-1$

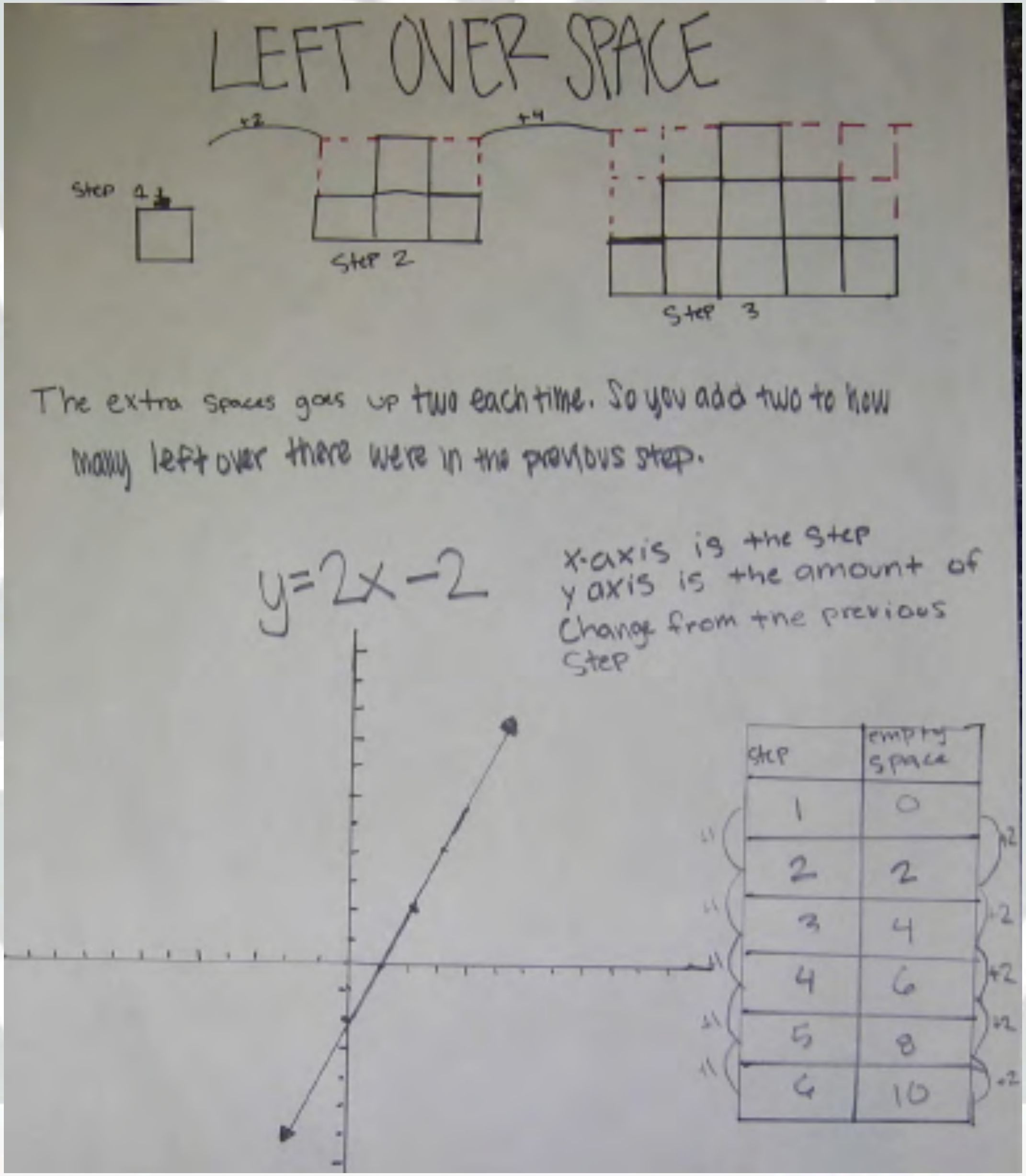
X	Y
1	1
2	3
3	5
4	7
5	9
6	11

Change in left over space.

