

# 2015 NCTM Annual Conference

Andy Clark  
andrew.m.clark.1@gmail.com



## Visual Models to Solve Word Problems and Non-routine Problems

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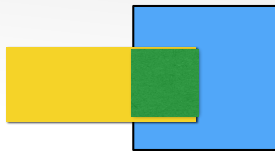
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Can you see it?



The green square is created by overlapping  $\frac{1}{3}$  of the yellow rectangle with  $\frac{1}{4}$  of the blue square. What fraction of the whole figure is green?

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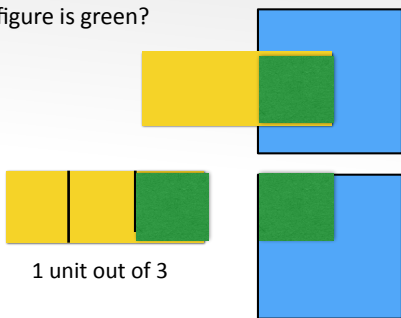
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The green square is created by overlapping  $\frac{1}{3}$  of the purple rectangle with  $\frac{1}{4}$  of the red square. What fraction of the whole figure is green?



1 unit out of 3

1 unit out of 4

But these green shaded squares overlap so there are only 6 units, so 1 out of 6 units are green.  $\frac{1}{6}$ th is green

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An important feature of the Singapore curriculum is its emphasis on visualization as a major strategy for developing understanding and problem solving.

One often hears teachers there saying: "Can You see it?"

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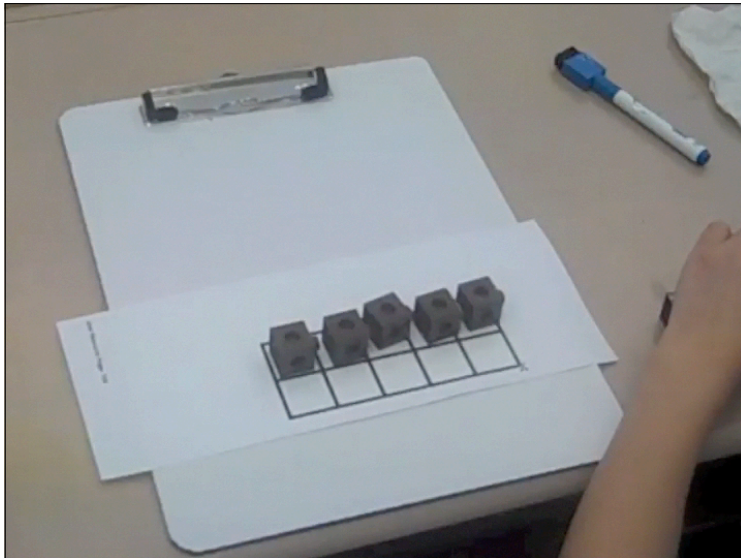
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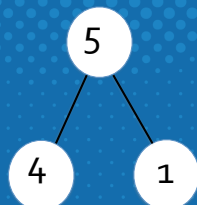
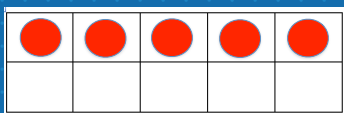
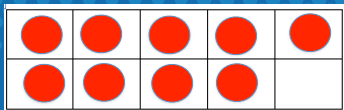
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## Visual Models

$$9 + 5 = 10 + 4$$



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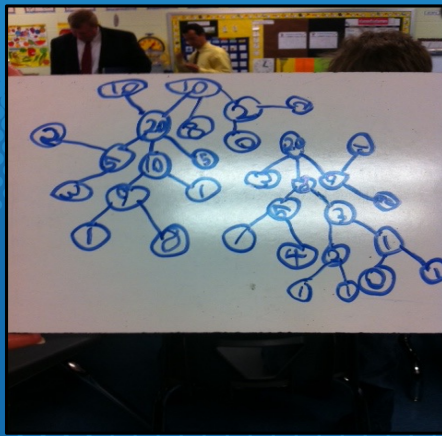
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Number Bond  
of 20  
with many parts!



