Engineering a Number Line by construction

A Hands-on, Integrative Project

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Purposes of today's talk

- To share a project that integrates geometry & number sense development, that engages students with hands-on learning, that introduces them to the classical compass-and-straightedge construction techniques, and that gives them some opportunities to grow in problem-solving grit.

Common Core State Standards

- 5-NF-B.3 Interpret a fraction as the division of the numerator by the denominator…
- 6-NS-C.5 Understand that positive and negative numbers are used together to describe quantities having opposite directions or values
- 6-NS-C.6 Understand a rational number as a point on the number line.
- 6-NS-C.7 Understanding ordering and absolute value of rational numbers
- 6-NS-C.8 Include use of coordinates and absolute value to find distances between points
- 7-NS-A Apply and extend previous understandings of operations with fractions…[including] understanding that integers can be divided , provided that the divisor is not zero, and every quotient of integers (with non-zero divisor) is a rational number.
- 8-NS-A.2 Use rational approximations of irrational numbers to compare the size of irrational numbers, locate them approximately on a number line diagram, and estimate the value of expressions.
- 8-G-A Understand congruence and similarity…[including understanding] about the angles created when parallel lines are cut by a transversal
- 8-G-B Understand and apply the Pythagorean theorem

Standards of Mathematical Practice

- **MP1: Make sense of problems and persevere in solving them.**
- MP2: **Reason** abstractly and **quantitatively**.
- MP3: Construct viable arguments and critique the reasoning of others.
- **MP5: Use appropriate tools strategically.**
- **MP6: Attend to precision.**
- MP7: Look for and make use of structure.
- **MP8: Look for and express regularity in repeated reasoning.**

This project engages…

- Number types
  - Whole
  - Integral
  - Rational
  - Irrational
- Geometrical properties
  - Ruler Postulate
  - Segment Addition Postulate
  - Triangle Similarity
  - Parallel Line Properties
  - Pythagorean Theorem
- Your students!
  - Personalized unit
  - Hands-on activity
  - Personal choice of numbers to include

Empowered learners: what do they look like?

Student Feedback

- "I liked how this class was **a lot more hands on and it helped remember** what we learned in class. Thanks for teaching us."
- "This was really cool! [Mr. Ferguson] was really fun and easy to understand. I have never understood geometry as much as I do now!!! I **will definitely continue to use these skills**."
- "I was **interested particularly in the trisection constructions** because I think I could have figured out on my own (if I was trying to) to construct square roots and bisections. The trisections were more complex."
- "I really liked learning Geometry. I learned a lot. I liked how **you showed us everything on the board and gave us questions to answer so we could interact**. I am now excited about Geometry!!!"
- "I found learning about how to put irrational #s on a number line was really interesting. Everything seems so complicated at first, but the second time it was so much easier to understand. I also **liked having my own cubit**."
- "I learned so much in these four lessons. I didn't know there was a way to measure irrational numbers exactly. It was **really fun to do the hands-on work.** Thank you!"
- "I loved how you used a lot of **examples with everyday things such as pens**. After these 4 classes I feel like I understand geometry a lot better."
- "I **now know how to construct for example $\sqrt{2}$ inches**. I also know how to make perfect perpendicular lines without graph paper."
- "Before this lesson, I knew almost nothing about Geometry, but now I probably know more than most people my age. **Nothing was old for me, so all of the lessons were very compelling**. Thank you."
- "I was interested in how to learn how to **graph objects without ruler measurements**. Graphing using random measurement and being able to graph almost any number was truly amazing."
- "This lesson was so interesting and **it was very cool to be able to use the pythagorean theorem**. I had a great time **demonstrating the things we learned for my brother and parents**."

Sequence

- Introduction/revisitation of number types
  - An aside: an opportunity to introduce some language
- Integer construction
  - An aside: can we count 'all of' ($\aleph_o$)them?
- Rational number construction
  - Ok, but can we count 'all of' **these** (are there $\aleph_o$ as well)?
- Irrational number construction
  - How about these we're constructing? How about *all* irrationals?