Step	Left-over Space
1	0
2	2
3	6
4	12
5	20
6	30

+2 +4 +6 +8 +10

+2 +2 +2 +2



LEFT OVER SPACE

The extra spaces goes up two each time. So you add two to how many left over there were in the previous step.

$y=2x-2$

x-axis is the step
y axis is the amount of Change from the previous Step

step	empty space
1	0
2	2
3	4
4	6
5	8
6	10

What Mathematics Could Students Learn by Engaging with this Problem?

- Behavior of differences
  - Linear behavior has constant first differences.
  - Quadratic behavior has linear first differences.
  - Quadratic behavior has constant second differences.
- The change of quantities measured in two directions is typically quadratic in nature.
- The change of quantities measured in one directions is typically linear in nature.

What Mathematics Could Students Learn by Engaging with this Problem?

- The sum of the first *n* odd numbers is equal to n^2 and there is a geometric explanation of that relationship.

What makes this problem "Intellectually Engaging"?

- Open Ended
  - There are multiple correct answers
  - This is a different type of open-endedness than the last problem because that one had multiple paths to the same correct solution.
- Students were allowed to pick the quantity they wanted to investigate.
- There are interesting similarities and differences in the resulting functions depending on the quantity being measured.

Detail Matters

- What would the impact be if the original task prescribed which quantity to measure?
- Does the prompt of "use a table, graph and equation to describe the change" lower the cognitive demand?

For More Information

Peterson, Blake E. (2006) Linear and Quadratic Change: A Problem from Japan. *Mathematics Teacher*, 100(3), 206-212

Attributes of an Intellectually Engaging Problem

Open Ended

- Multiple paths to the same answer
- Multiple viable solutions

Mathematically Rich

- Focused on central concepts
- Multiple representations allow for more connections to be made

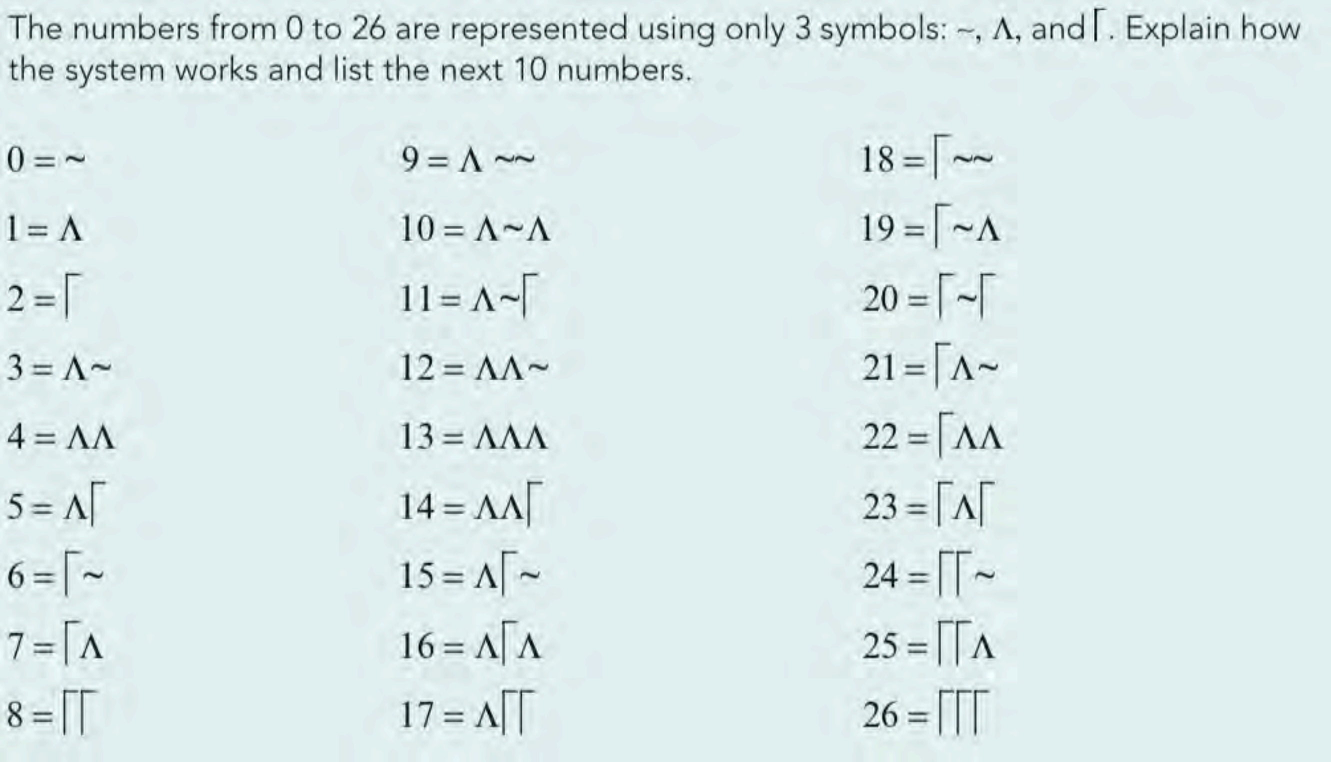
The multiplicity of strategies, solutions, and representations provide