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Diezmann, C. M. (2000). The difficulties students experience in generating diagrams for novel problems. In T. Nakahara & M. Koyama (Eds.), *Proceedings of the 25th Annual Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 241-248). Hiroshima, Japan: PME.

Diezmann, C. M., & English, L. D. (2001). Promoting the use of diagrams as tools for thinking. In A. A. Cuoco (Ed.), *2001 National Council of Teachers of Mathematics Yearbook: The role of representation in school mathematics* (pp.77-89). Reston, VA: National Council of Teachers of Mathematics.

Novick, L. R. (2001). Spatial diagrams: Key instruments in the toolbox for thought. In D. L. Merlin (Ed.), *The psychology of learning and motivation*, 40, 279-325.

van Essen, G., & Hamaker, C. (1990). Using self-generated drawings to solve arithmetic word problems. *Journal of Educational Research*, 83(6), 301-312.

Lowrie, T., & Kay, R. (2001). Relationship between visual and nonvisual solution methods and difficulty in elementary mathematics. *Journal of Educational Research*, 94(4), 248-255.

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## Why do students struggle with word problems?

- Linguistic issues: reading and understanding the problems
- Difficulty with the context
- Struggle with writing an equation to represent the problem
- Are scared of word problems
- Don't have real entry points into problems
- Can't visualize the relationships
- Look for "key" words which don't help
- Too quick to compute and not to represent the situation

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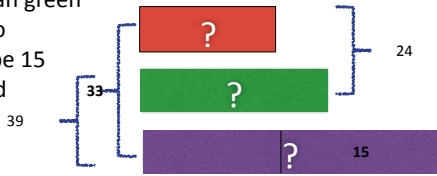


### Non routine problem or algebraic

I am thinking of three numbers. The sum of two of them is 24, the sum of two others is 33 and the sum of two others is 39.

What are the numbers?

green and red is 15 smaller than green and purple, so purple must be 15 more than red

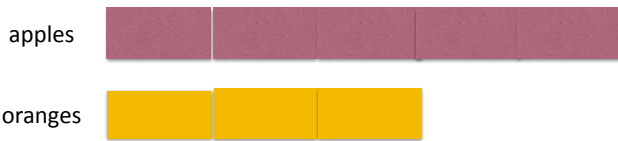


$$\begin{aligned} 2 \text{ bars} + 15 &= 33 & \text{red} &= 9 \\ 2 \text{ bars} &= 18 & \text{purple} &= 9 + 15 = 24 \\ 1 \text{ bar} &= 9 & \text{green} &= 24 - 9 = 15 \end{aligned}$$

9, 15, 24

The ratio of the number of apples to the number of oranges at a store was 5:3. After an equal number of apples and oranges were sold, the ratio became 7:1. how many oranges were there at first if 32 apples were sold. (6th grade)

Before



After



### Model Method: Definition

Pictorial or Diagrammatic method of representing quantities and their relationship to solve both arithmetic and algebraic word problems

Students ask themselves:

How do I represent or visualize the problem?

What is the question or questions?

## Model Method: Definition

- “It involves the construction of pictorial models, namely the part-whole model and the comparison model, to help students visualize abstract mathematical relationships and various problem structures through pictorial representations. It is a powerful visual aid for solving complex problems involving fractions, ratio and percentage.”

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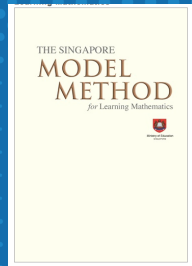
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## Model Method: Definition

“Above all, it is closely related to the algebraic method for solving algebra word problems”

The Singapore Model Method for Learning Mathematics



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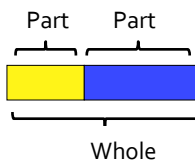
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## 2nd grade: Part /Whole Problems

- Two or more subsets make up a whole
  - Represented with two or more connected bars to make a whole



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## Part -Whole Problems

Shawn orders 75 boxes of fruit on the first day. He orders another 84 boxes of fruit on the second day. How many boxes of fruit does he order on both days?