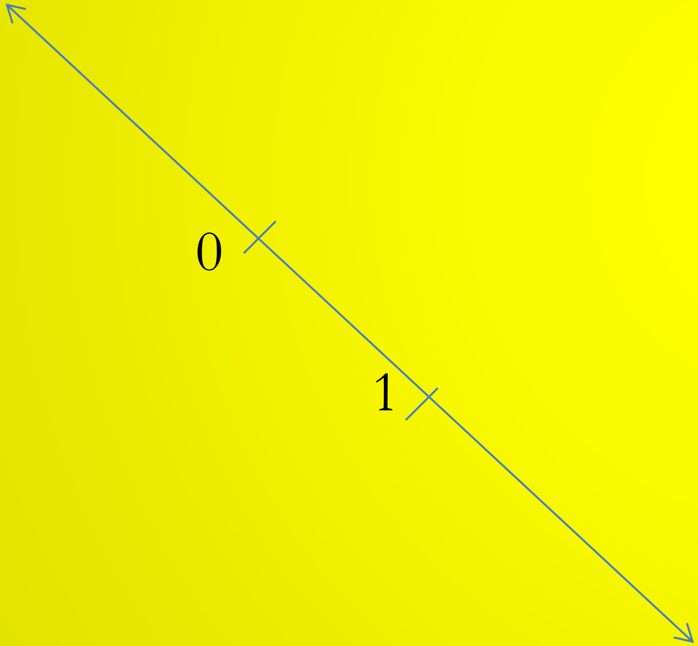
Number Types: an analogy

Numbers	Writing Instruments
Imaginaries	Invisible pens
Irrational numbers	Pens of all kinds (unerasable)
Transcendentals	Highlighters/markers…still pens
Rational numbers	Pencils of all kinds (erasable)
Integers	'Old school*', require sharpening
Natural numbers	Graphite, grey only

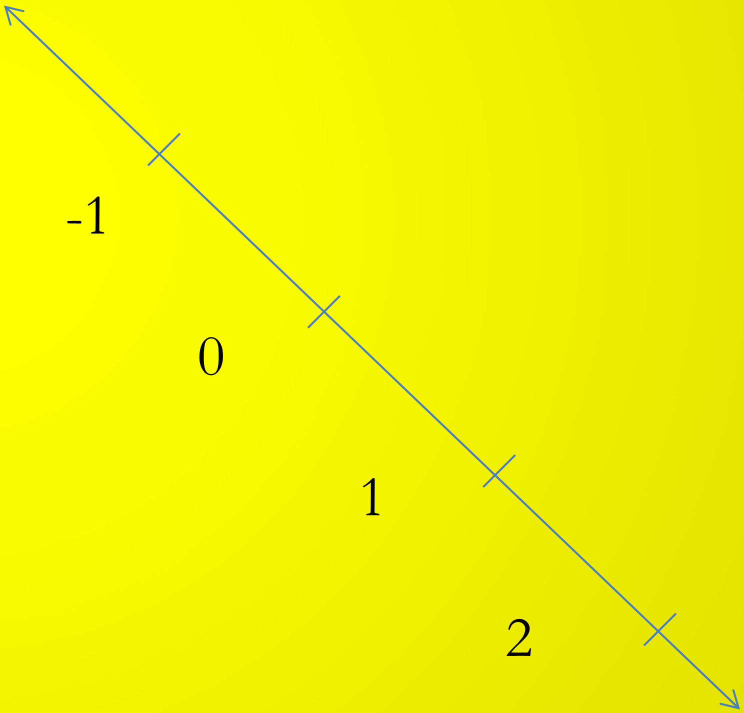
* on 'originals,' Kronecker's quote: "God created the integers, all else is the work of [humanity]."