- Multiple access points
- Opportunities for creativity

The problem doesn't need to be about pizza and cell phones to be intellectually engaging!

Detail Matters

How might the layout of the numbers in these two tasks impact the way students will think about the mathematics?

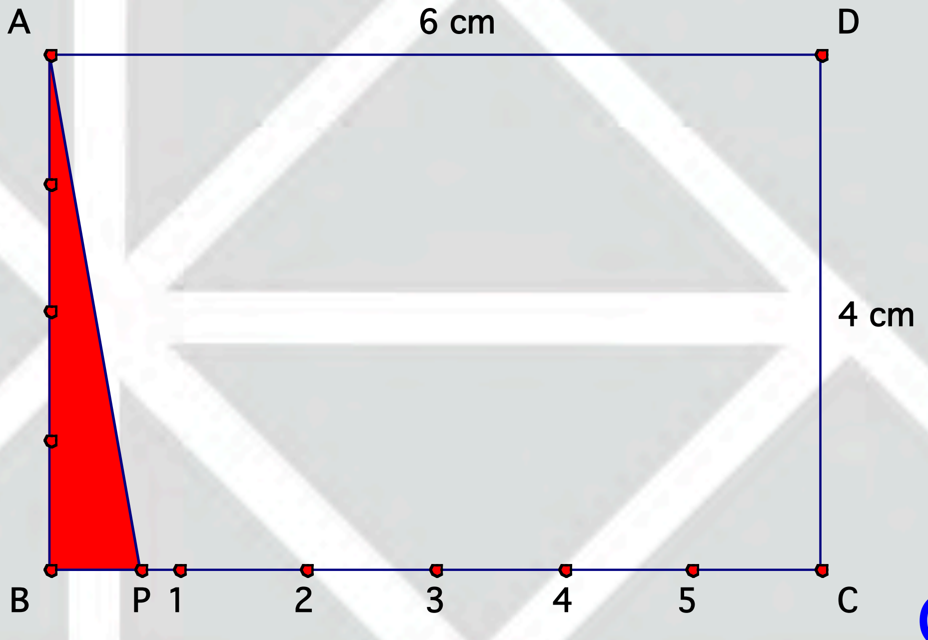


The numbers from 0 to 26 are represented using only 3 symbols: ~, Λ, and ⌈. Explain how the system works and list the next 10 numbers.

0 = ~
1 = Λ
2 = ⌈
3 = Λ~
4 = ΛΛ
5 = Λ⌈
6 = ⌈~
7 = ⌈Λ
8 = ⌈⌈
9 = Λ ~~
10 = Λ~Λ
11 = Λ~⌈
12 = ΛΛ~
13 = ΛΛΛ
14 = ΛΛ⌈
15 = Λ⌈~
16 = Λ⌈Λ
17 = Λ⌈⌈
18 = ⌈~~
19 = ⌈~Λ
20 = ⌈~⌈
21 = ⌈Λ~
22 = ⌈ΛΛ
23 = ⌈Λ⌈
24 = ⌈⌈~
25 = ⌈⌈Λ
26 = ⌈⌈⌈

