The Mandelbrot Fractal in Pre-Calculus

Dan Anderson
Queensbury High School
NY Master Teacher - Capital District
dan@recursiveprocess.com
@dandersod

All resources found here: `bit.ly/mandelbrotfractal`

Motivation

- How I got started with the Mandelbrot Fractal
- Why I use the Mandelbrot Fractal as a teaching tool in PreCalculus

What topics are addressed?

- Complex numbers
- Arithmetic with Complex numbers
- Complex plane and Argand diagrams
- Recursive sequences
- Polar Form of Complex numbers
- Graphing using the Polar plane
- DeMoivre's Theorem

(Presenter hat) Administrivia

How do you use these materials with a class?

This is approximately 3-4 days of material.
I wouldn't use an accelerated presentation like this;
I'd make sure that the students are active for each step, trying out examples and doing a whole bunch of thinking/talking/calculating (in that order).

Start with the Basics - Complex Numbers

- The complex plane is a modified Cartesian plane, where the *real* part of a complex number is graphed on the x-axis and the *imaginary* part is graphed on the y-axis.
- What is the size (modulus) of ***a+bi***?

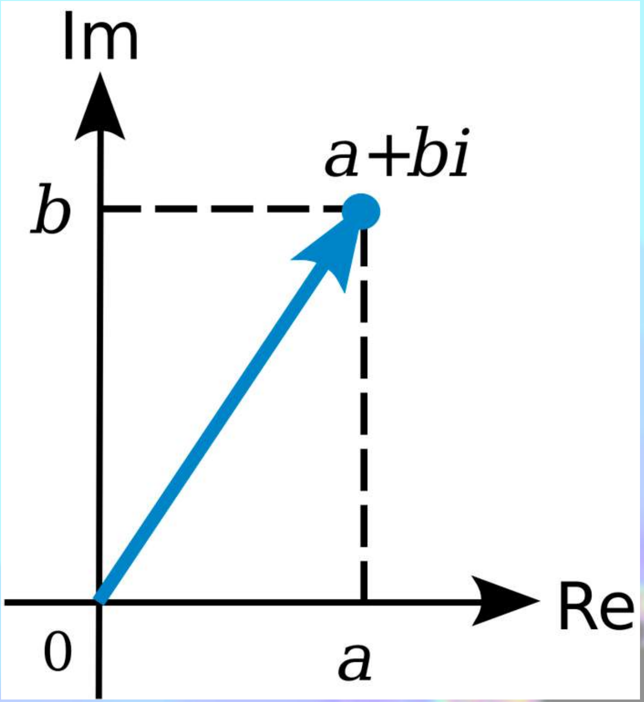
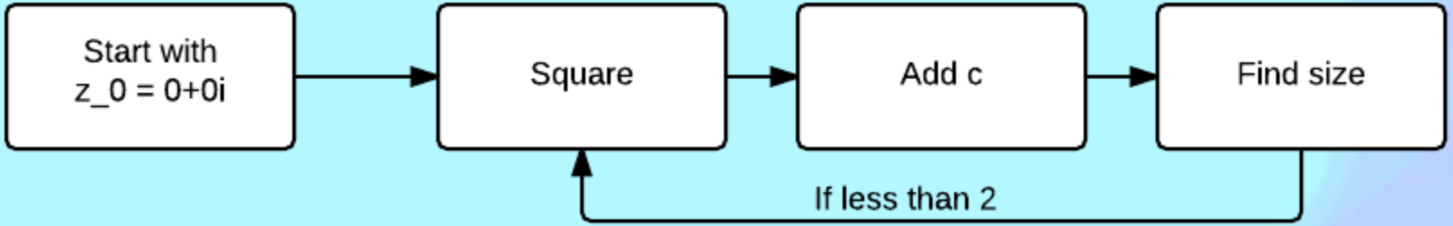


Diagram from Wikipedia

The Mandelbrot Set - Definition

- The Mandelbrot set is defined as the set of all complex numbers, *c*, where the following (infinite) task is **bounded** (the size doesn't "blow up").



- All points who are bounded (size < 2) are in the set, otherwise the point is out of the set.

Next step

- What points are interesting?
- Is $c = 5+12i$ interesting in this context?