Now to it…first, the **integers**

- Ruler Postulate



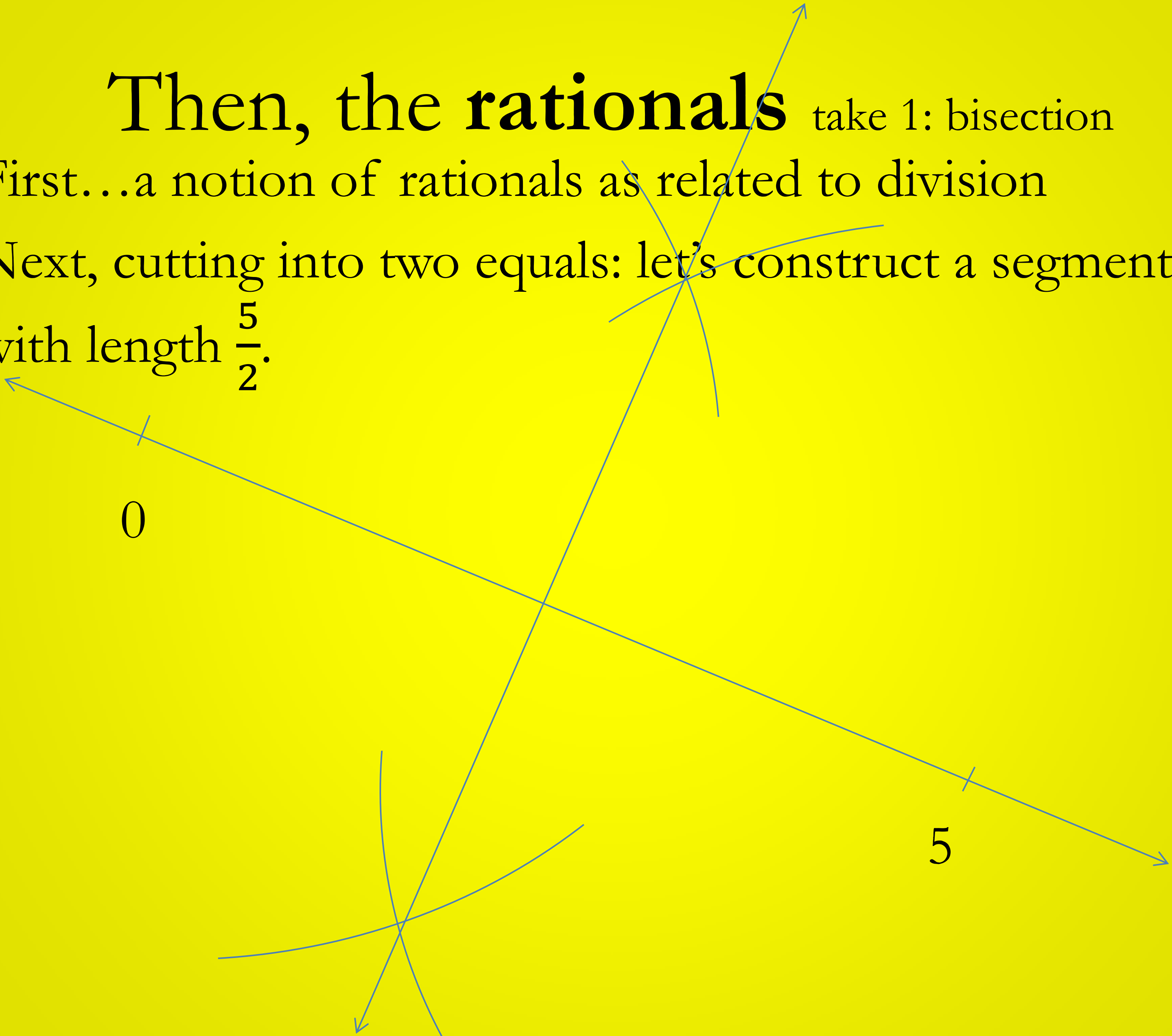
- Segment Addition



By the way…how many are there?