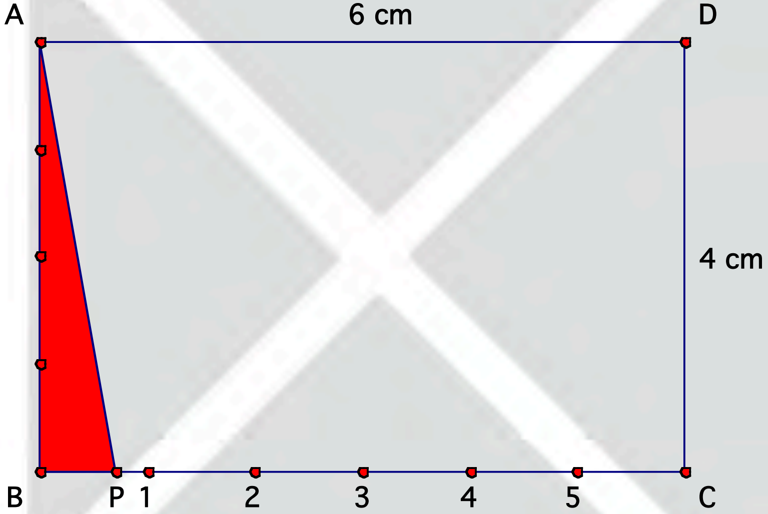
0 = ~
1 = Λ
2 = ⌈
3 = Λ~
4 = ΛΛ
5 = Λ⌈
6 = ⌈~
7 = ⌈Λ
8 = ⌈⌈
9 = Λ~~
10 = Λ~Λ
11 = Λ~⌈
12 = ΛΛ~
13= ΛΛΛ
14 = ΛΛ⌈
15 = Λ⌈~
16 = Λ⌈Λ
17 = Λ⌈⌈
18 = ⌈~~
19 = ⌈~Λ
20 = ⌈~⌈

In rectangle ABCD, point P moves from B to A through C and D at a rate of 1 cm per second. Using a table, graph and equation, describe the area of triangle ABP as it changes over time.