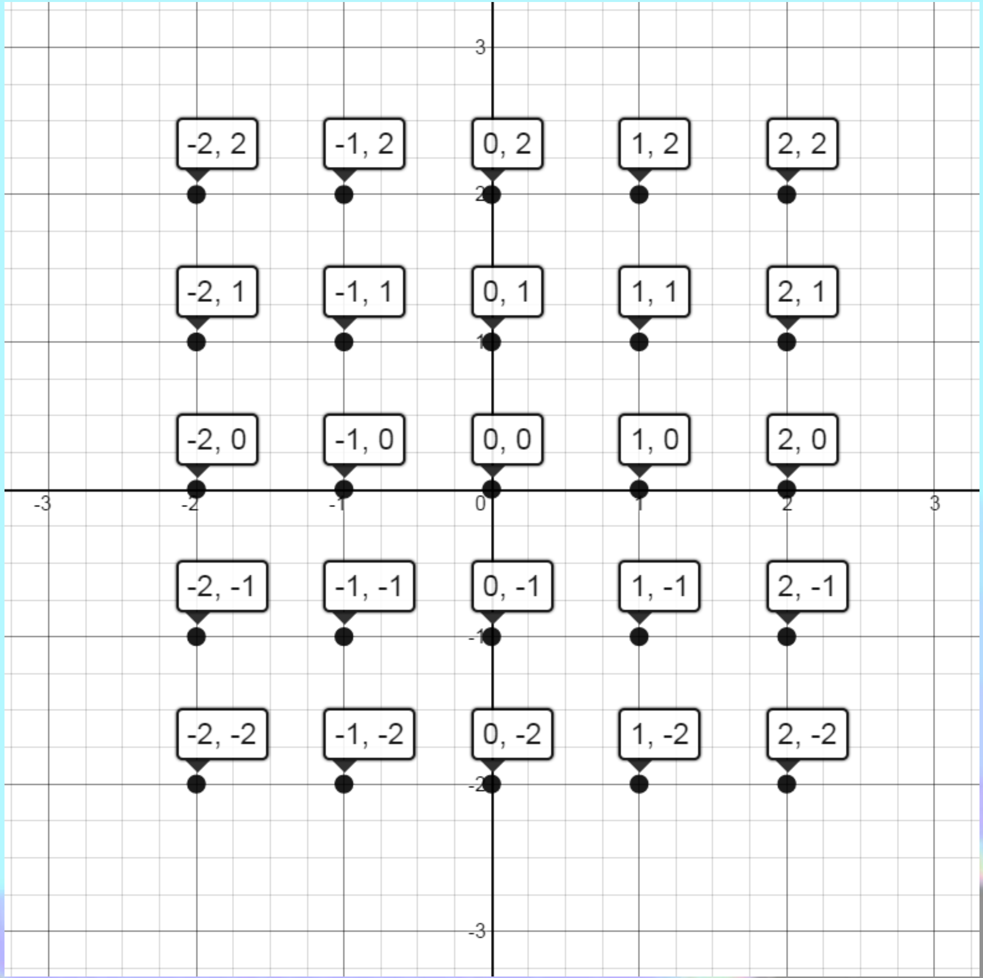
    - z_0 = 0 + 0i -> size of 0, continue
    - z_1 = (z_0)^2 + c = (0+0i)^2 + (5 + 12i) -> size of 13 (out of set)