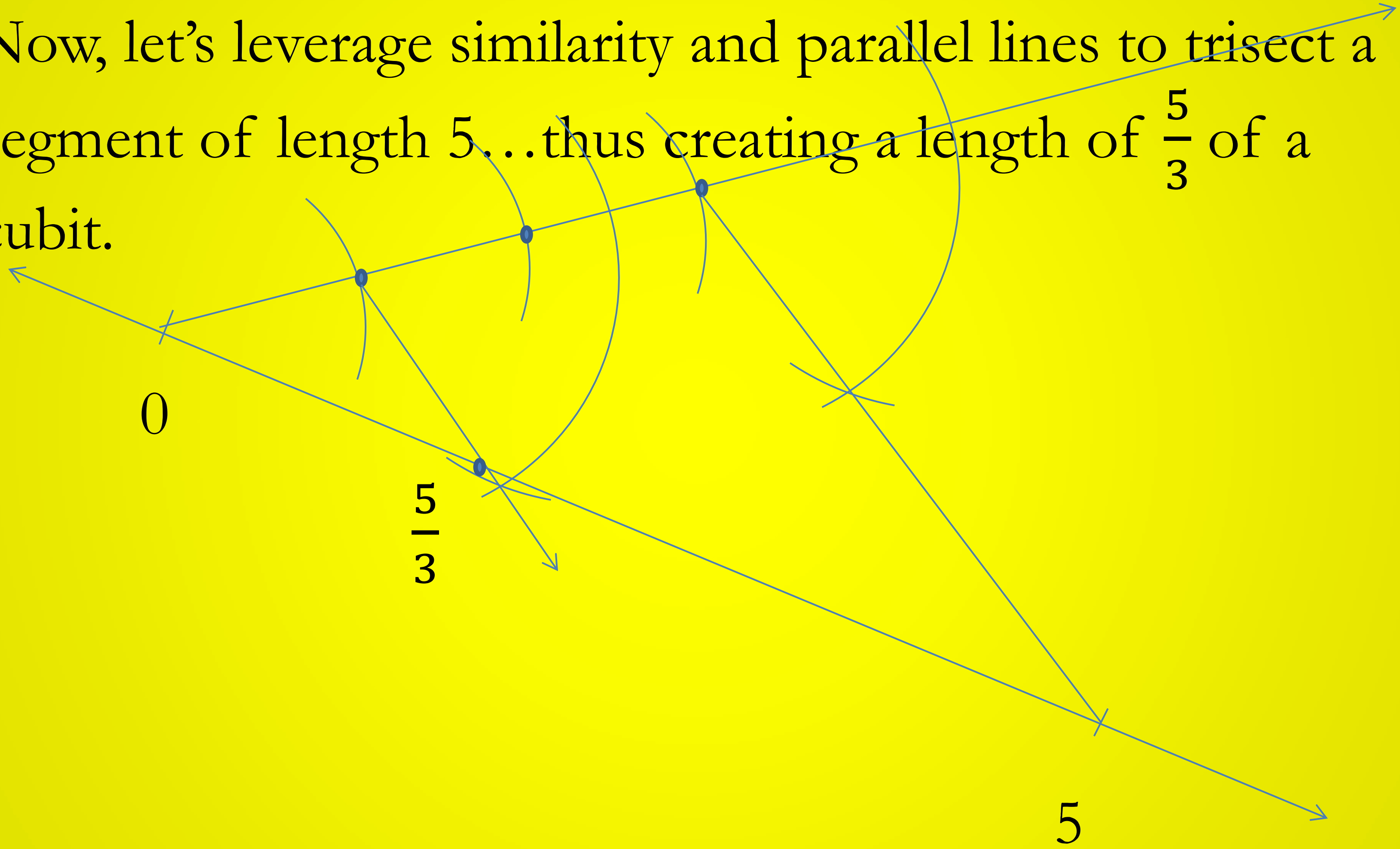
Then, the **rationals** take 1: bisection

- First…a notion of rationals as related to division
- Next, cutting into two equals: let's construct a segment with length $\frac{5}{2}$.



