A Better Way to Teach Radian Measure

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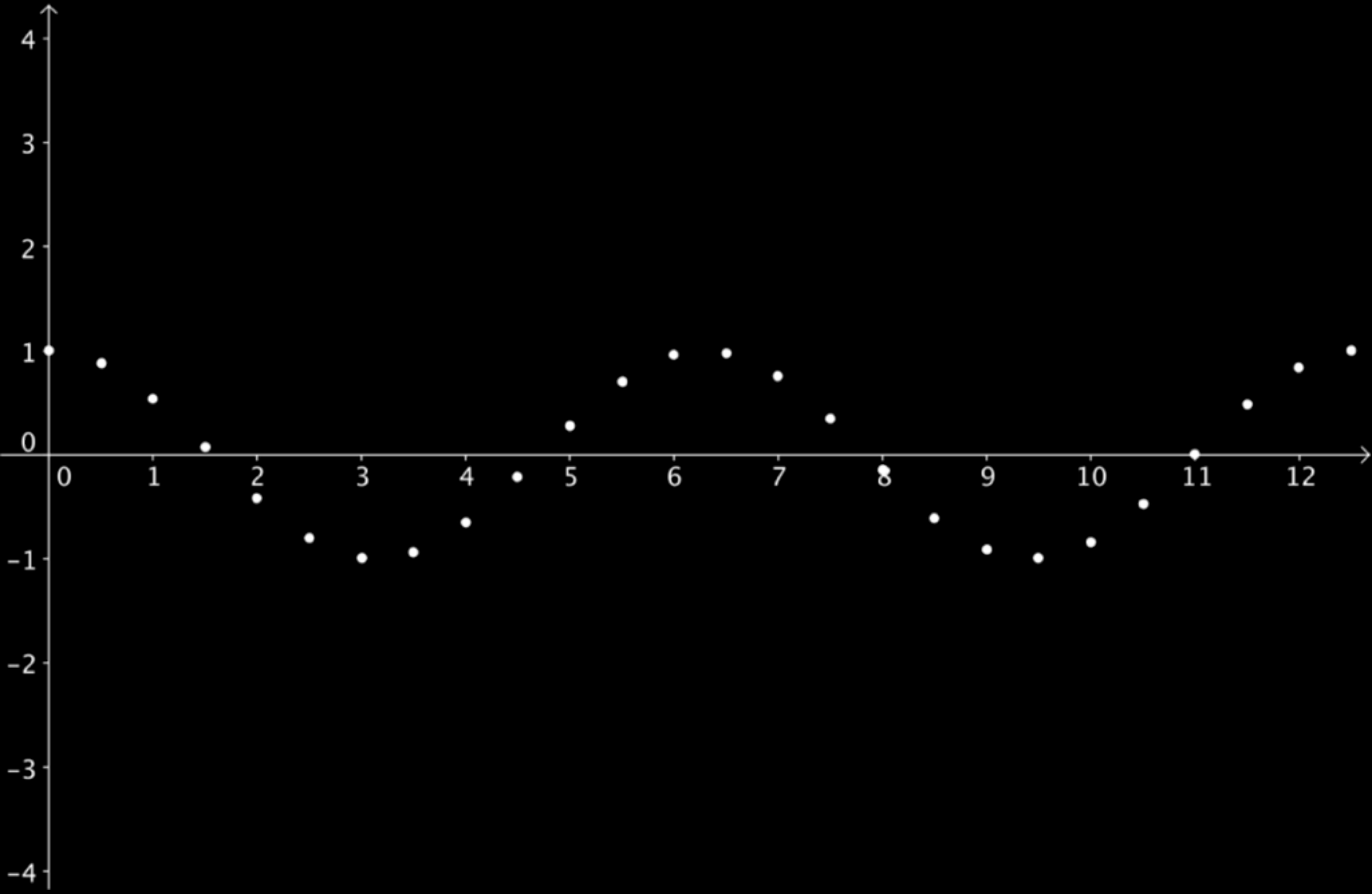
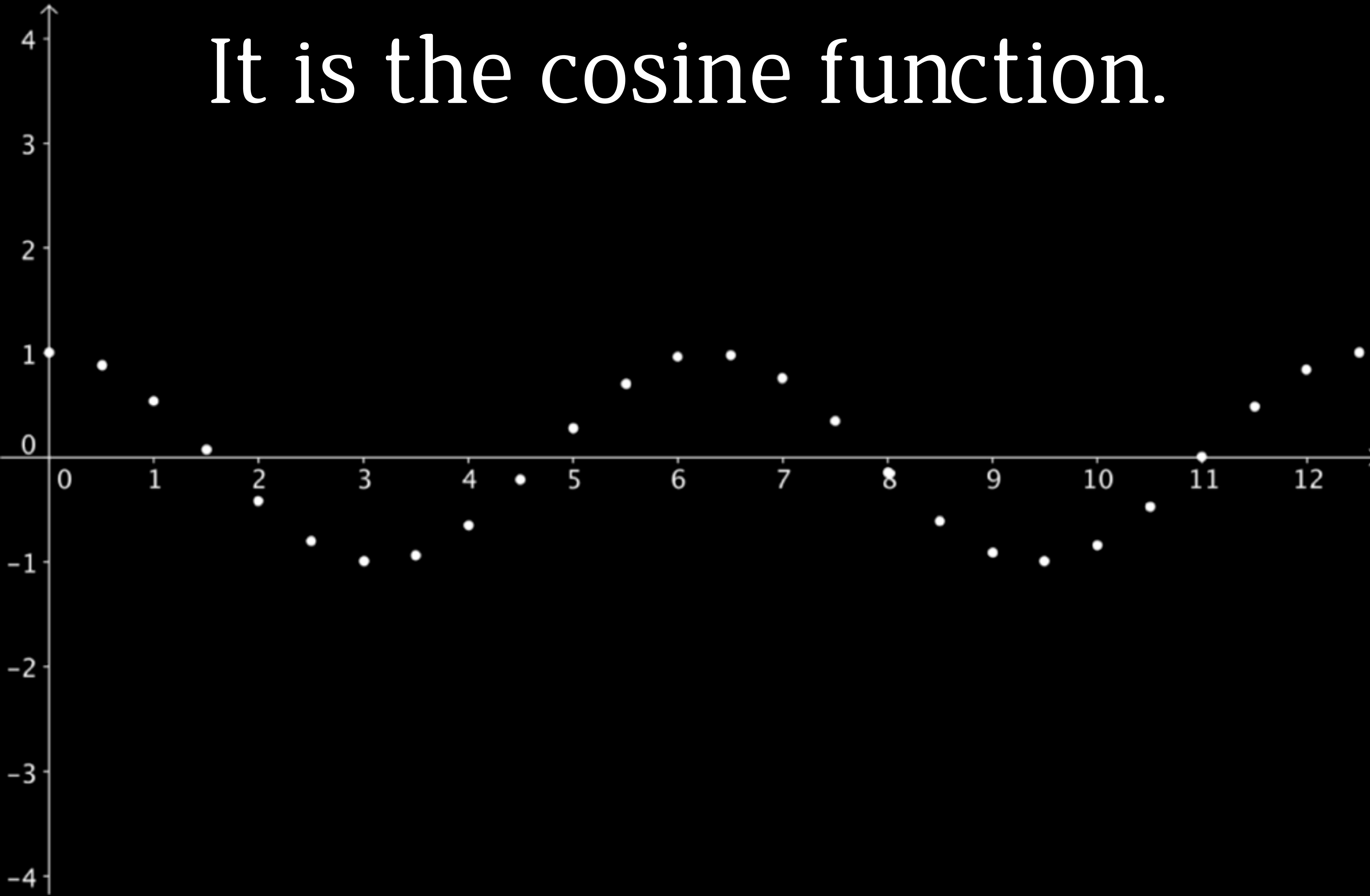
How many students would recognize what this table of values is?

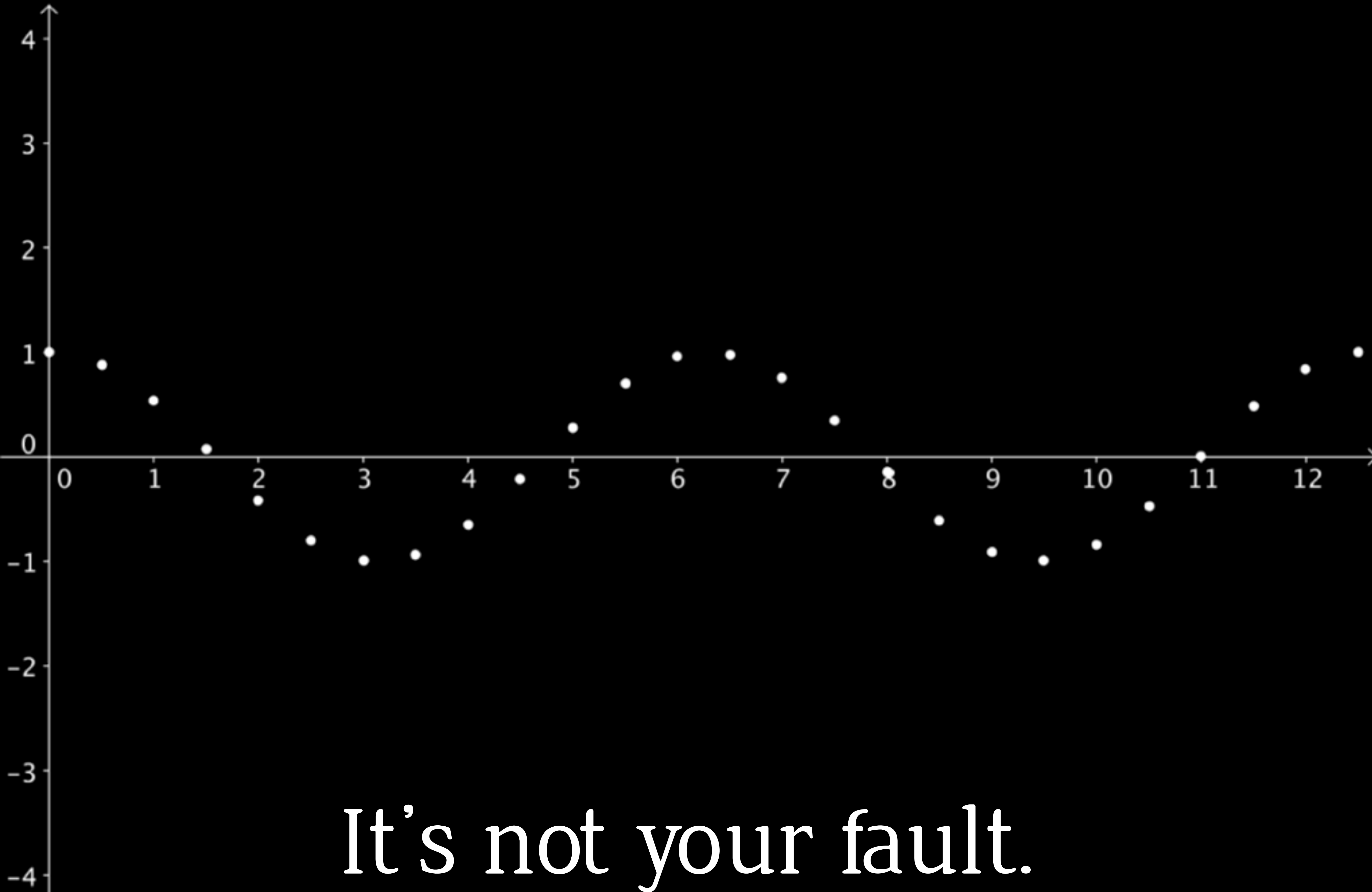
0	1		6.5	0.98
0.5	0.88		7	0.75
1	0.54		7.5	0.35
1.5	0.07		8	–0.15
2	–0.42		8.5	–0.6
2.5	–0.8		9	–0.91
3	–0.99		9.5	–0.997
3.5	–0.94		10	–0.84
4	–0.65		10.5	–0.48
4.5	–0.21		11	0.004
5	0.28		11.5	0.48
5.5	0.71		12	0.84
6	0.96		12.5	0.998

How many students would recognize what this graph is?


4
3
2
1
0
−1
−2
−3
−4
0
1
2
3
4
5
6
7
8
9
10
11
12

It is the cosine function.





It's not your fault.

History Lesson

This is why we have 360° in a circle...