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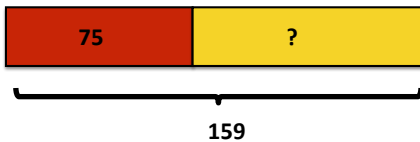
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## Part -Whole Problems

OR  
Shawn orders 159 boxes of fruit in two days. If he bought 75 on the first day, how many boxes did he buy on the second day?



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## 2nd grade Additive Comparison Problems



green is \_\_ less than blue  
or blue is \_\_ more than green  
or what is the difference of the two?

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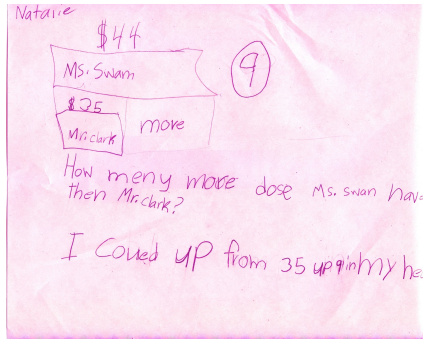
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Additive Comparison:  
Mr. Clark has \$35 dollars. Ms. Swan has \$44.  
How much more does Ms. Swan have?




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**COMMON CORE STATE STANDARDS INITIATIVE**  
PREPARING AMERICA'S YOUTH FOR COLLEGE & CAREER

TABLE 1. Common addition and subtraction situations.<sup>a</sup>

	Result Unknown	Change Unknown	Start Unknown
<b>Add to</b>	Two bunnies sat on the grass. Three more bunnies hopped there. How many bunnies are on the grass now? $2 + 3 = ?$	Two bunnies were sitting on the grass. Some more bunnies hopped there. Then there were five bunnies. How many bunnies hopped over to the first two? $2 + ? = 5$	Some bunnies were sitting on the grass. Three more bunnies hopped there. Then there were five bunnies. How many bunnies were on the grass before? $? + 3 = 5$
<b>Take from</b>	Five apples were on the table. I ate two apples. How many apples are on the table now? $5 - 2 = ?$	Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat? $5 - ? = 3$	Some apples were on the table. I ate two apples. Then there were three apples. How many apples were on the table before? $? - 2 = 3$
<b>Put Together/ Take Apart<sup>b</sup></b>	<b>Total Unknown</b> Three red apples and two green apples are on the table. How many apples are on the table? $3 + 2 = ?$	<b>Addend Unknown</b> Five apples are on the table. Three are red and the rest are green. How many apples are green? $3 + ? = 5$ , $5 - 3 = ?$	<b>Both Addends Unknown<sup>c</sup></b> Grandma has five flowers. How many can she put in her red vase and how many in her blue vase? $5 = 0 + 5$ , $5 = 5 + 0$ $5 = 1 + 4$ , $5 = 4 + 1$ $5 = 2 + 3$ , $5 = 3 + 2$
<b>Compare<sup>d</sup></b>	<b>Difference Unknown</b> (“How many more?” version): Lucy has two apples. Julie has five apples. How many more apples does Julie have than Lucy?  (“How many fewer?” version): Lucy has two apples. Julie has five apples. How many fewer apples does Lucy have than Julie? $2 + ? = 5$ , $5 - 2 = ?$	<b>Bigger Unknown</b> (Version with “more”): Julie has three more apples than Lucy. Lucy has two apples. How many apples does Julie have?  (Version with “fewer”): Lucy has 3 fewer apples than Julie. Lucy has two apples. How many apples does Julie have? $2 + 3 = ?$ , $3 + 2 = ?$	<b>Smaller Unknown</b> (Version with “more”): Julie has three more apples than Lucy. Julie has five apples. How many apples does Lucy have?  (Version with “fewer”): Lucy has 3 fewer apples than Julie. Julie has five apples. How many apples does Lucy have? $5 - 3 = ?$ , $? + 3 = 5$

<sup>a</sup>These take apart situations can be used to show all the decompositions of a given number. The associated equations, which have the total on the left of the equal sign, help children understand that the unknowns can be in either position.