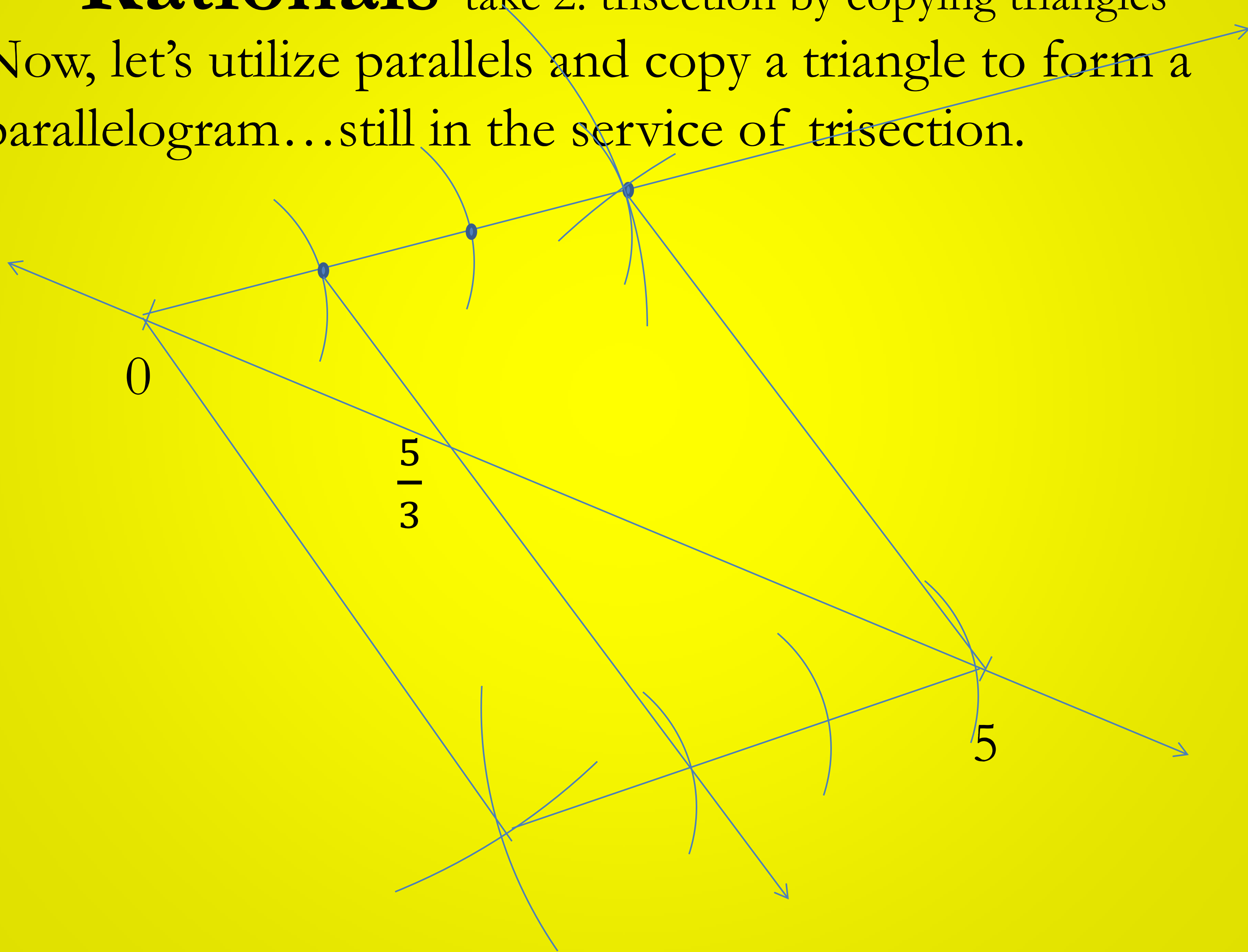
Rationals take 2: trisection (and multisection)

- Now, let's leverage similarity and parallel lines to trisect a segment of length 5…thus creating a length of $\frac{5}{3}$ of a cubit.