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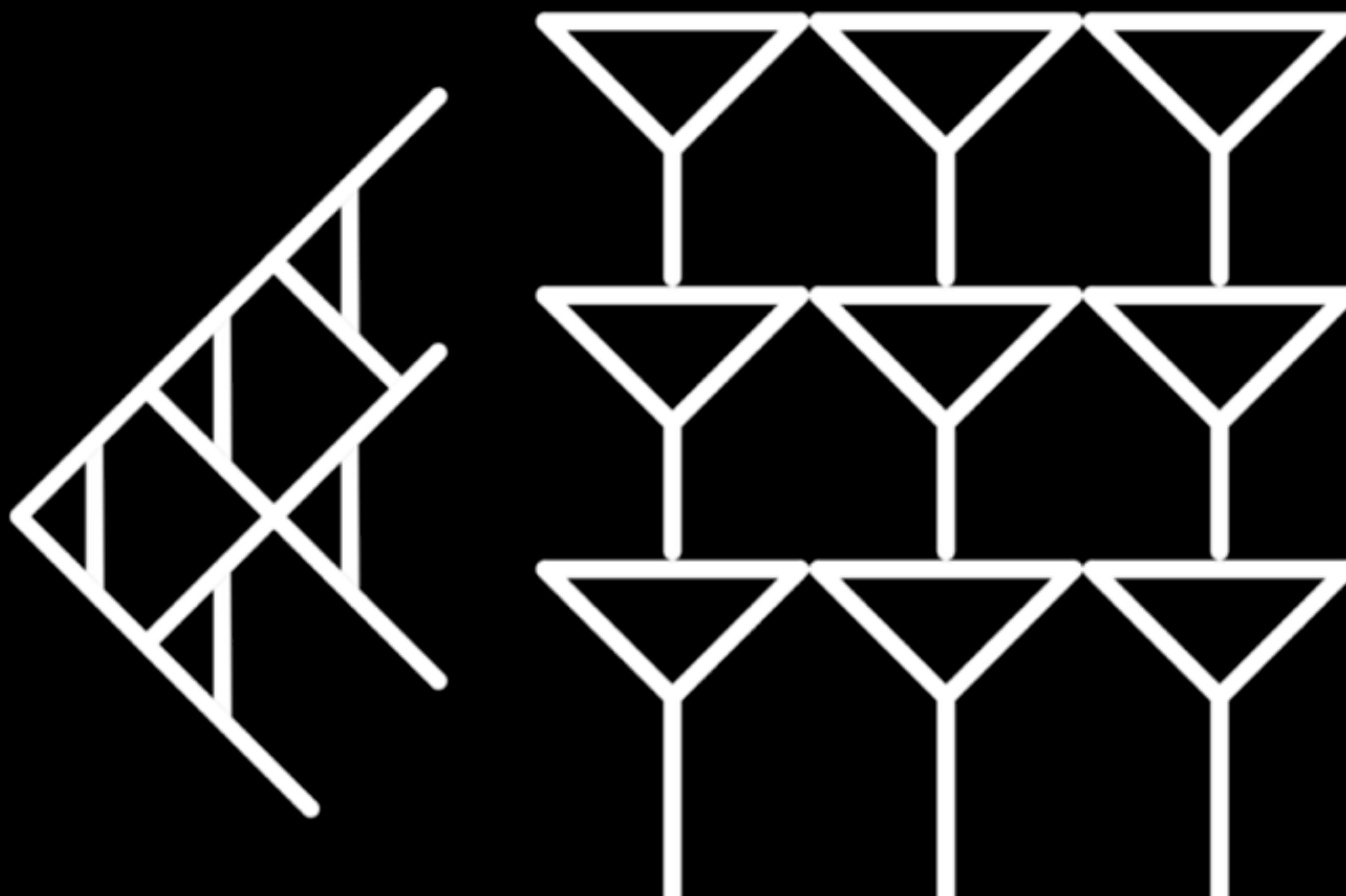
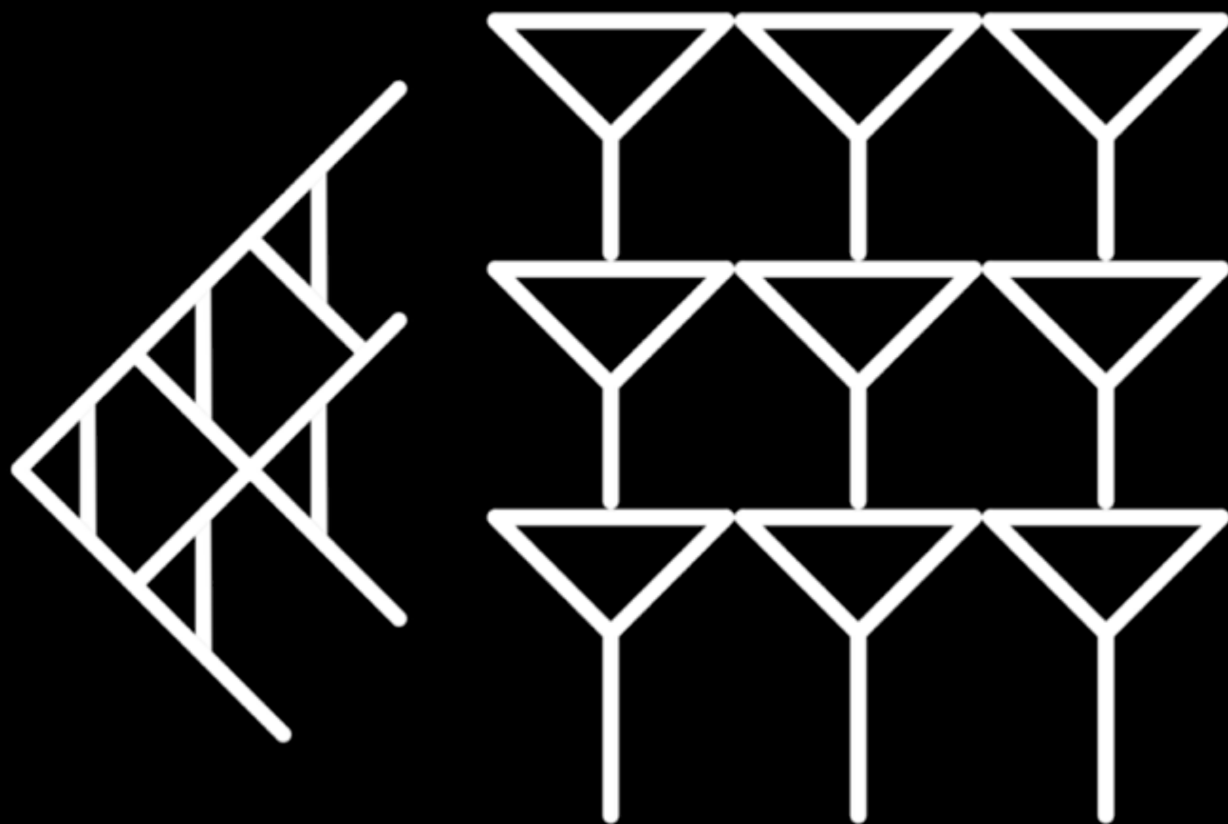
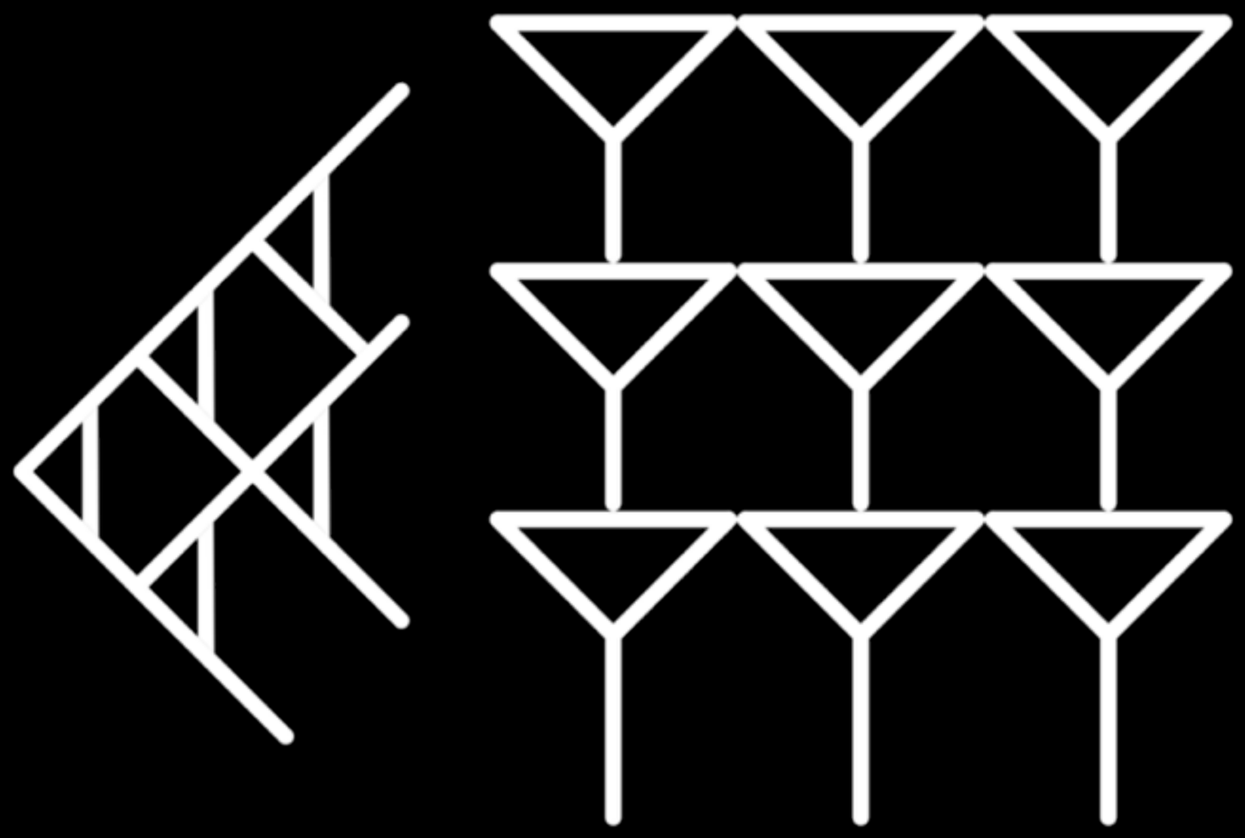
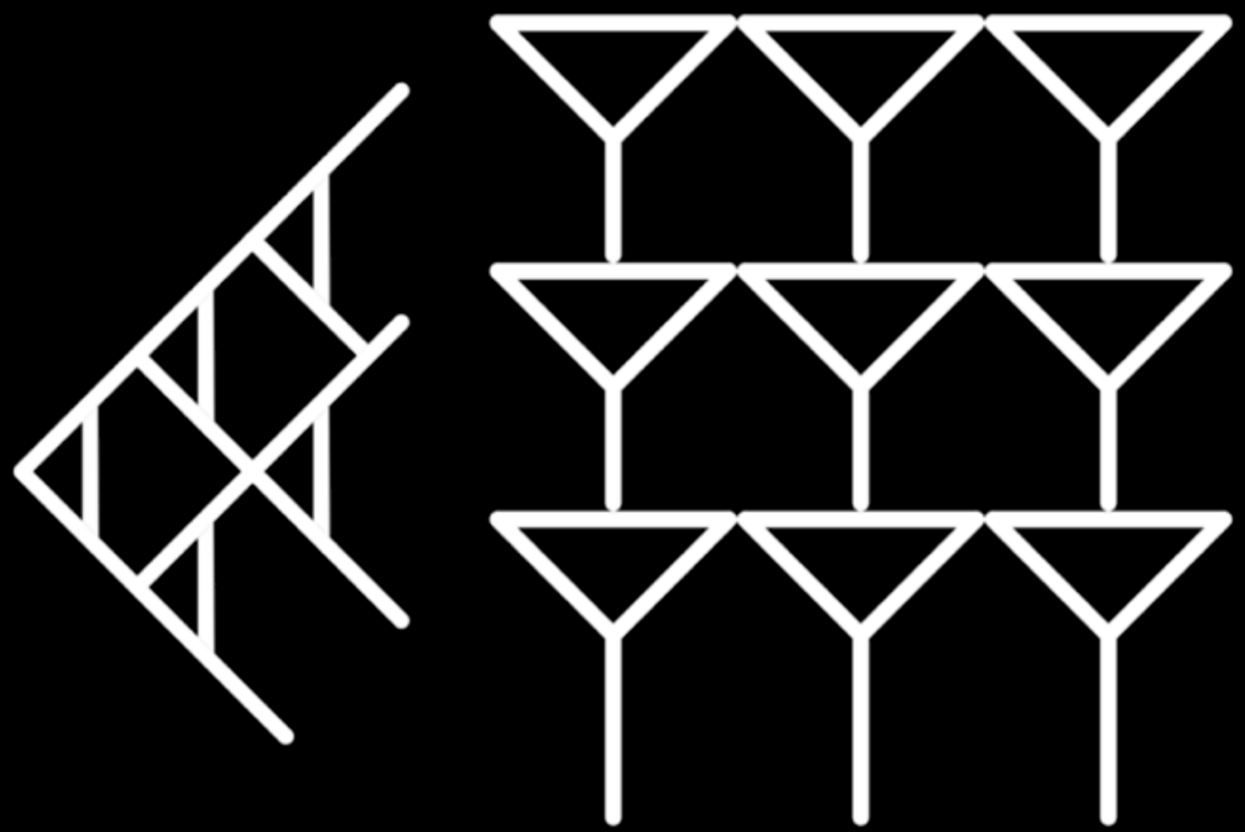