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**COMMON CORE STATE STANDARDS INITIATIVE**  
PREPARING AMERICA'S YOUTH FOR COLLEGE & CAREER

There are 10 biscuits on a plate.  
Luis takes some.  
6 biscuits are left.  
How many biscuits does he take?

$6 + ? = 10$   
 $10 - 6 = ?$   
He takes 4 biscuits

Five apples were on the table. I ate some apples. Then there were three apples. How many apples did I eat?  
 $5 - ? = 3$

Change Unknown

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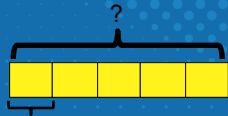
Table 2. Common multiplication and division situations.<sup>2</sup>

	Unknown Product $3 \times 6 = ?$	Group Size Unknown ("How many in each group?" Division) $3 \times ? = 18$ , and $18 \div 3 = ?$	Number of Groups Unknown ("How many groups?" Division) $? \times 6 = 18$ , and $18 \div 6 = ?$
Equal Groups	There are 3 bags with 6 plums in each bag. How many plums are there in all? <i>Measurement example.</i> You need 3 lengths of string, each 6 inches long. How much string will you need altogether?	If 18 plums are shared equally into 3 bags, then how many plums will be in each bag? <i>Measurement example.</i> You have 18 inches of string, which you will cut into 3 equal pieces. How long will each piece of string be?	If 18 plums are to be packed 6 to a bag, then how many bags are needed? <i>Measurement example.</i> You have 18 inches of string, which you will cut into pieces that are 6 inches long. How many pieces of string will you have?
Arrays, <sup>3</sup> Area <sup>4</sup>	There are 3 rows of apples with 6 apples in each row. How many apples are there? <i>Area example.</i> What is the area of a 3 cm by 6 cm rectangle?	If 18 apples are arranged into 3 equal rows, how many apples will be in each row? <i>Area example.</i> A rectangle has area 18 square centimeters. If one side is 3 cm long, how long is a side next to it?	If 18 apples are arranged into equal rows of 6 apples, how many rows will there be? <i>Area example.</i> A rectangle has area 18 square centimeters. If one side is 6 cm long, how long is a side next to it?
Compare	A blue hat costs \$6. A red hat costs 3 times as much as the blue hat. How much does the red hat cost? <i>Measurement example.</i> A rubber band is 6 cm long. How long will the rubber band be when it is stretched to be 3 times as long?	A red hat costs \$18 and that is 3 times as much as a blue hat costs. How much does a blue hat cost? <i>Measurement example.</i> A rubber band is stretched to be 18 cm long and that is 3 times as long as it was at first. How long was the rubber band at first?	A red hat costs \$18 and a blue hat costs \$6. How many times as much does the red hat cost as the blue hat? <i>Measurement example.</i> A rubber band was 6 cm long at first. Now it is stretched to be 18 cm long. How many times as long is the rubber band now as it was at first?
General	$a \times b = ?$	$a \times ? = p$ , and $p \div a = ?$	$? \times b = p$ , and $p \div b = ?$

<sup>2</sup>The language in the array examples shows the easiest form of array problems. A harder form is to use the terms rows and columns: The apples in the grocery window are in 3 rows and 6 columns. How many apples are in there? Both forms are viable.  
<sup>3</sup>Area involves arrays of squares that have been pushed together so that there are no gaps or overlaps, so array problems include these especially important measurement situations.

## Model Types: Multiplicative Parts/Whole

There are 5 boxes of pencils.  
Each box has 12 pencils. How many pencils are there in all?



$5 \times 12 = 60$  There are 60 pencils in all

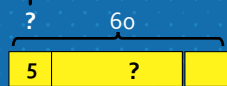


## Division Models

60 peaches divided equally between 5 baskets



Modeling two kinds of division



60 peaches put 5 in a basket, how many baskets?