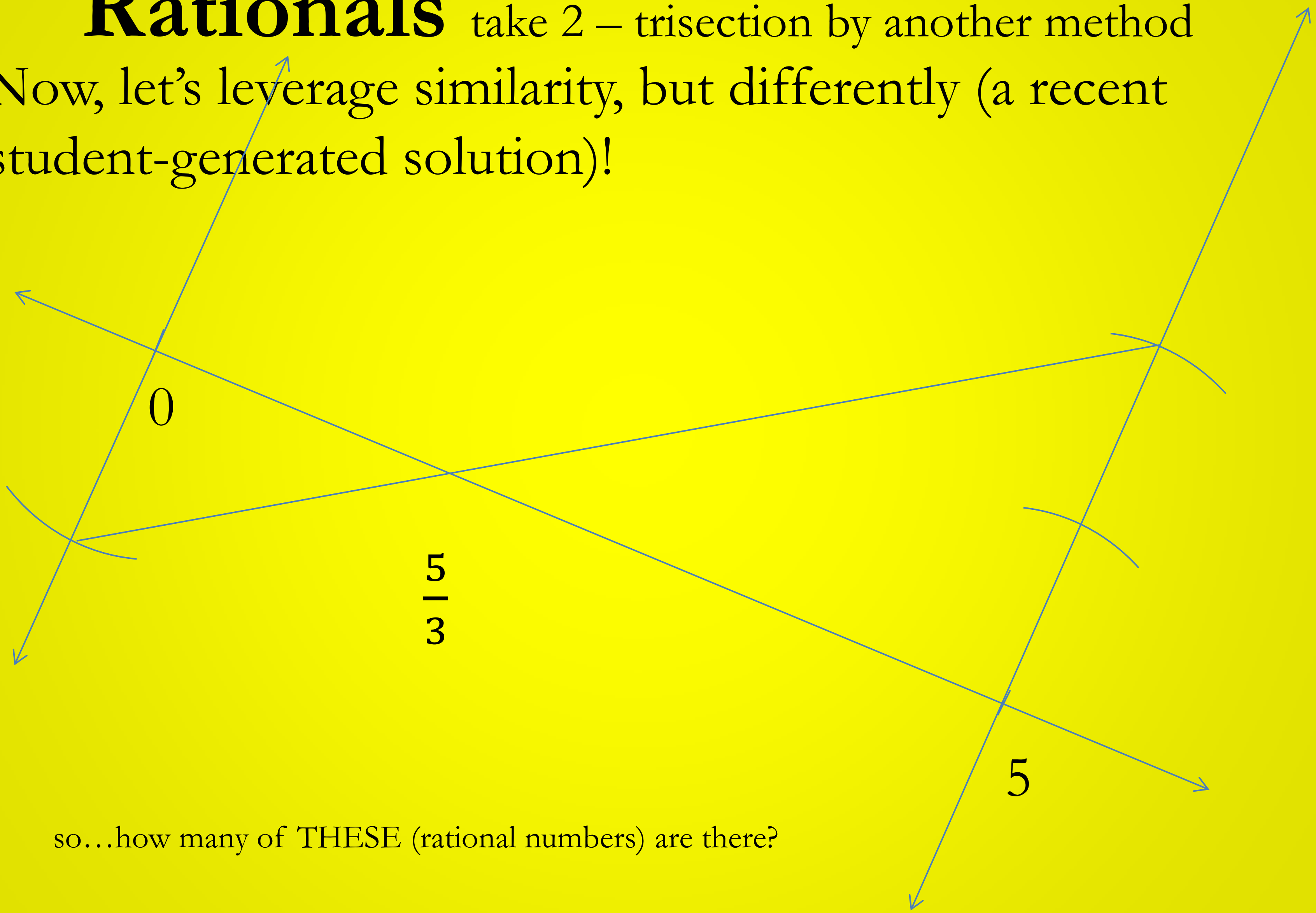
Rationals take 2: trisection by copying triangles

- Now, let's utilize parallels and copy a triangle to form a parallelogram…still in the service of trisection.



Rationals take 2 – trisection by another method

- Now, let's leverage similarity, but differently (a recent student-generated solution)!



so…how many of THESE (rational numbers) are there?

On to the irrationals

Let's try to construct segments of length $\sqrt{2}, \sqrt{13}, \sqrt{7}$, and $\sqrt{41}$.

Since considering square roots are tricky, let's consider squares. W.W.P.D.? (What would Pythagoras do?)

$$l_1{}^2 + l_2{}^2 = h^2$$

1+1=2… so $\sqrt{1}^2 + \sqrt{1}^2 = \sqrt{2}^2$, or $1^2 + 1^2 = \sqrt{2}^2$.

In other words, if we have legs of length 1, our hypotenuse would have length $\sqrt{2}$.

Are there leg lengths that would help you construct $\sqrt{13}$?

4+9=13

What number sentence would help you construct $\sqrt{41}$?