GSP

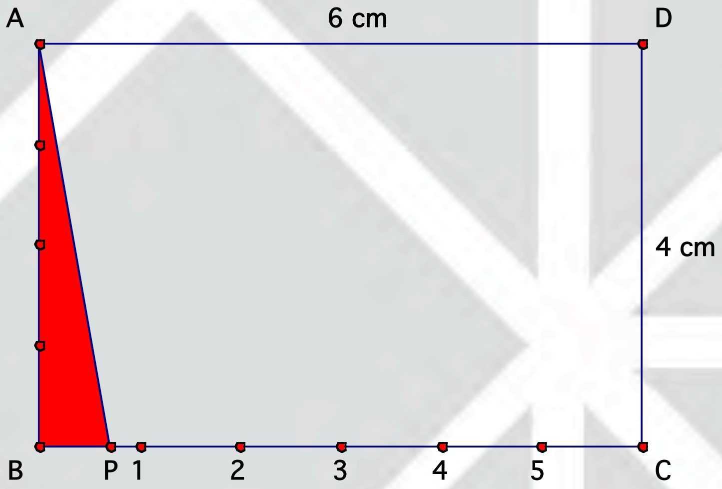
Area versus time in a table

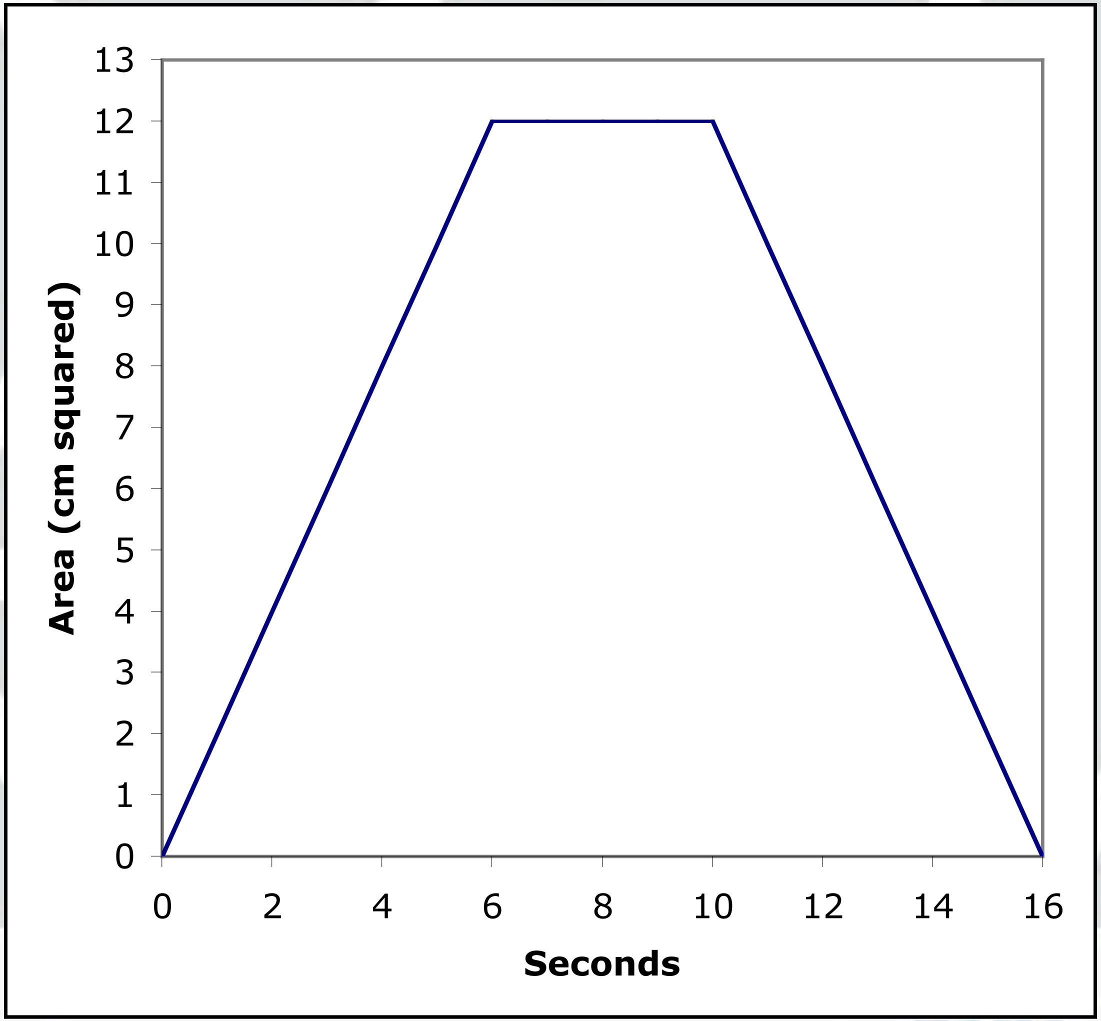


Seconds	0	1	2	3	4	5	6	7	8
Area	0	2	4	6	8	10	12	12	12

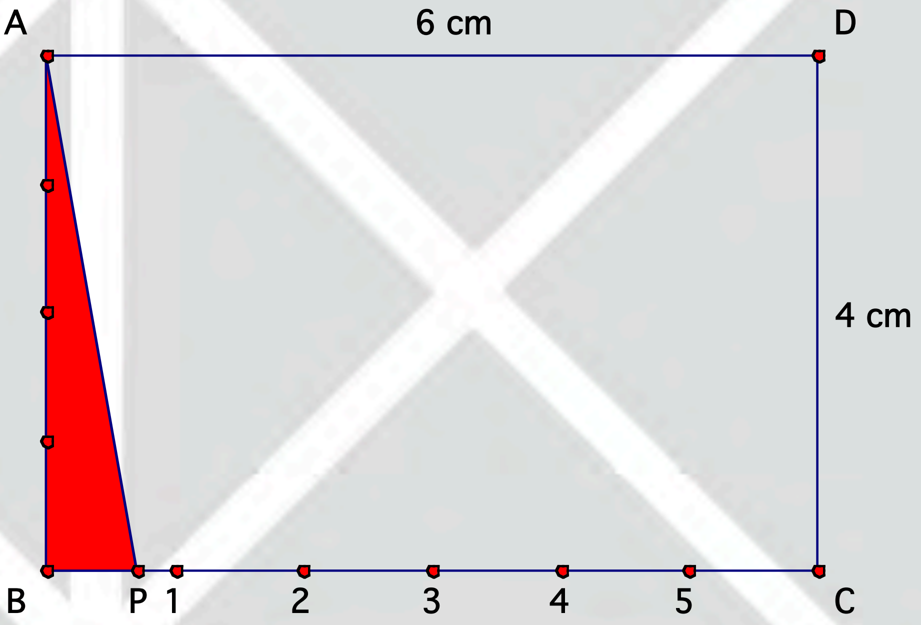
Seconds	9	10	11	12	13	14	15	16
Area	12	12	10	8	6	4	2	0