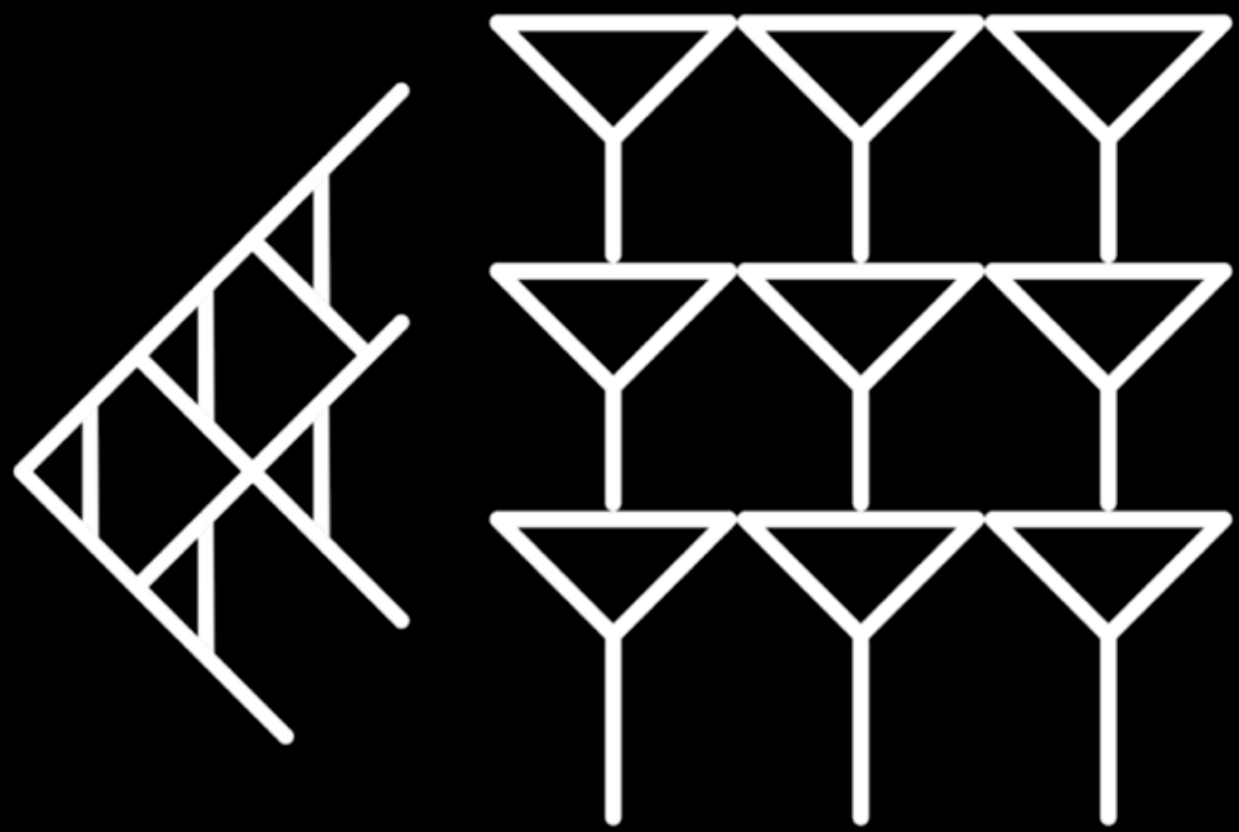


50 + 9 =
largest
Babylonian
numeral



50 + 9 = largest Babylonian numeral

What's our largest numeral?



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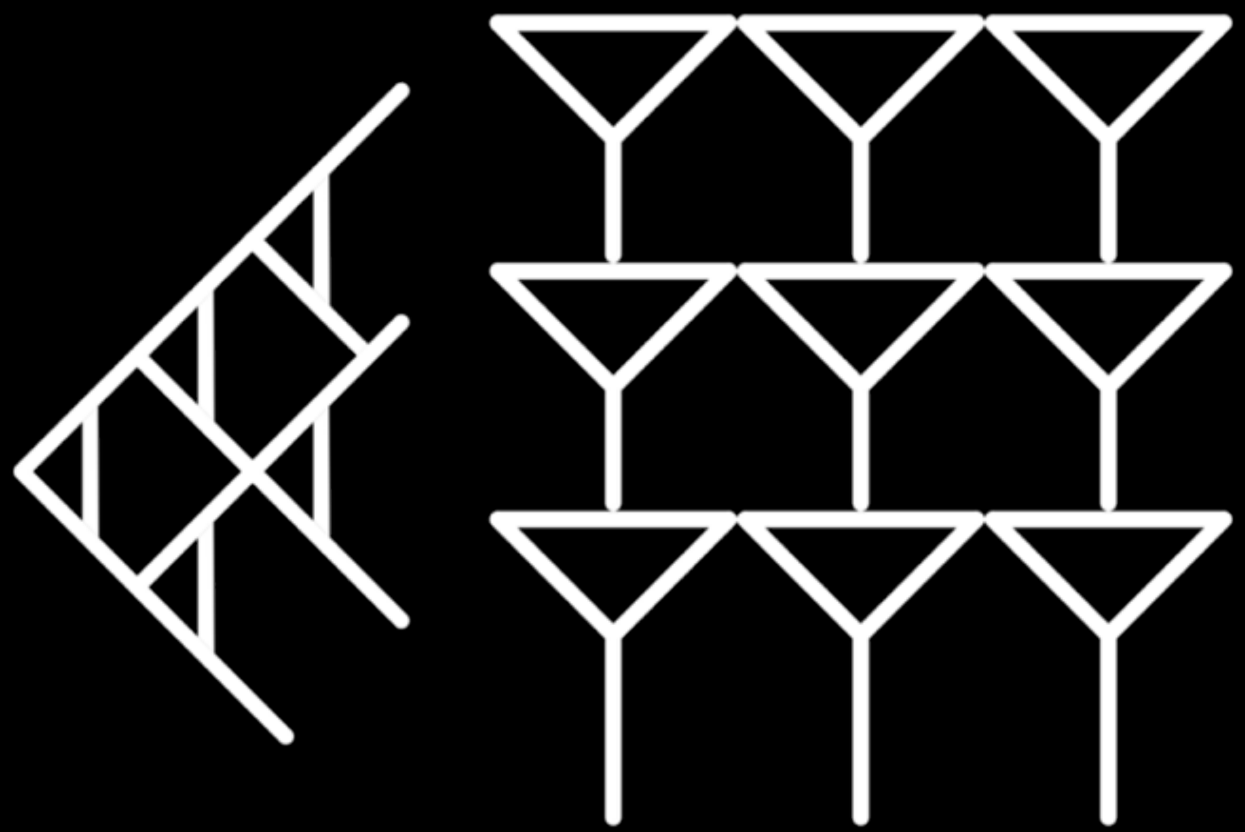
What's our largest numeral? 9



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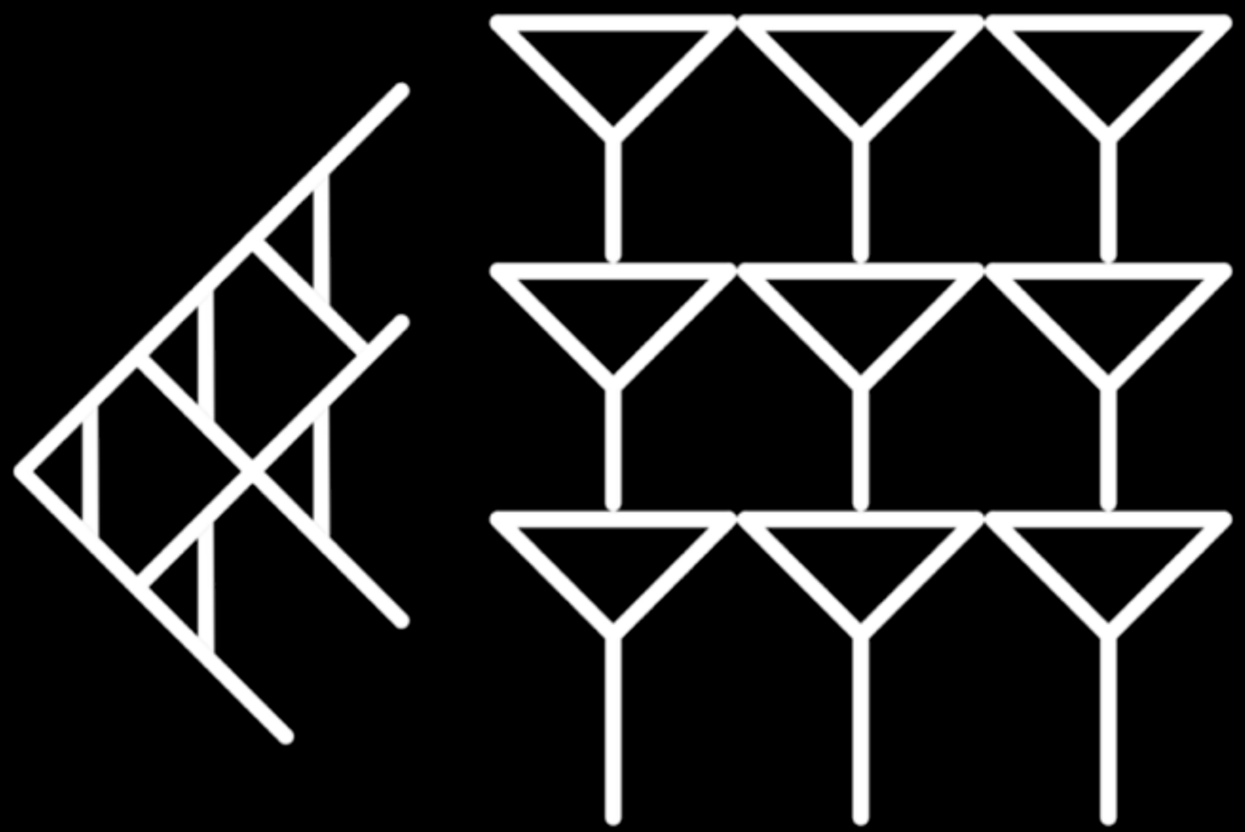
What's our largest numeral? 9

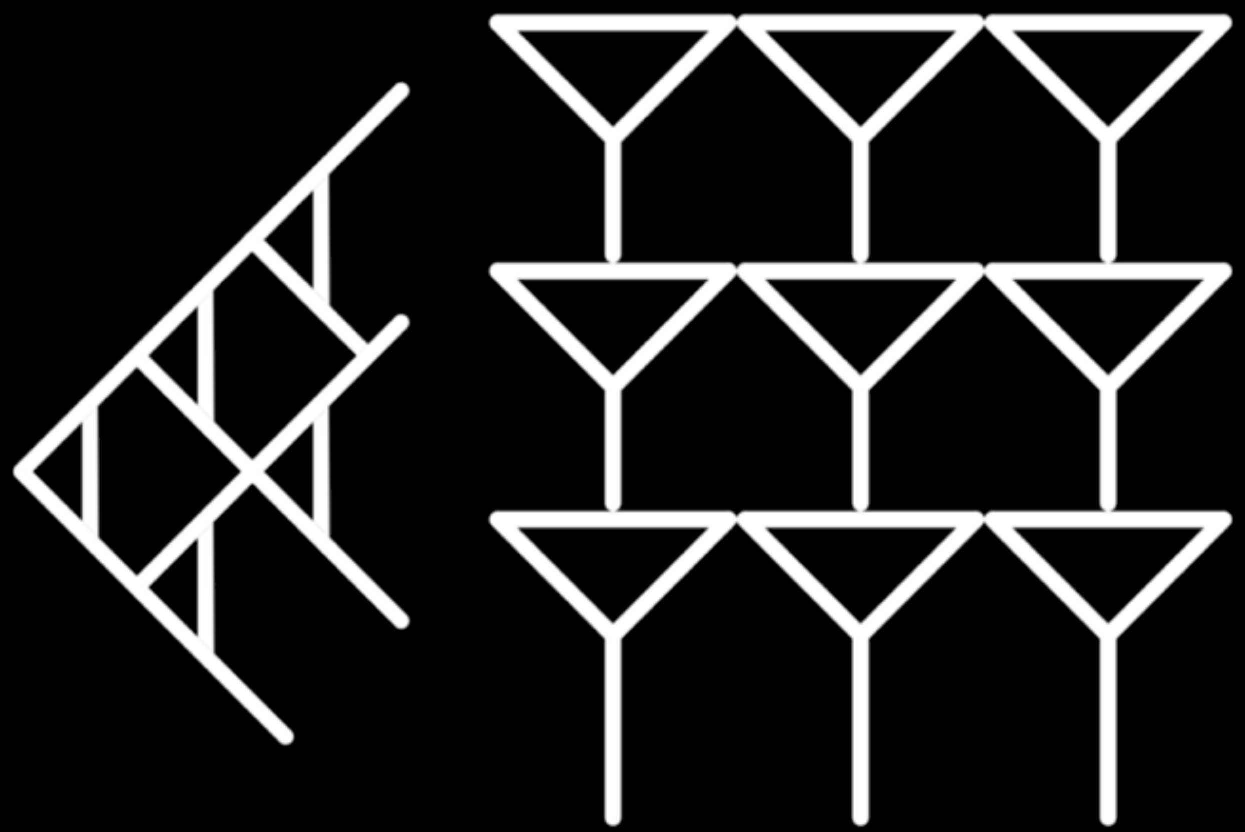
What's our number base?