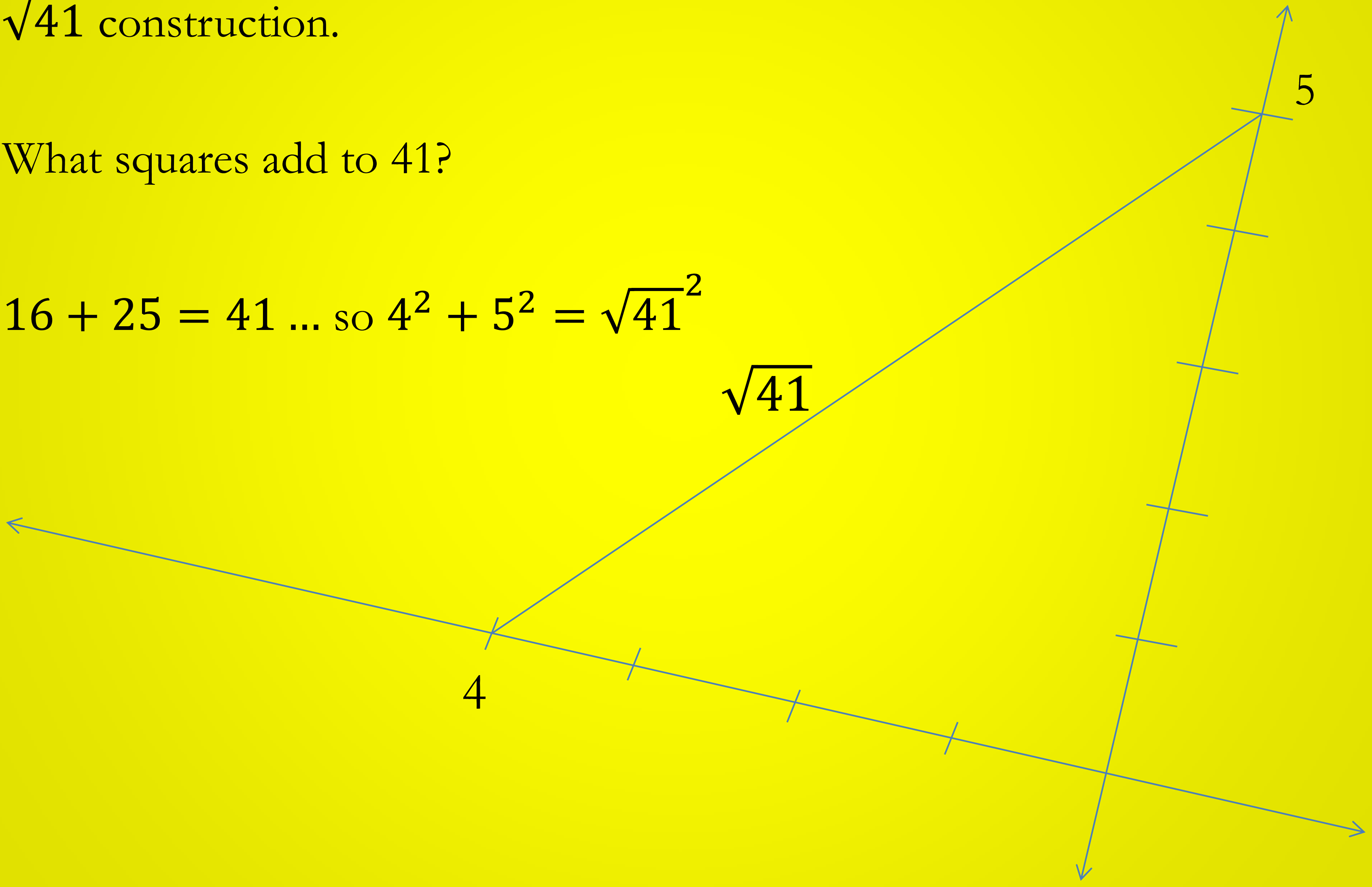
16+25=41

Hmmm…how about $\sqrt{7}$?