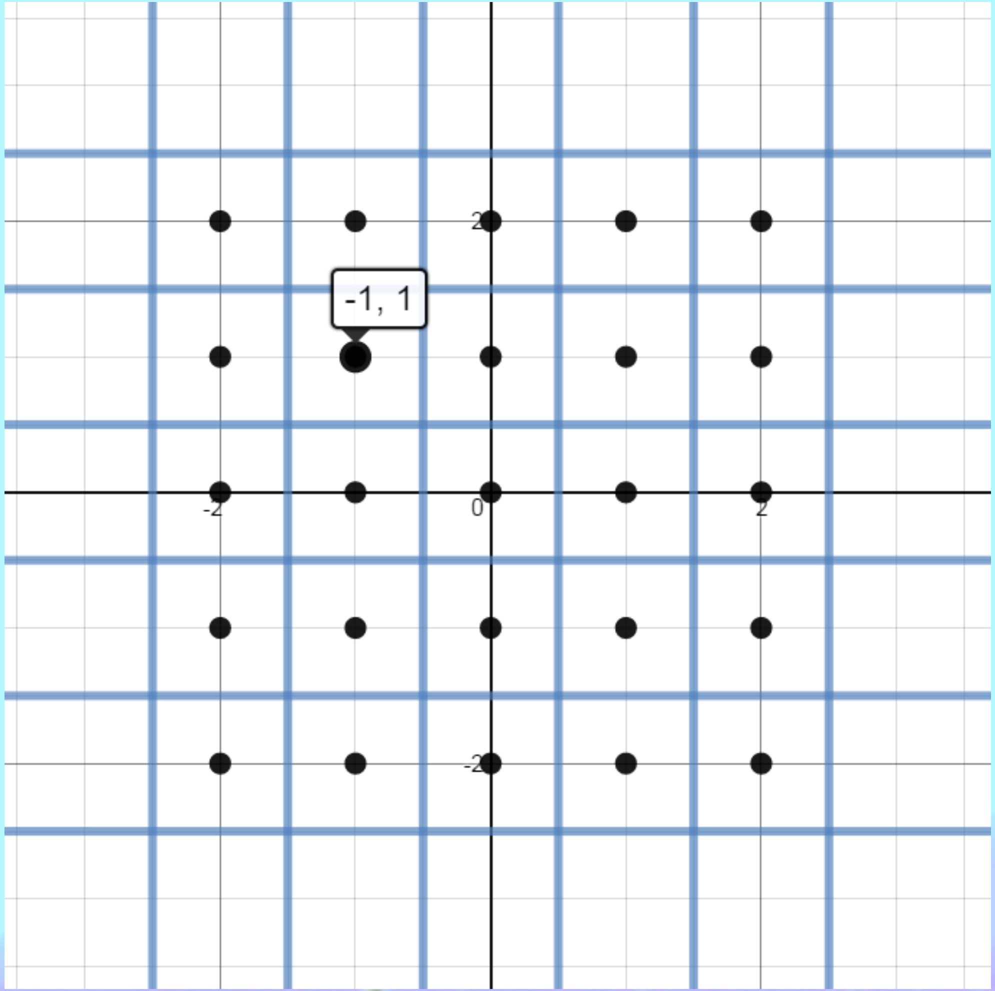
So what points do we have to check?

Let's consider the following points:

We'll interpret each of these points as a square.

Note: (-2,1) represents -2 + 1i on the Argand Plane





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(Teacher hat)
Have the kids do some work

Assign each student a constant (there are 25 to handle).

If you have less than 25 students, you can assign the leftovers later.

(Presenter hat)
Your Turn

You'll be assigned a constant based on what day of the month you were born.

Yes, *you* will be doing some calculations!

There is a clicker app to keep track of the results on the resources page.

(Teacher) Work through a *c* together

Especially for reluctant learners, it can help to build confidence by working through an example together.

Take $1+0i$ and reassign the student(s) that had that as their constant.

1+0i

- $z_0 = 0 + 0i$ -> Size is 0, so continue process
- $z_1 = (z_0)^2 + c = (0+0i)^2+(1+0i) = 1+0i$ -> Size is 1, continue process
- $z_2 = (z_1)^2 + c = (1+0i)^2+(1+0i) = 2+0i$ -> Size = 2 which is not <2 so process **stops** at the 2nd step. The "escape velocity" is **2**.

Now it's your turn to do some math

- Take your constant and calculate z_1
- Then find the size (magnitude) of z_1.
- If it's less than 2,
  - then your square is black,
  - else your square is white (outside mandelbrot set).

(Teacher hat) For those who finish early, what shape should this have? Why?

Next Step

- *Note: If your c is out already, pick a different square and start verifying people's answers.*

- Calculate z_2 (from your z_1 and your c). Size? In Mandelbrot set?
- Calculate z_3 (from z_2 and your c). Size? In Mandelbrot set?

Step 4+

- Calculate $|z_4|$
- z_5?
- Infinite process right? Are we getting a better picture?