50 + 9 = largest Babylonian numeral

What's our largest numeral? 9

What's our number base? 10

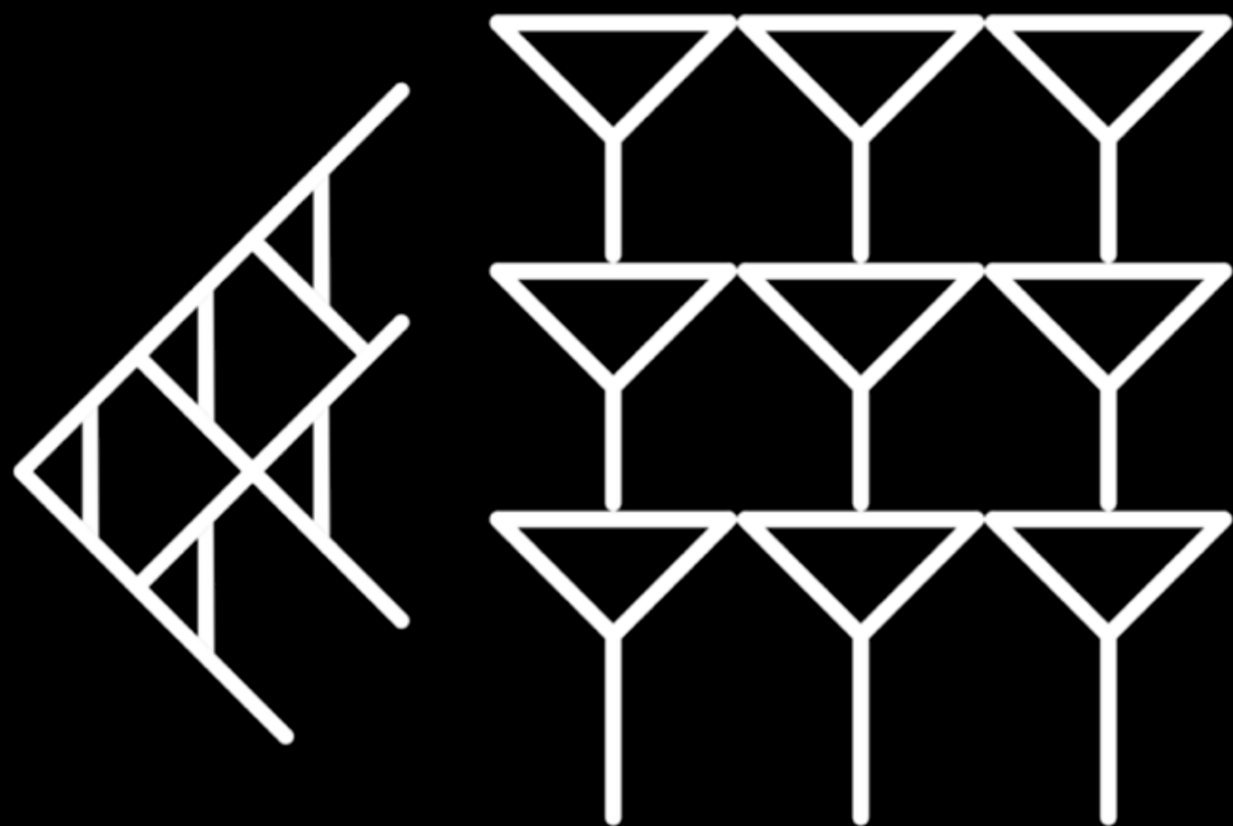


50 + 9 = largest Babylonian numeral

What's our largest numeral? 9

What's our number base? 10

What's their number base?



50 + 9 =
largest
Babylonian
numeral

What's our largest numeral? 9

What's our number base? 10

What's their number base? 60

Why Use Base 60 Numbers?

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You can count to 12 on one hand and keep track of the 12's on the other.

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1 2 3 4 5 6 10 12 15 20 30 60

6 groups of 60 coincides closely with the length of the year.

But what does a degree have to do with circles?

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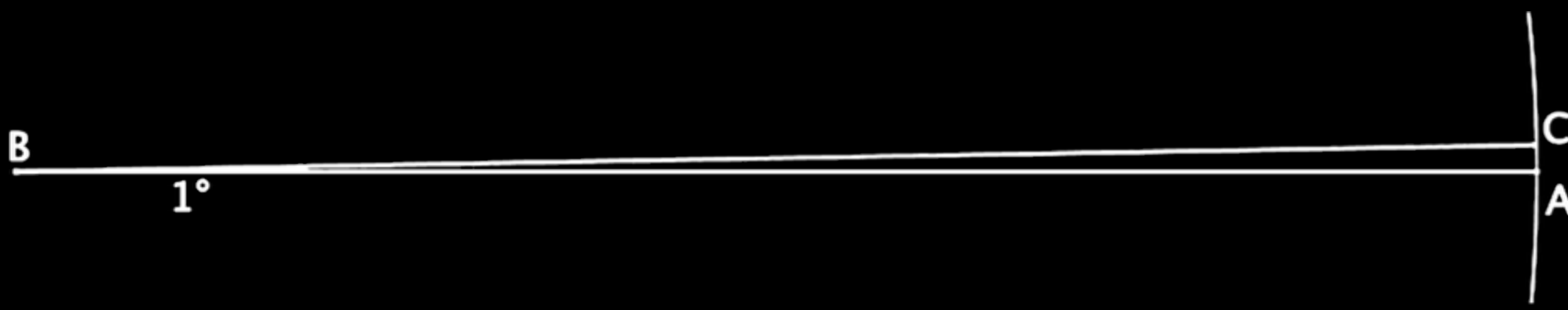

Not much.

But what does a degree have to do with circles?

But, it stuck.

So, we teach our students to measure in degrees, and give them tools to help them learn.

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Like this:

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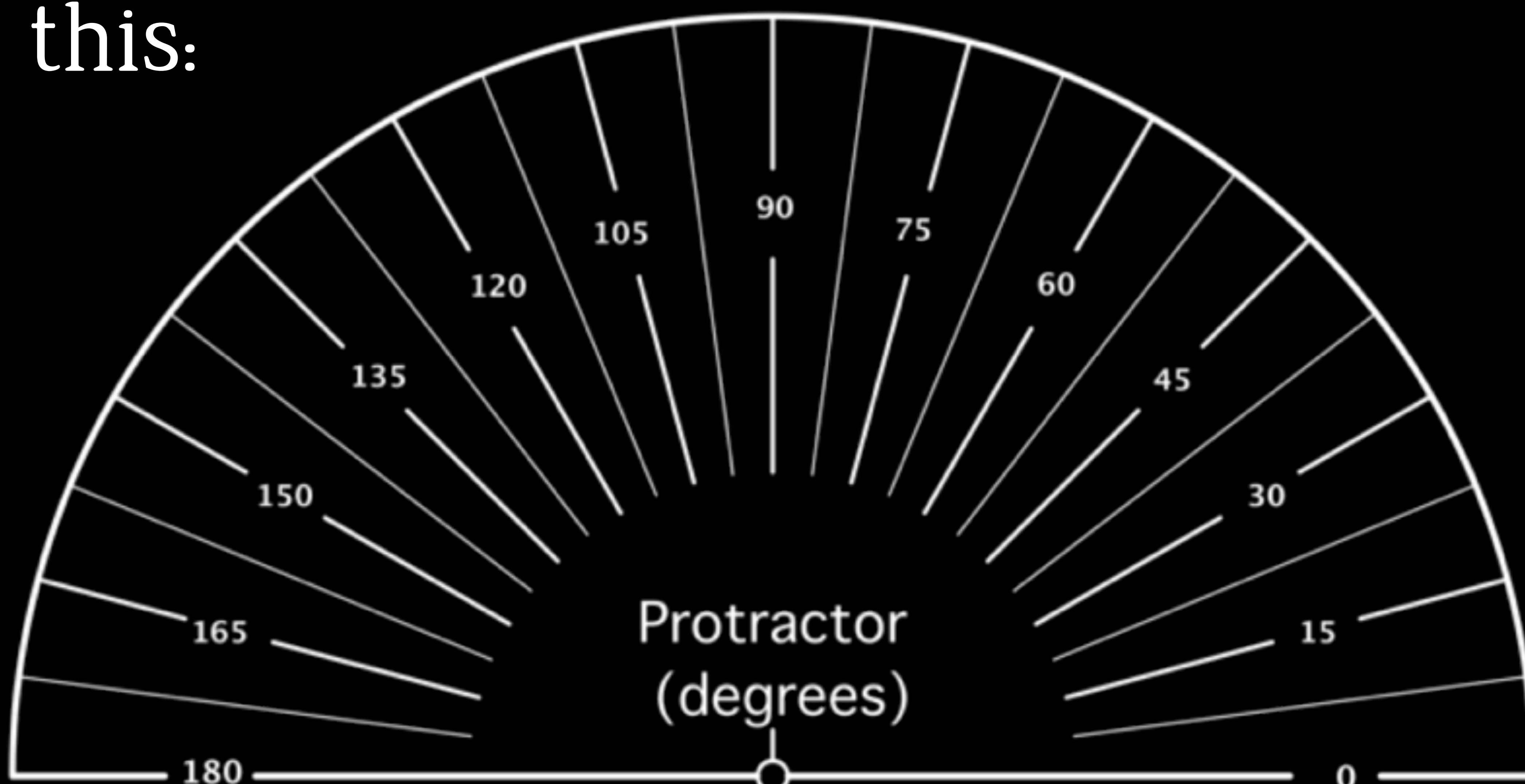

And life is good; they learn all kinds of things, content and secure in their knowledge of angle measure.

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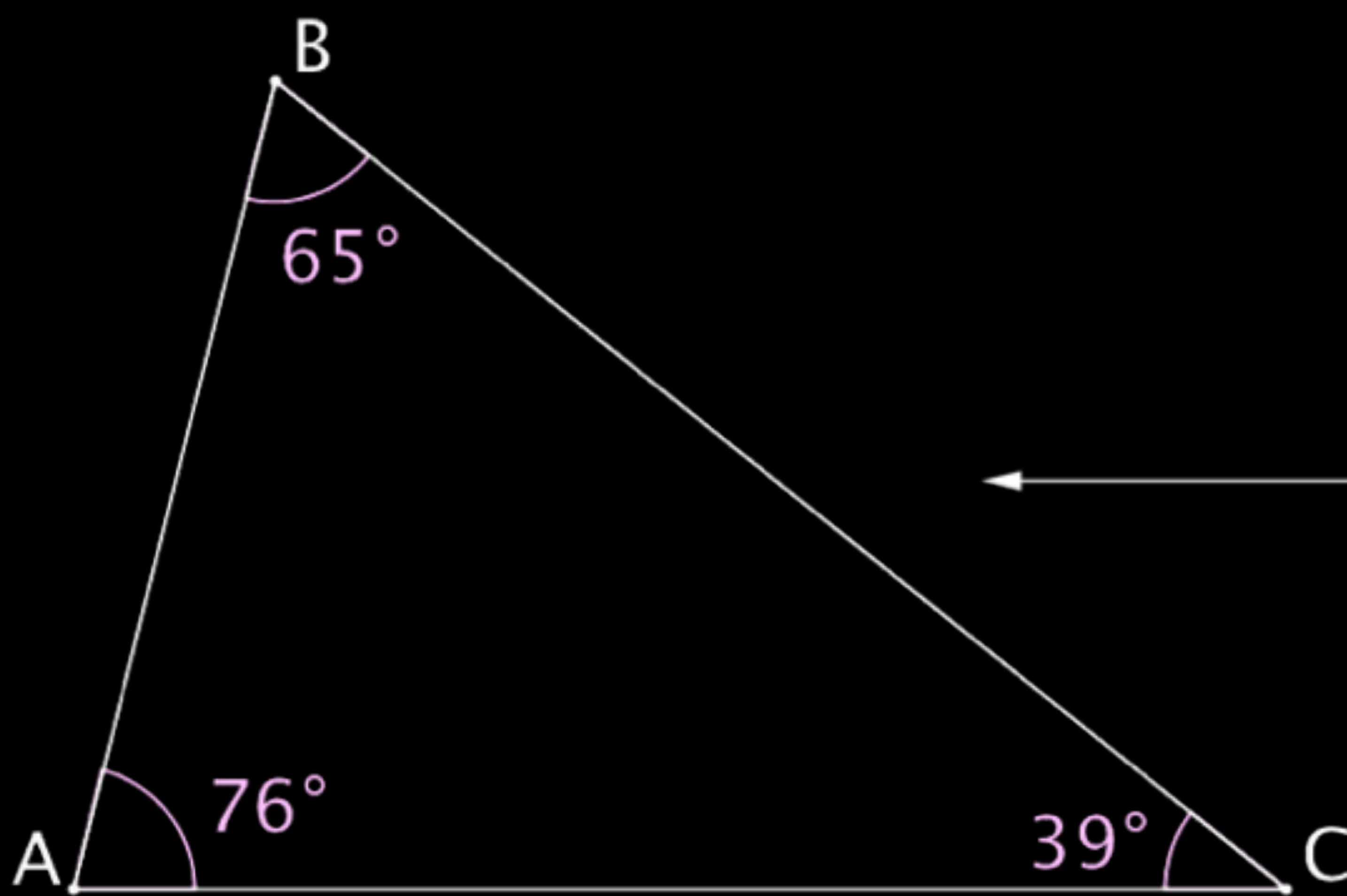