16-9=7………so 9+7=16

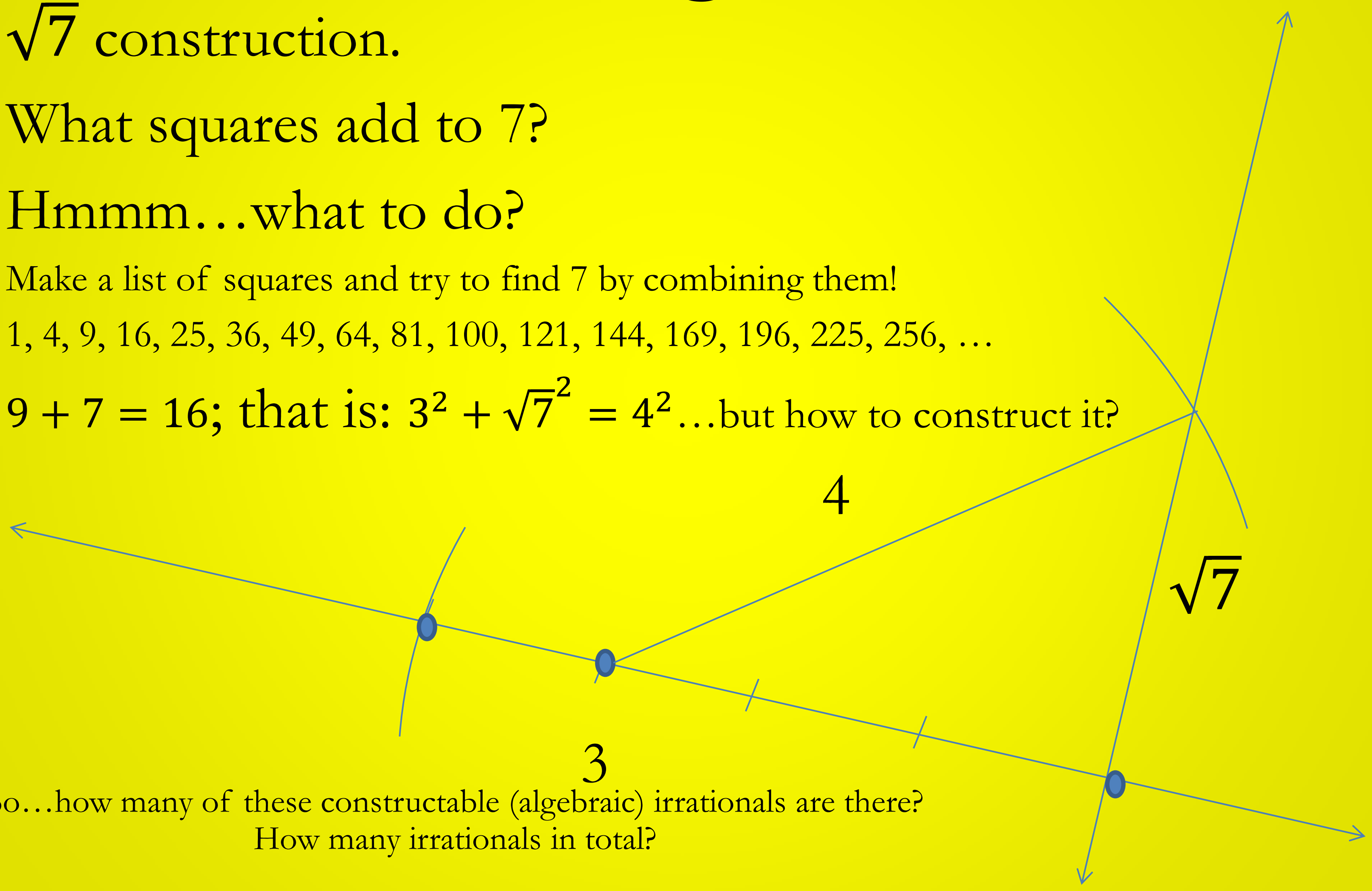
Constructing irrationals

- $\sqrt{41}$ construction.
- What squares add to 41?
- $16 + 25 = 41$ ... so $4^2 + 5^2 = \sqrt{41}^2$



Constructing irrationals

- $\sqrt{7}$ construction.
- What squares add to 7?
- Hmmm…what to do?
- Make a list of squares and try to find 7 by combining them!
- 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, …
- $9 + 7 = 16$; that is: $3^2 + \sqrt{7}^2 = 4^2$…but how to construct it?



So…how many of these constructable (algebraic) irrationals are there?
How many irrationals in total?

Further resources/authors for study:

- On developing mathematical understanding (a classic):

  Courant and Robbins: *What is Mathematics?*

- Four books for every math teacher to have on the shelf:

  Richard Philips: Numbers: *Facts, Figures, and Fiction*
  Steven Strogatz: *The Joy of X*
  David Wells: *The Penguin Dictionary of Curious and Interesting Numbers*
  David Wells: *The Penguin Dictionary of Curious and Interesting Geometry*

- On developing empowerment, belonging, and identity in students by addressing cognitive and non-cognitive factors:

  Daniel Willingham: *Why Don't Kids Like School?*
  Carol Dweck: *Mindset*
  Claude Steele: *Whistling Vivaldi*

A Message on ‘Fit’

“Good teachers join self and subject and students in the fabric of life.”

–Parker Palmer, p.11, *The Courage to Teach*

Thank you for your interest and for participating in this NCTM session.

Please give feedback (email to Bferguson@Lawrenceville.org)– I'd welcome ANY comments you have about this lesson, or about this talk. Thanks! Please also take a book list (on purple paper)

Visit rosenthalprize.momath.org for a link to the entire lesson plan, including teacher notes.

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