(Teacher hat)
How do we make this better?

How can we improve the picture?

How do we make this automatic?

- The computer programming languages don't know about complex numbers. Can you teach them how to square a complex number?
- What are the **Real** and **Imaginary** portions of (a+bi)^2?
- You tell me! Expand and separate.

Automatic - Mandelblocks Program

The Mandelblocks program is linked on the resources page.

- First jump into the code to show where the *(a+bi)^2* code is.
- Talk about coloring mode
- How can we do even better?

Mandelbrot Program

Treat each pixel as a coordinate on the Complex Plane.

The Mandelbrot program is linked on the resources page.

- Step 1. A circle? Why??
- Symmetry?

Mandelbrot Zoom

We can do better, let's zoom in and see the detail.

The Mandelbrot Zoom program is linked on the resources page.

- Is there a limit to how far we can zoom in?
- Note the window width as we zoom.
- Note the resolution required as we zoom in.

What's next?

- How can we expand on this Mandelbrot set?
- What if we consider $z_{(n+1)} = z_n^3 + c$? How will cubing the number change the picture of the set?
- Time for you to get to work and expand $(a+bi)^3$ and separate the real and imaginary parts.

Let's put in the code and see the fractal!

All resources found here: `bit.ly/mandelbrotfractal`

What's next continued?

What about $z_{n+1} = z_n^4 + c$?

Fifth power?

Sixth power?

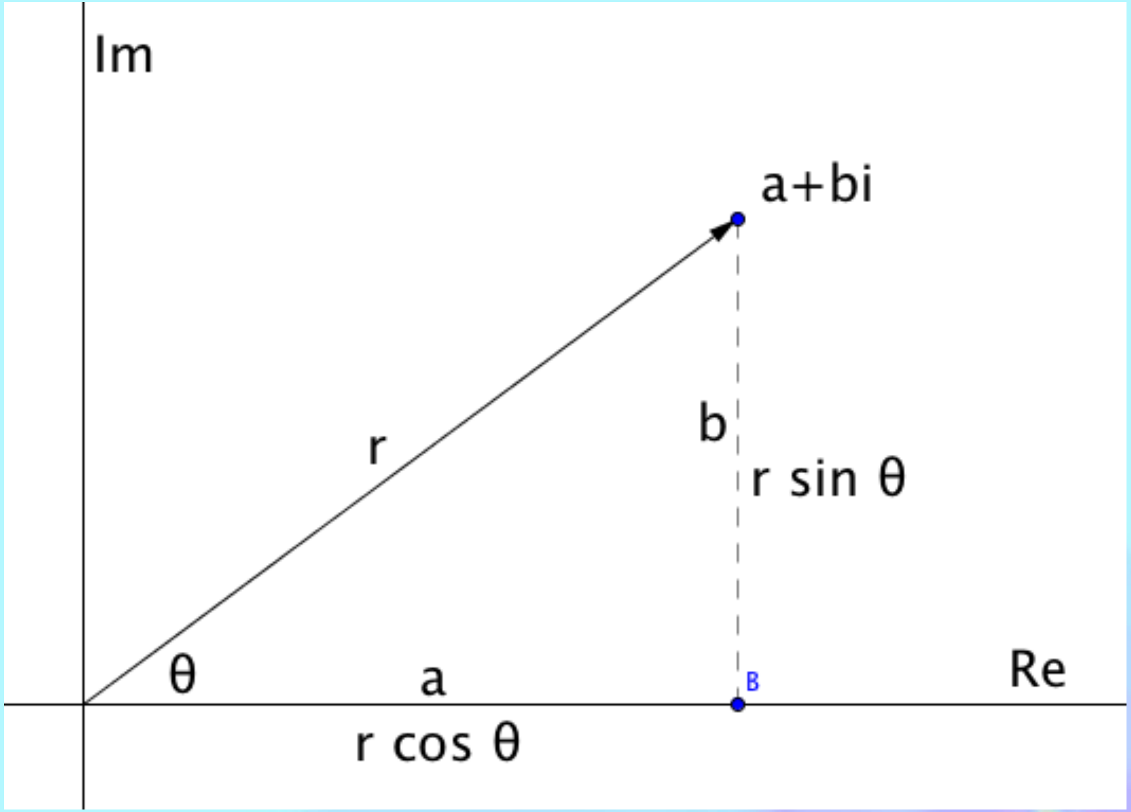
Tenth power? (Binomial Expansion right?)