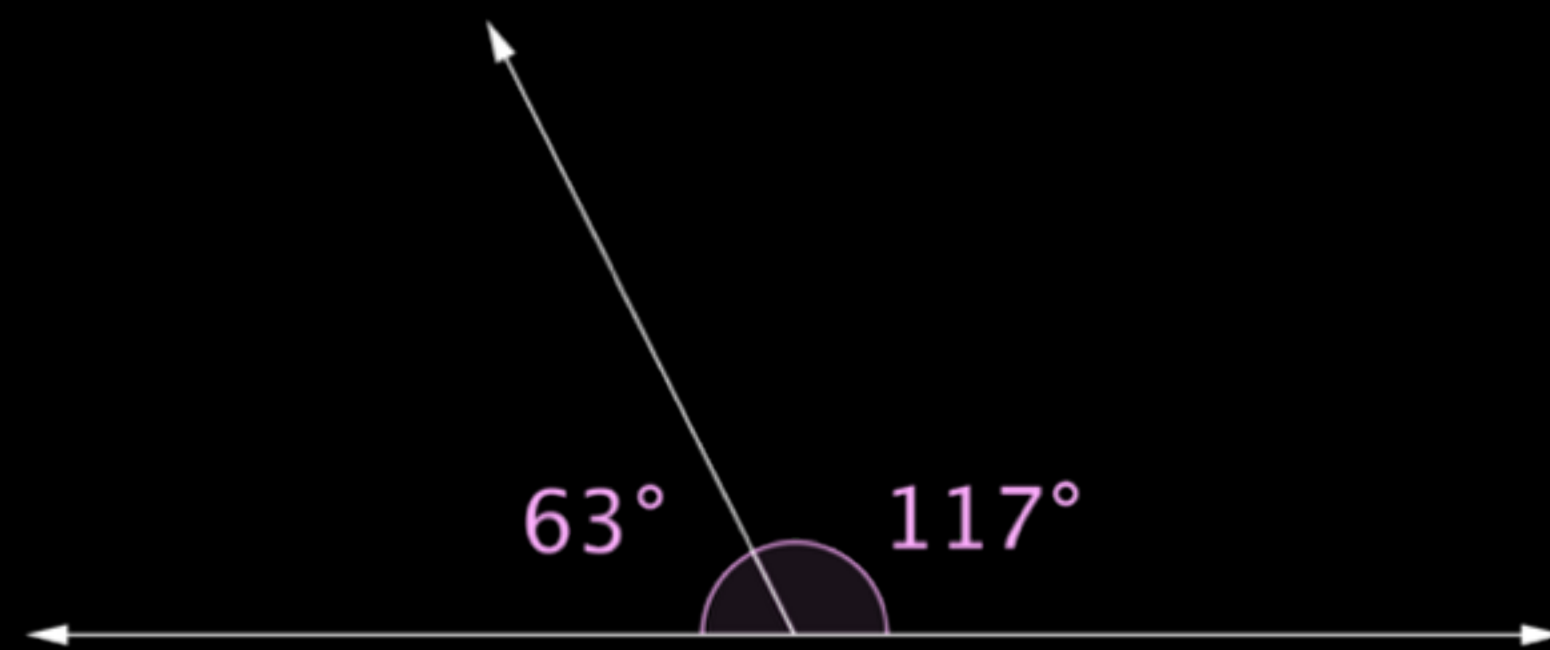

Until…

Until...

$$sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

Imagine precalculus students, encountering their first transcendental functions...

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with a new unit of measure...

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with irrational inputs...

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and irrational outputs...

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CONFUSION!

And this doesn't help much...

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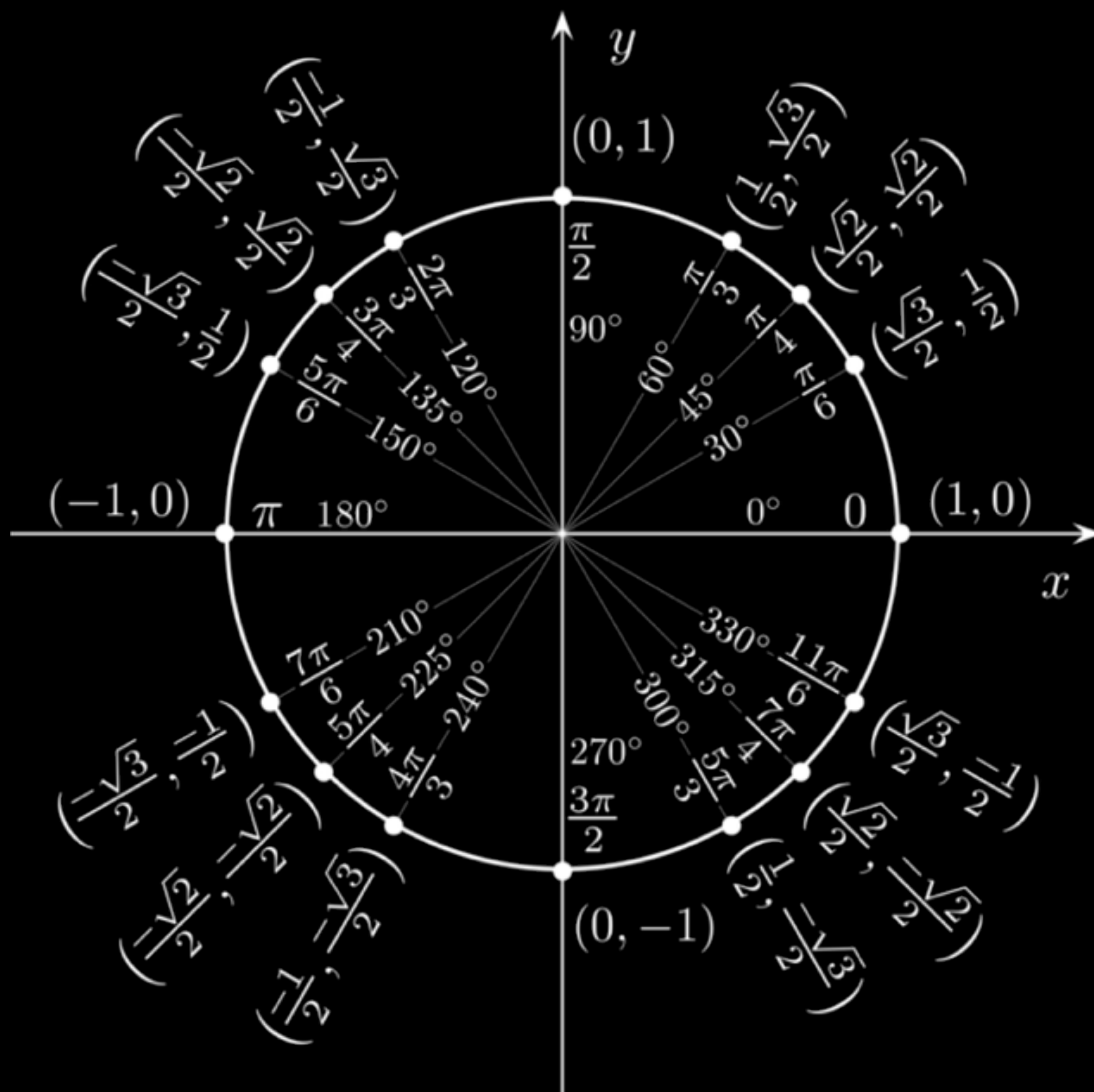

In the Spring of 2011, I was teaching a widely-diverse, public school precal class.

I thought that my students needed practice <u>measuring</u> "in radians," so I looked online and in all the catalogs for radian scale protractors...

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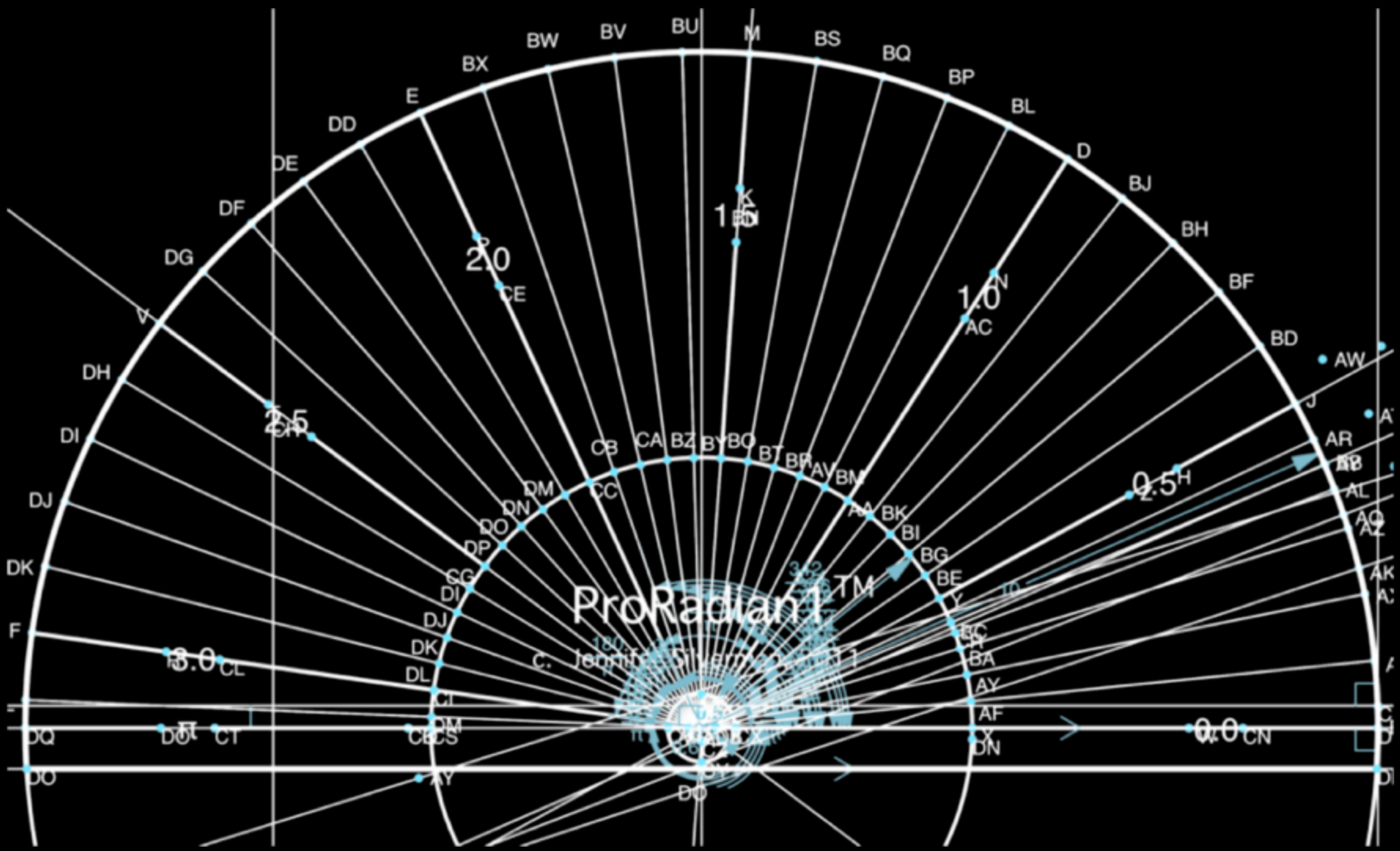
but they didn't exist.