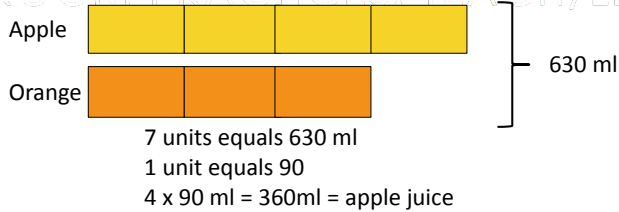


## Model For Ratio

**Learn** Draw models to solve problems involving ratios.

Megan prepares a fruit punch using apple juice and orange juice in the ratio 4 : 3.

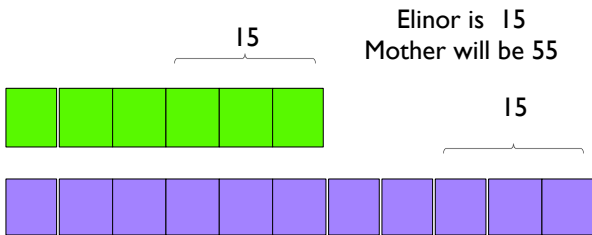
- a) If the total volume of the fruit punch is 630 milliliters, find the volume of apple juice Megan uses.



## Ratio Problems

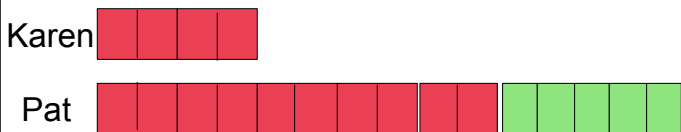
- 1a) Today the ratio of Elinor's age to her mother's age is 3 : 8. After 15 years, the ratio will become 6 : 11.

- a) Find Elinor's age today.  
b) Find her mother's age after 15 years.



## Ratio problems

- The ratio of the number of beads Karen had to # of beads Patricia had was 2:5. After Patricia bought another 75 beads, the ratio became 4:15. How many beads did each girl have at first?



Karen = 60  
Patricia had 150

$75 \div 5 = 15$   
[Green box] = 15

## Eventually students just see it.

It takes 5 pounds of meat to feed 2 tigers,  
how many pounds to feed 11 tigers?

Or

It takes 5 pounds of meat to feed 2 tigers,  
how many pounds to feed 14 tigers?

Or it takes 5 pounds of meat to feed 2 tigers, how many  
pounds to feed 72? or 77?

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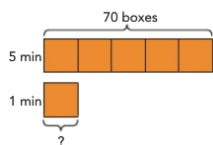
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## From ratios to rates

A machine can pack 70 boxes of spaghetti in 5 minutes.  
At this rate, how many boxes of spaghetti can it pack in 8 minutes?

The machine can pack the same number of boxes every minute.



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## Percent Problems

Given a quantity and its percent, find the whole

Ana has 8% of her CD collection in a box. If there are  
96 CDs in the box, how many CDs are in Ana's whole collection?



96 CDs (8%)

$$\begin{aligned} \text{Whole bar is } 100\% \\ 8\% &= 96 \\ 1\% &= 96 \div 8 = 12 \\ 100\% &= 100 \times 12 = 1200 \end{aligned}$$

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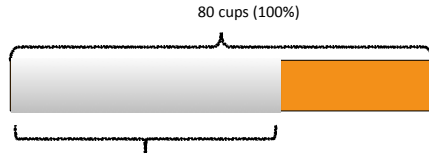
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## Percent Problems

Mr. Jefferson is making 80 cups of fruit punch for the grand opening of his bakery. He uses 52 cups of fruit juice and the rest is sparkling water. What percent of the punch is fruit juice.



$$\begin{aligned} 52/80 \text{ fraction of fruit juice} \\ = 13/20 \\ 13/20 \times 100\% = 65\% \end{aligned}$$

$$\begin{aligned} 80 \text{ cups} &= 100\% \\ 1 \text{ cup} &= 100/80\% = 5/4\% \\ 52 \text{ cups} &= 52 \times 5/4\% \\ &= 65\% \end{aligned}$$