Area versus time in a graph.



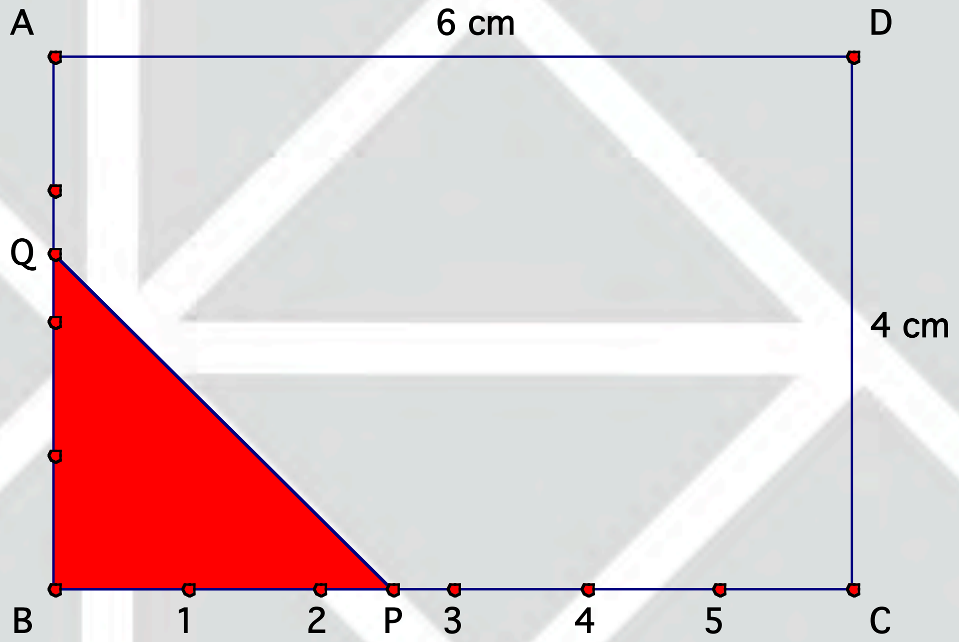


Area versus time in an equation



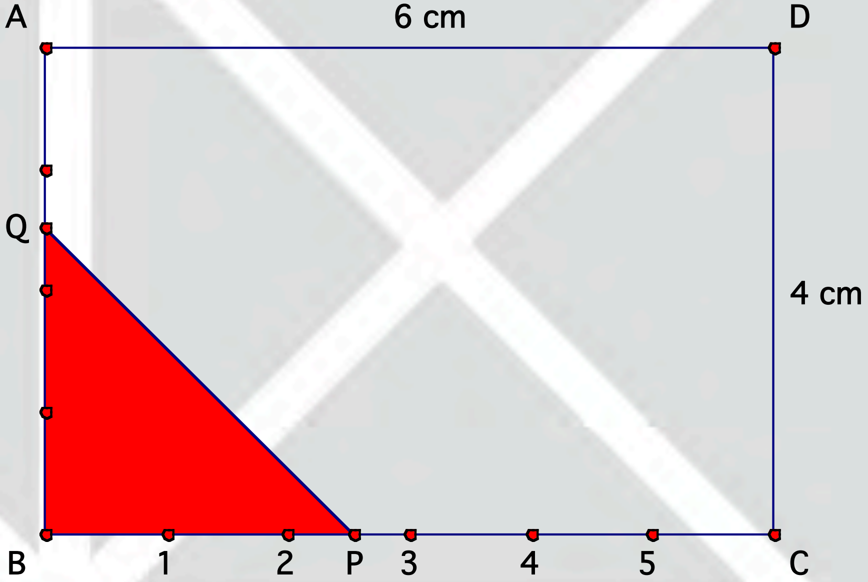
$$A(t)=\begin{cases}2t & 0\le t\le 6\\ 12 & 6\le t\le 10\\ -2t+32 & 10\le t\le 16\end{cases}$$