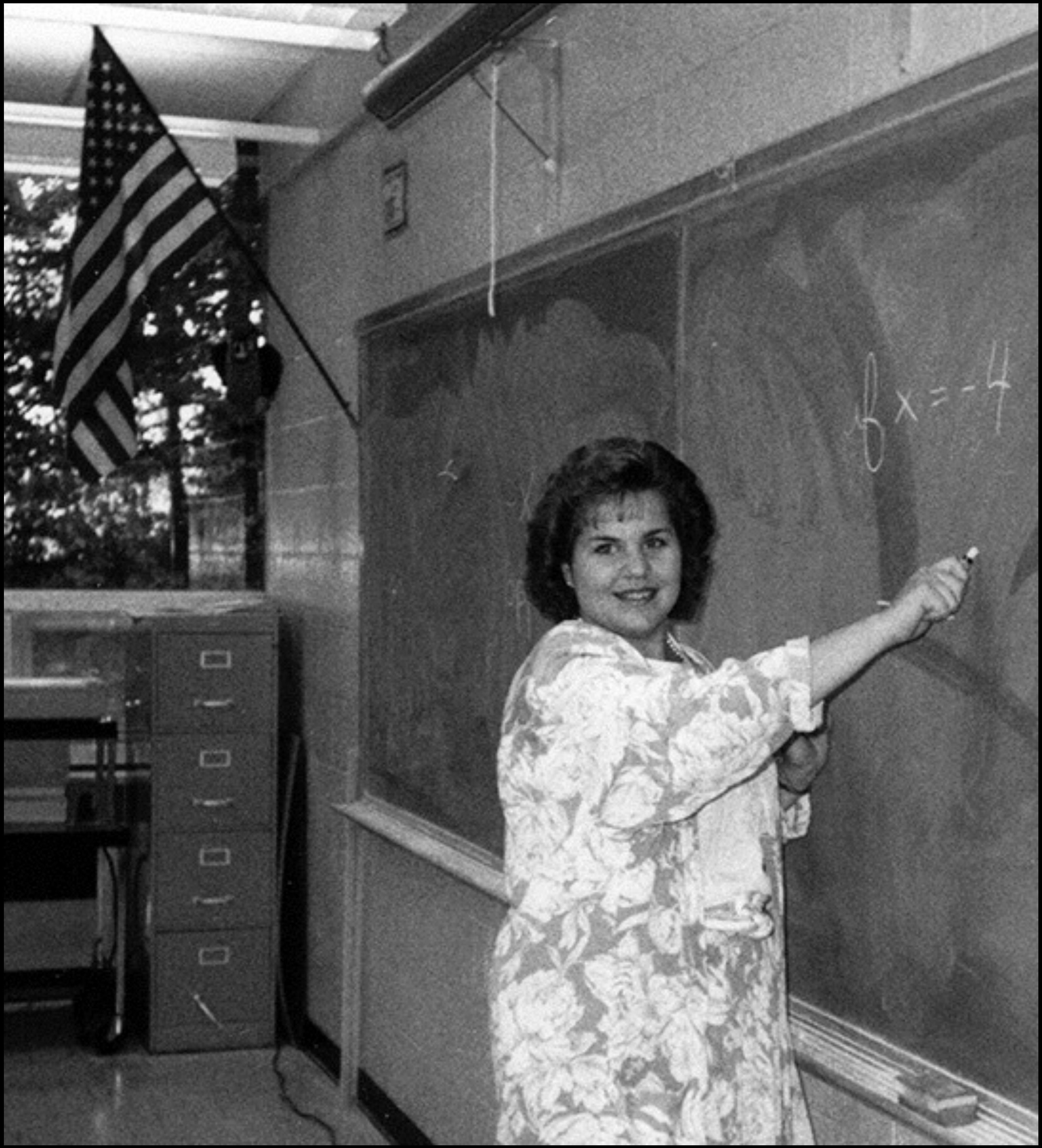
So, I made them...



The next day, I brought in my first prototypes...

Mind you, I'd been teaching math for a long time...

OK, not
that long!



But that long!

...and I had never seen so many lightbulbs going off at once!

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I knew I was on to something.

I found a company in NY State, willing to take-on my small job. Shout out to everyone at Plastic-Craft!

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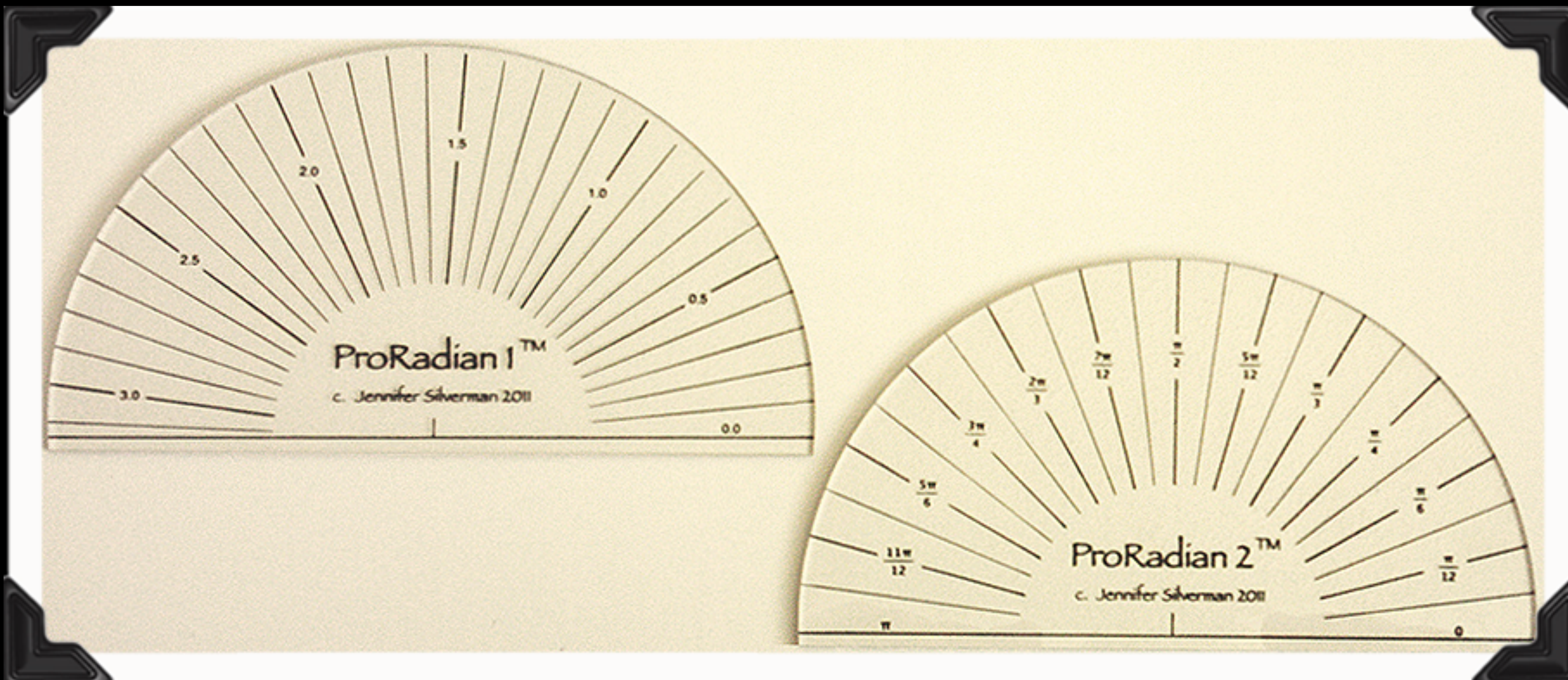
Pretty soon...they arrived!



Look - Twins! Aren't they cute?

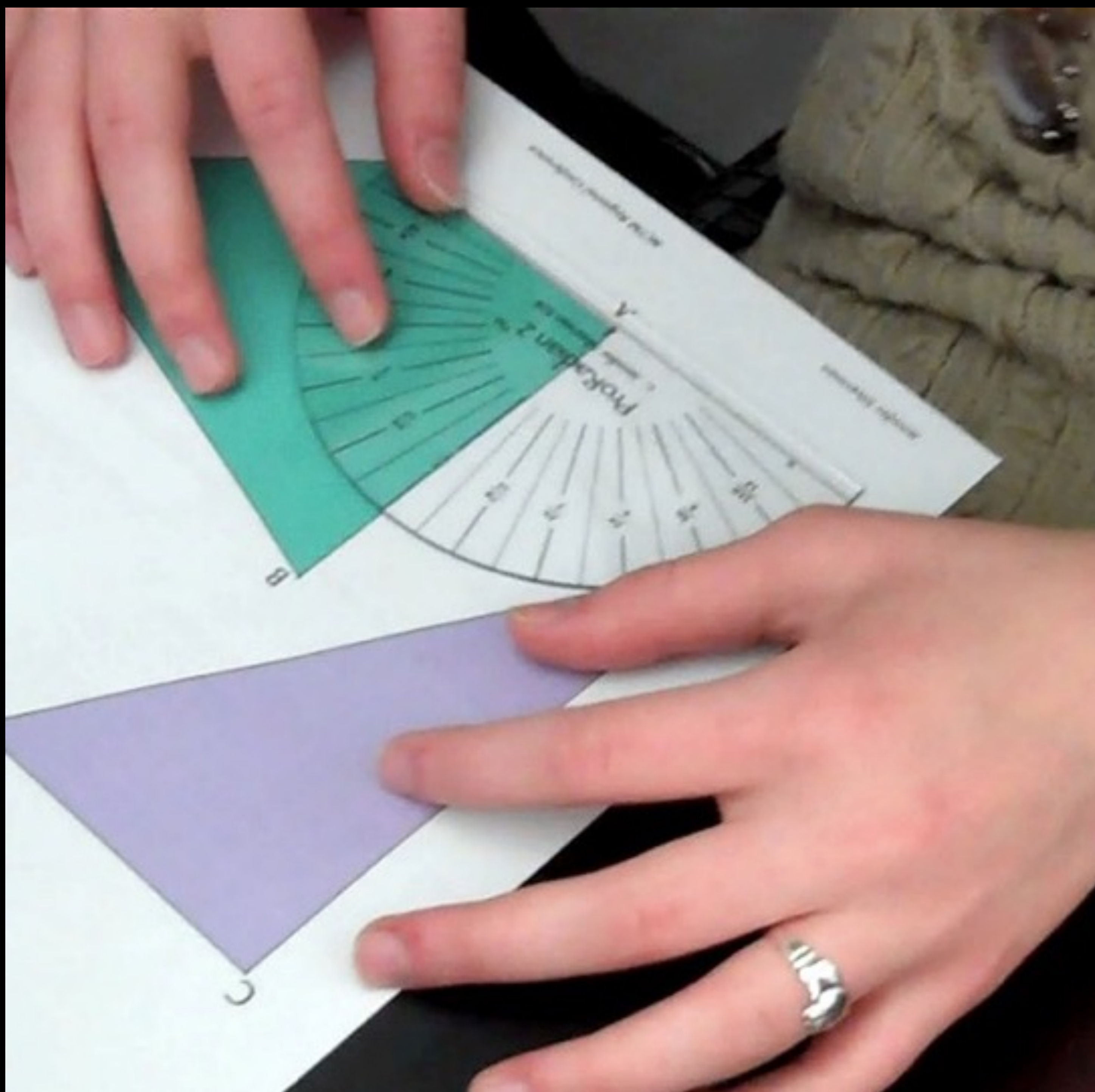
I met with another group of students and asked them to try ProRadian® protractors.

Here is what the students said…

To me it was the greatest math invention! If I had this last year during precalculus I would have further solidified my trigonometry skills which are crucial in AP Calculus!

Thank You, Simran

Here is what the students said...

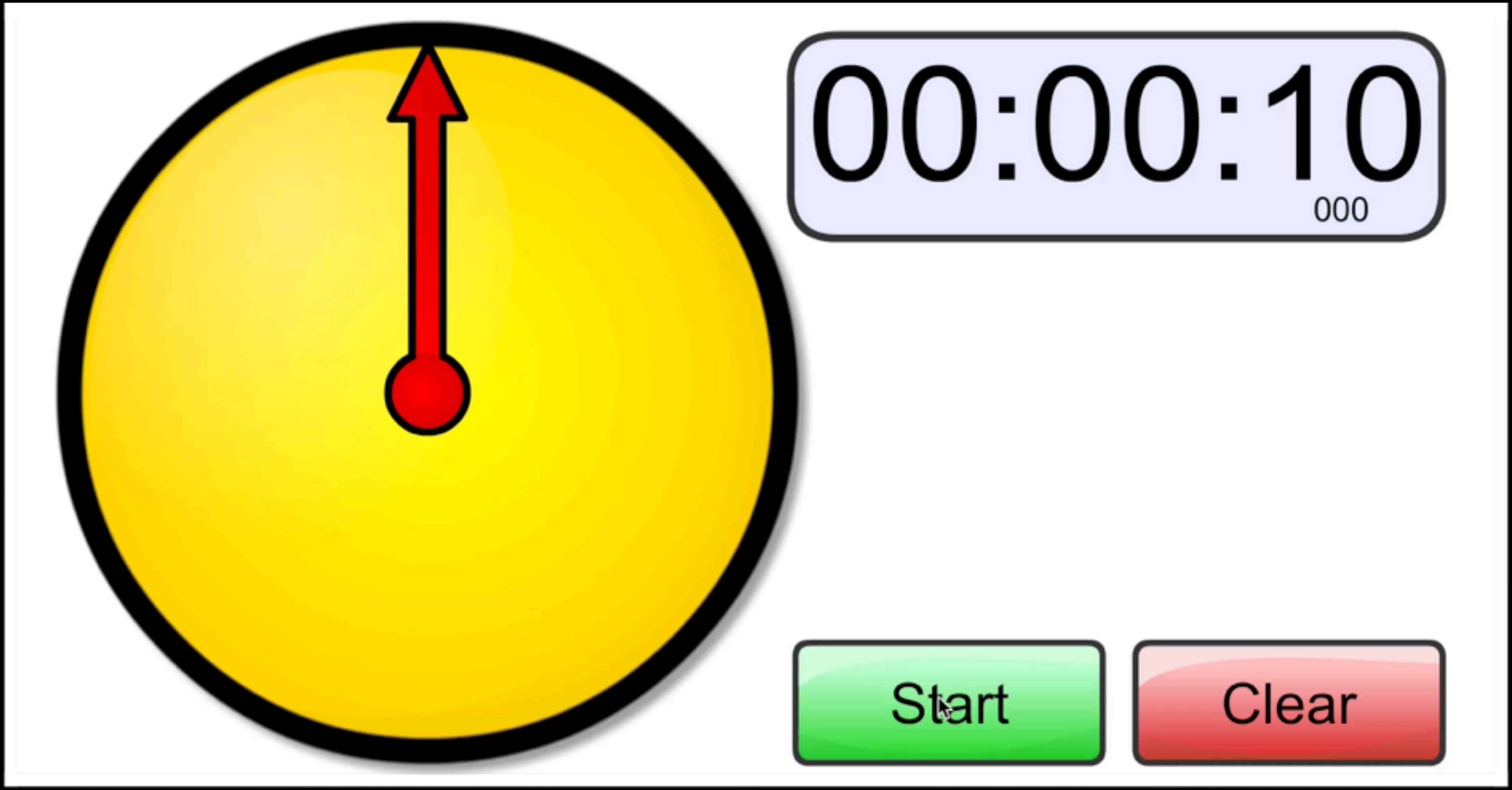


I truly believe that this product can help people better understand this concept. Best wishes, you have a great product!