3/2 power? What does that even mean in this context?

Polar to the rescue!

If we convert from rectangular coordinates to polar coordinates then we can find the general solution for *any* power of z_n!

Polar Form



$$a = r\ cos(\theta)$$

$$b = r\ sin(\theta)$$

$$so$$

$$a + bi\ = r\ cos(\theta) + r\ sin(\theta)\ i = r\ cis(\theta)$$

Why Polar Form?

deMoivre's Theorem!

$$(a+bi)^n = (r\ cis\ \theta)^n = r^n\ cis(n\theta)$$

The math is so much easier, operations with real numbers instead of binomial expansion. *And n doesn't have to be an integer!*

So:

- Convert from rectangular to polar
- Use deMoivre's Theorem
- Convert back to rectangular

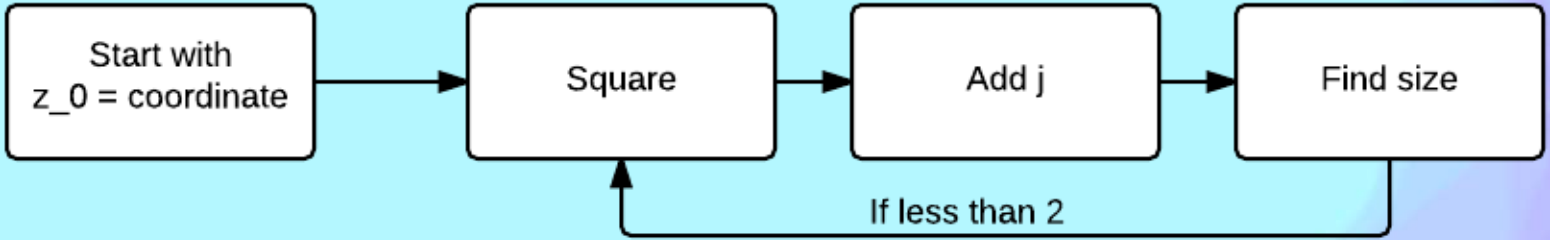
Mandelbrot Family Interactive

The Mandelbrot Family program is linked on the resources page.

Extensions: Julia Set

Let's consider the following rule (Julia Set).

Start with a complex constant *j*.



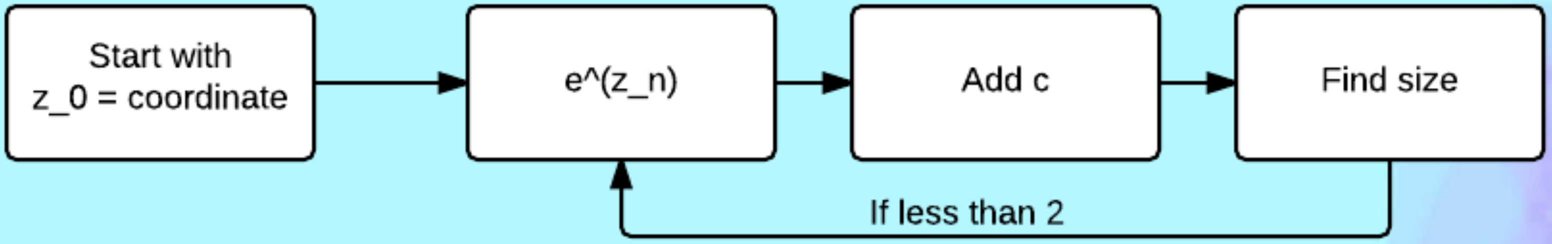
Julia Set Interactive

The Julia Set program is linked on the resources page.

How are the Julia and Mandelbrot Sets related? drawMandelbrotAndJulia program is also linked on resources page.

Experimental

What if you consider the following rule?



[The Experimental program is linked on the resources page.]

Questions? and Thanks!

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Source: https://www.youtube.com/watch?v=MVzGyAAtHiU