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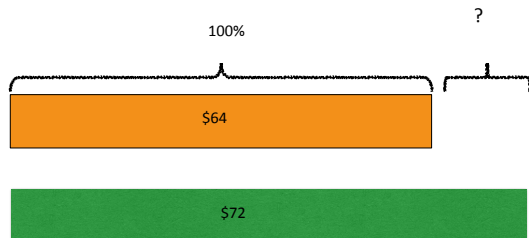
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## Percent Problems

Find percent increase or decrease

Denis bought an antique model train for 64\$. two years later he sold it for 72. What was the percent increase in the price of the model train train.



$$\begin{aligned} 8\$ \text{ increase} \\ 64\$ &= 100\% \\ 1\$ &= 100/64\% \\ 8\$ &= 8 \times 100/64 = 12.5\% \end{aligned}$$

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## Visualizing variable expressions and equations

Caleb is  $x$  years old. His sister is ten years older. She is  $y$  years old. Write an equation that relates their two ages.



$$y = x + 10$$

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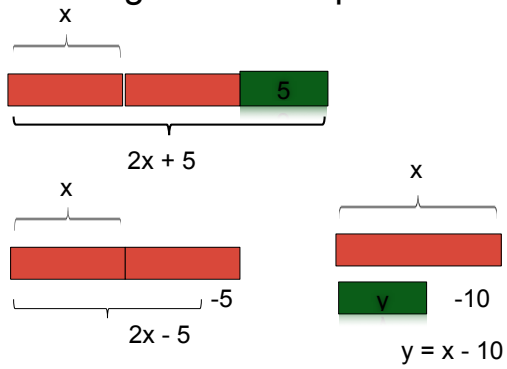
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## Visualizing variable expressions




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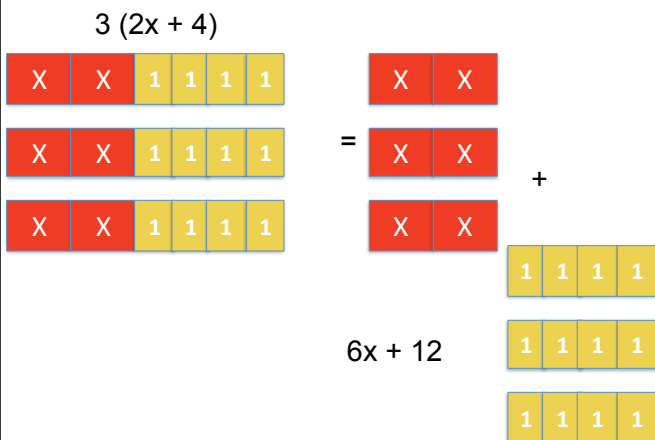
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## Visualize: Distributive property




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### • The Power of visualization

- To understand the problem: see relationships
- To simplify the problem
- To see connections to a related problem
- To cater to individual learning styles
- As a substitute for computation
- As a tool to check the solution
- To transform the problem into a mathematical form

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24. Albert and Bernard just become friends with Cheryl, and they want to know when her birthday is. Cheryl gives them a list of 10 possible dates.

May 15	May 16	May 19
June 17	June 18	
July 14	July 16	
August 14	August 15	August 17

Cheryl then tells Albert and Bernard separately the month and the day of her birthday respectively.

Albert: I don't know when Cheryl's birthday is, but I know that Bernard does not know too.

Bernard: At first I don't know when Cheryl's birthday is, but I know now.

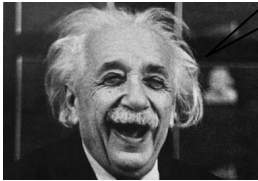
Albert: Then I also know when Cheryl's birthday is.

So when is Cheryl's birthday?

This primary school mathematics question circulating online is stirring up debate. -- PHOTO: KENNETH KONG/FACEBOOK

## Visualization: Enabling students to persevere & have entry points

Do you think a bar model will help?



## 2015 NCTM Annual Conference

Andy Clark  
[andrew.m.clark.1@gmail.com](mailto:andrew.m.clark.1@gmail.com)



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