Sincerely,

Cristina

Are you ready?



Are you ready?

It's time!

Time to check them out...

Time to check them out…

What do you notice?

Time to check them out…

What do you notice?

What do you wonder ?*

Time to check them out…

What do you notice?

What do you wonder ?*

*Thanks, Max Ray and MathForum at Drexel!

Plain Paper Activity 1

Use each protractor to draw a measured angle. Do NOT label them. Switch your paper with a neighbor. Measure his/her drawings.

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What do you notice?
What do you wonder ?

Middle School Activities

Reinforce Understanding of π

CCSS.MATH.CONTENT 7.G.B.4
Know the formulas for the area and circumference of a circle and use them to solve problems; give an informal derivation of the relationship between the circumference and area of a circle.

Reinforce Understanding of π

Know the formulas for the area and circumference of a circle

Start with an activity to discover π.

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circumference	diameter	ratio

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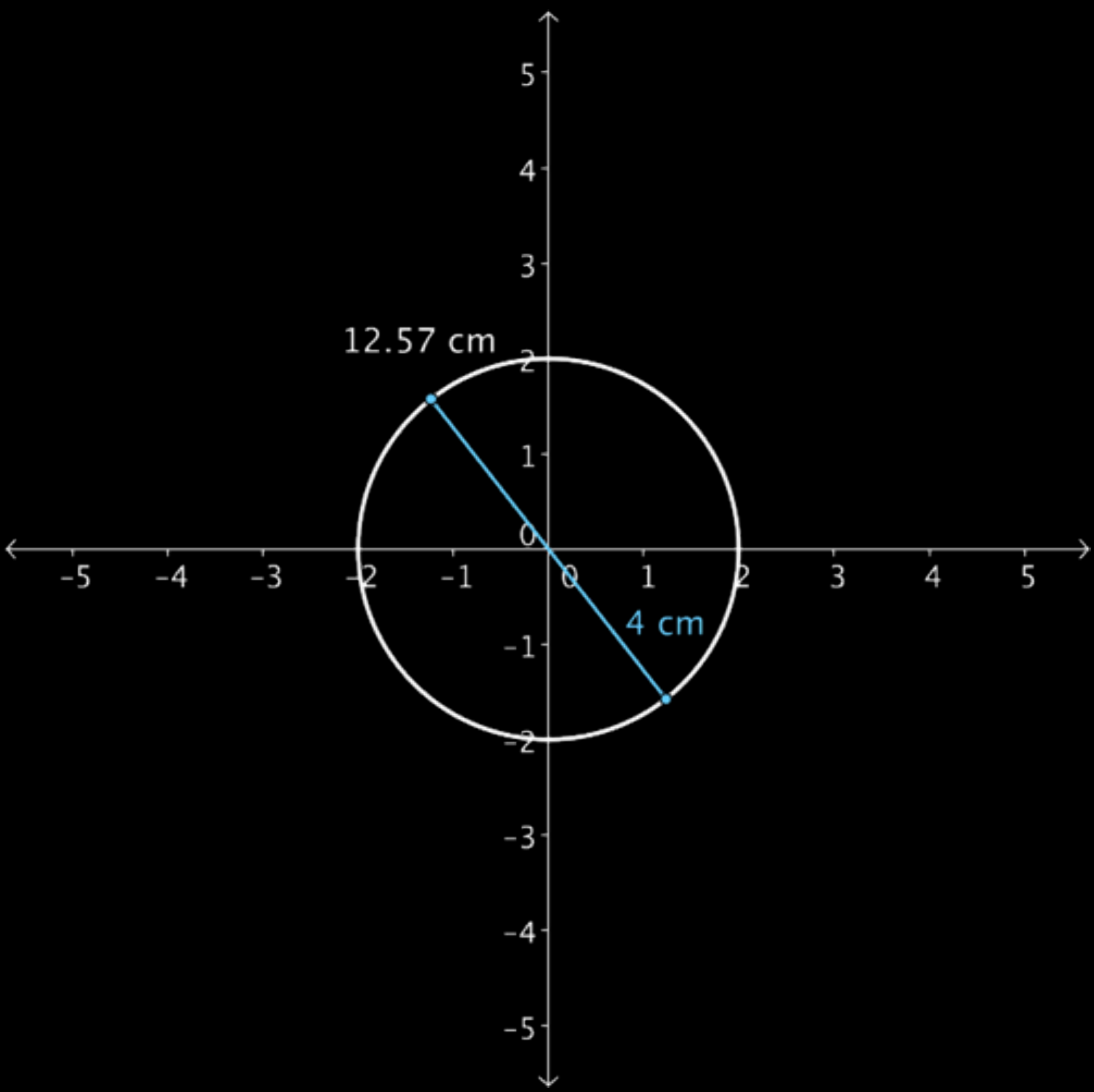
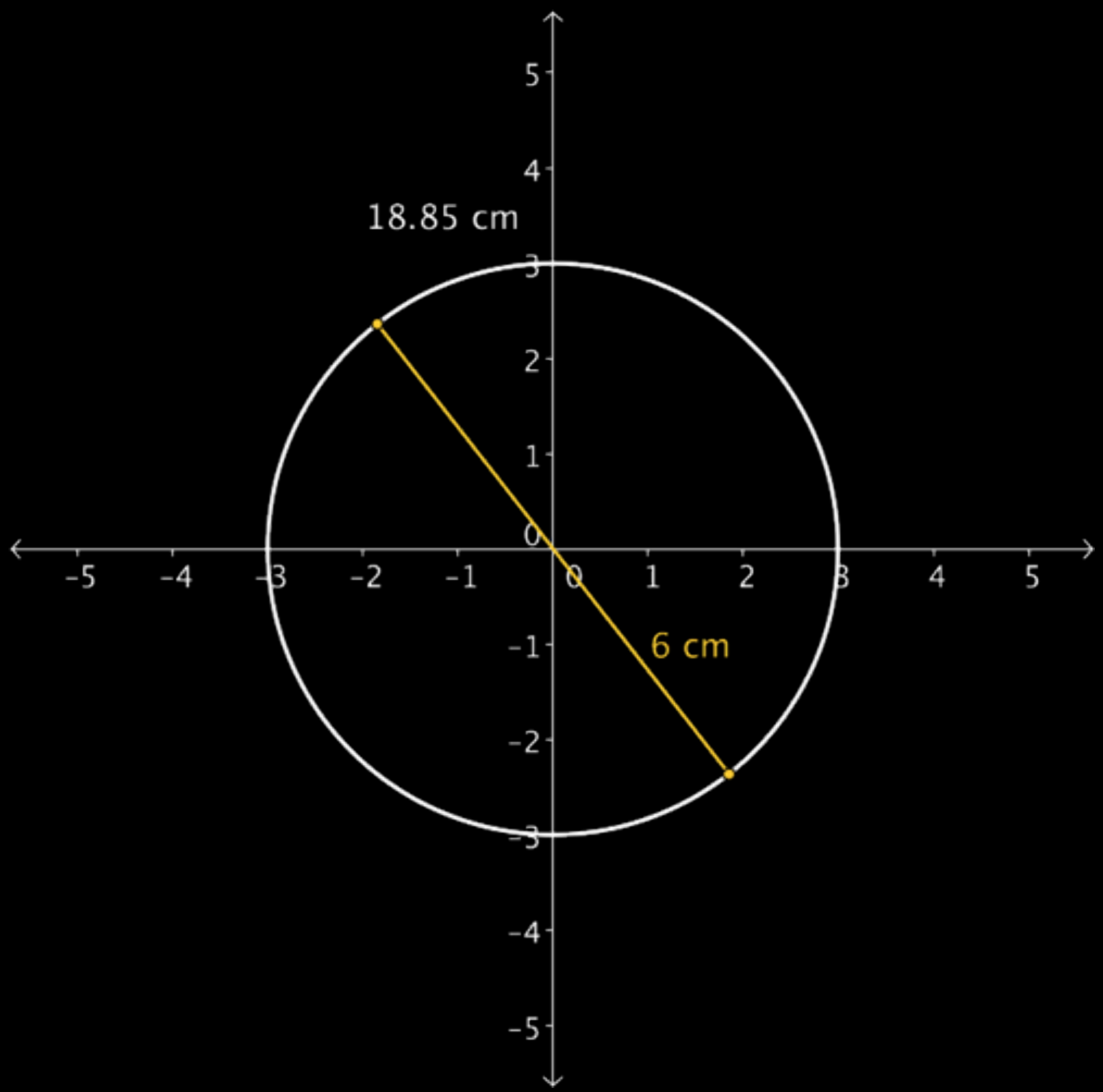


circumference	diameter	ratio
12.57	4	3.1425

Start with an activity to discover π.



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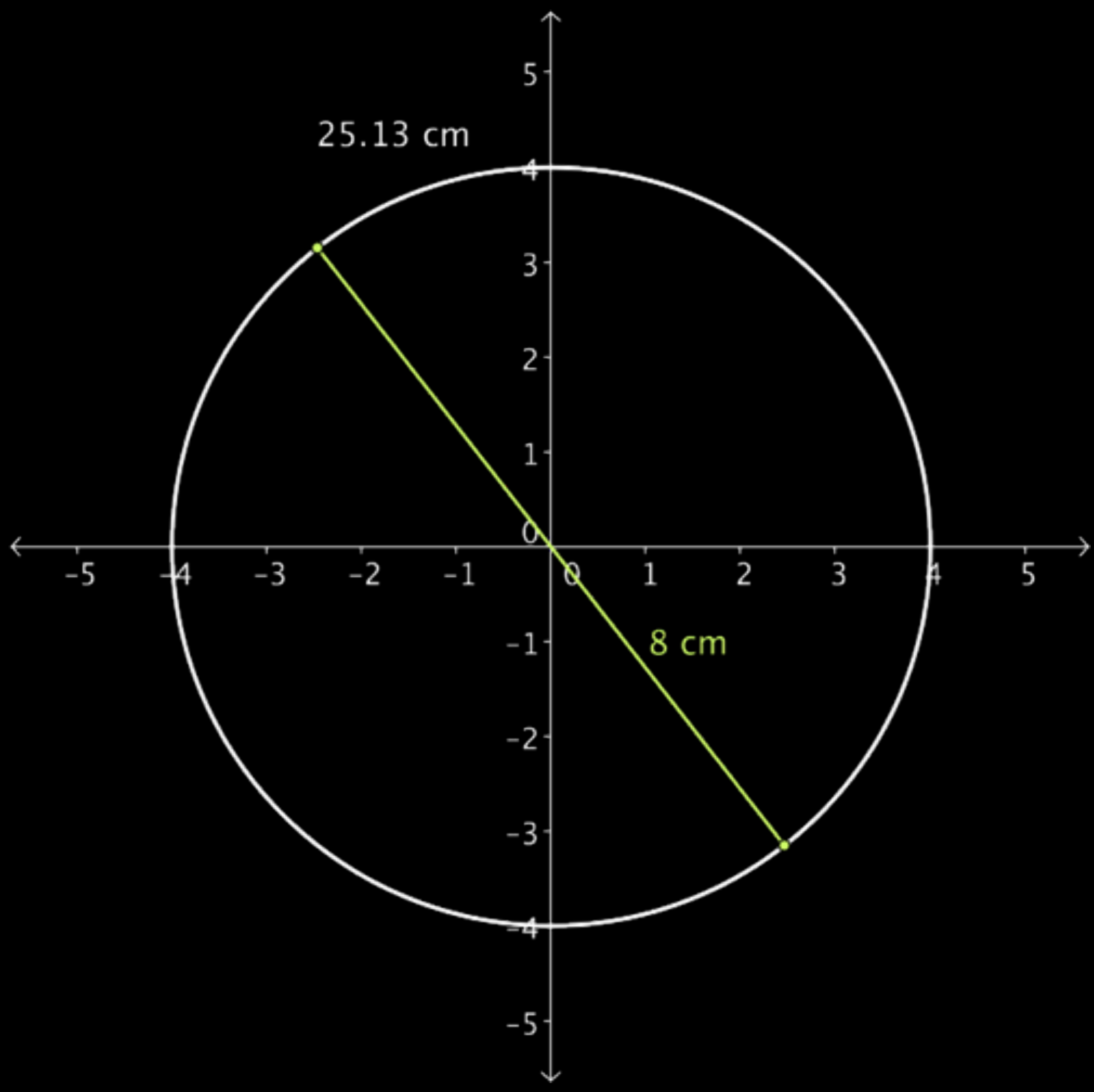