In rectangle ABCD, point P and Q move from B to D through C and A respectively at a rate of 1 cm per second. Using a table, graph and equation, describe the area of triangle BPQ as it changes over time.



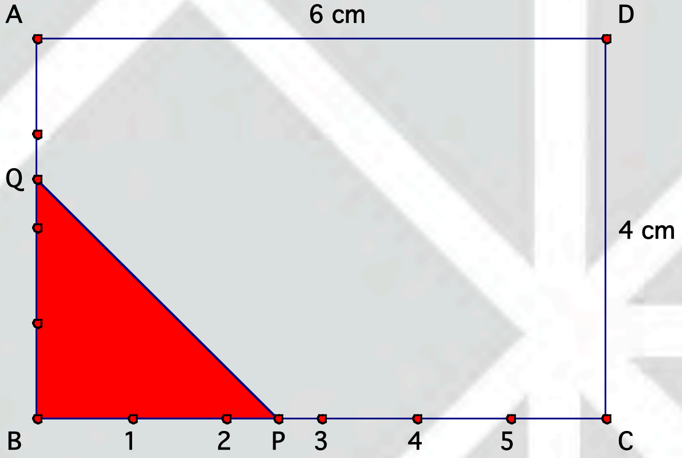
GSP

Area versus time in a table



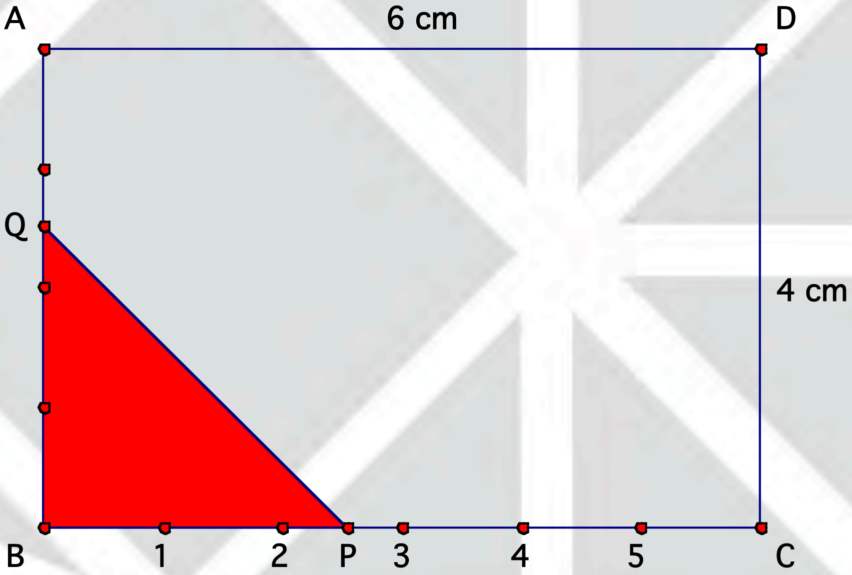
Seconds	0	1	2	3	4	5	6	7	8	9	10
Area	0	0.5	2	4.5	8	10	12	10.5	8	4.5	0

Area versus time in a graph.



Area versus time in an equation.

$$A(t)=\begin{cases}\dfrac{t^2}{2} & 0\le t\le 4\\ 2t & 4\le t\le 6\\ -\dfrac{t^2}{2}+5t & 6\le t\le 10\end{cases}$$



Attributes of an Intellectually Engaging Problem

Open Ended

Mathematically Rich

Multiple access points

Opportunities for creativity

Pizza and cell phones -> maybe

Be optimistic

Help students stretch and grow!