circumference	diameter	ratio
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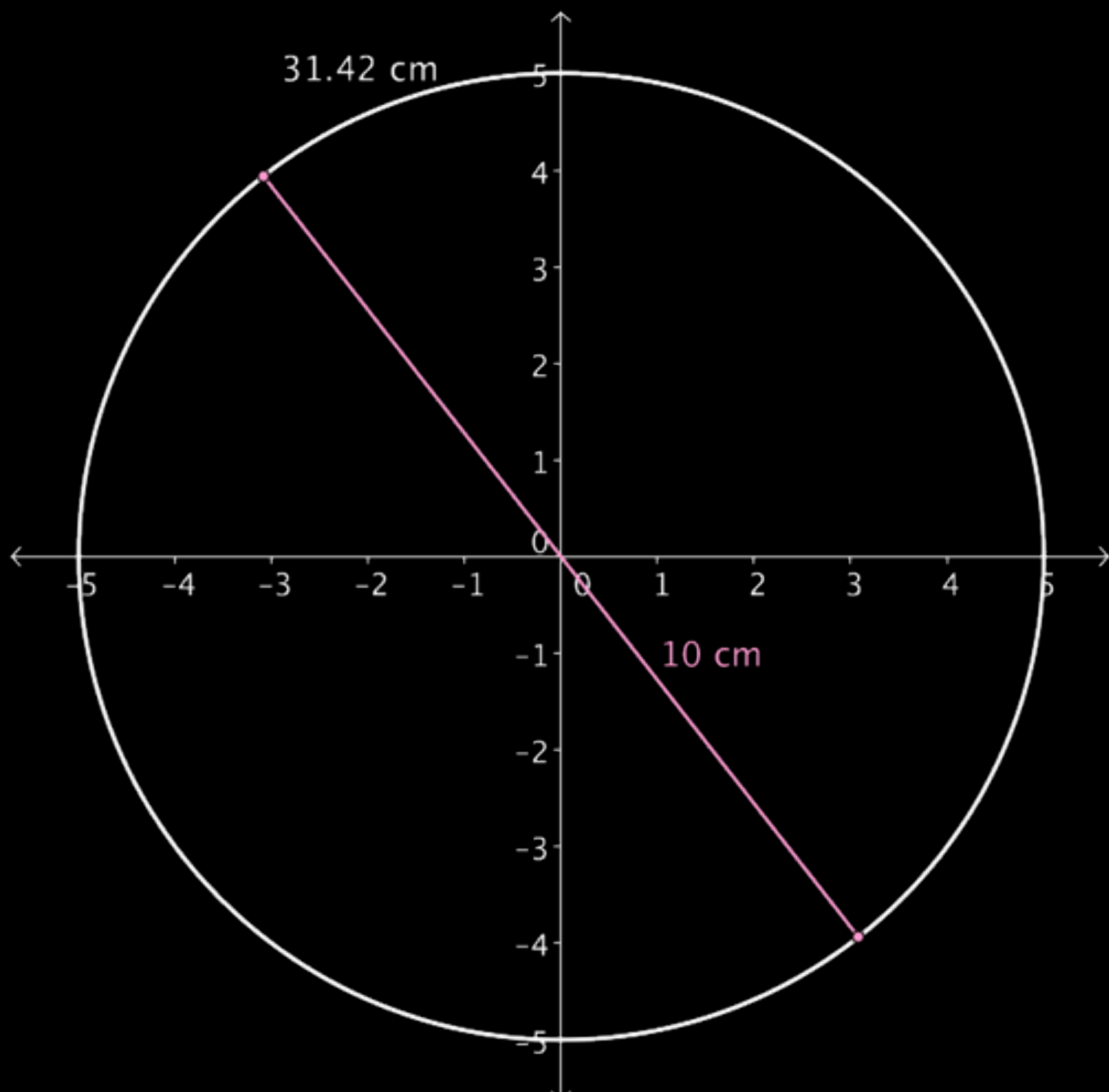
Start with an activity to discover π.



circumference	diameter	ratio
12.57	4	3.1425
18.85	6	3.1417
25.13	8	3.1413
31.42	10	3.1420

Name the ratio, π.

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Derive the circumference formula in terms of the radius.

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$$\pi = \frac{C}{D} \text{ and } D = 2r, \text{ so } \pi = \frac{C}{2r}.$$

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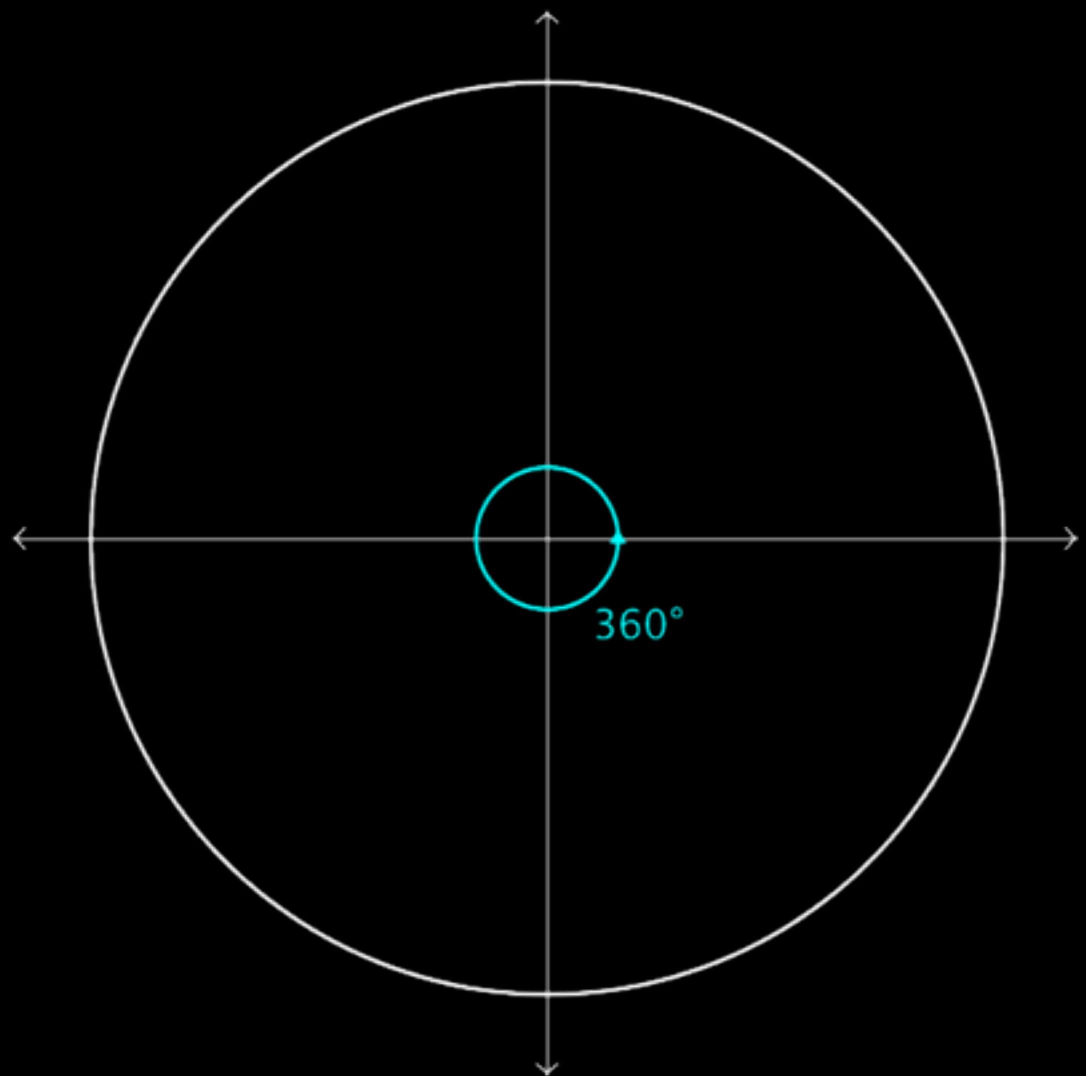
Derive the circumference formula in terms of the radius.

$$\pi = \frac{C}{D} \text{ and } D = 2r, \text{ so } \pi = \frac{C}{2r}.$$

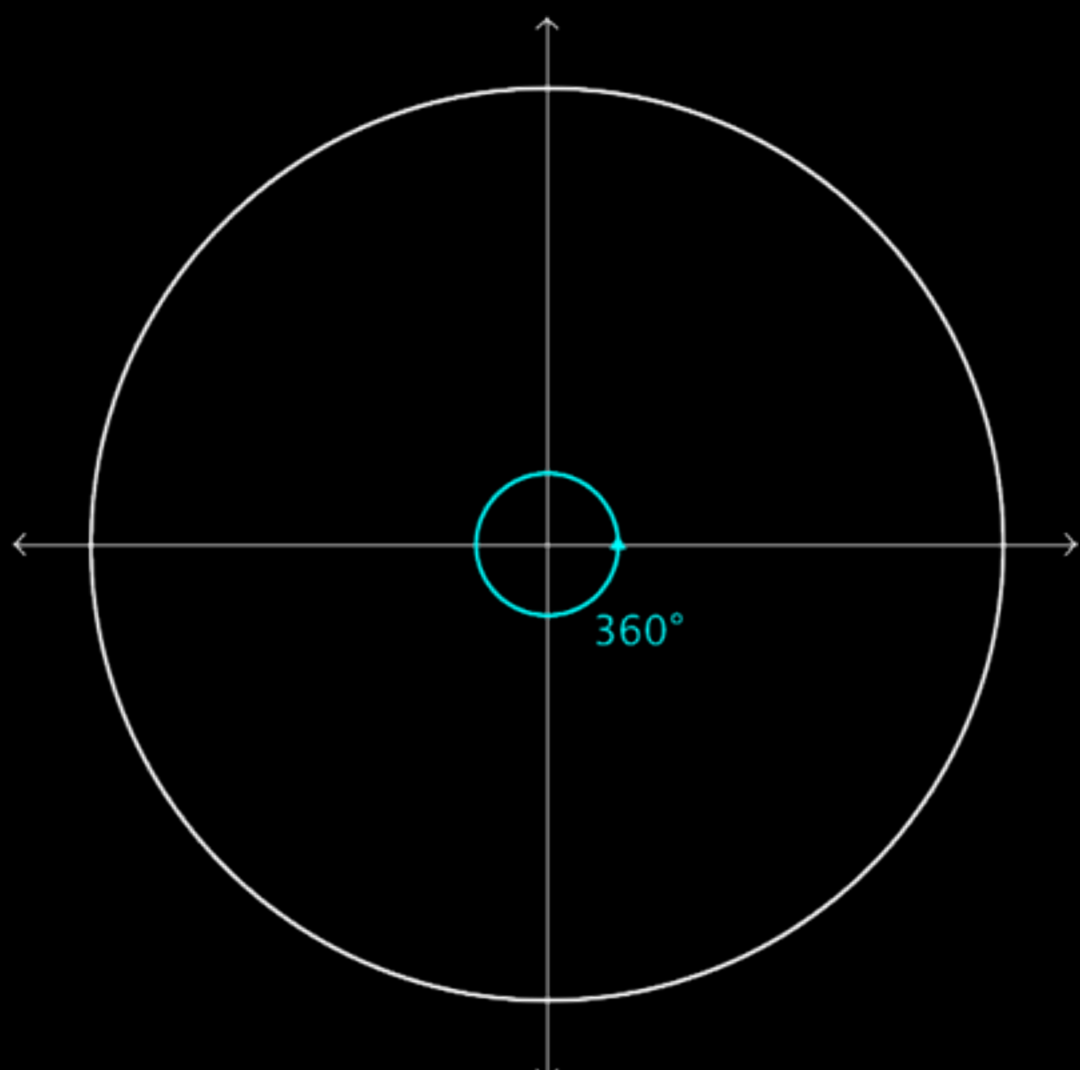
Then $2r\pi = C$ or $C = 2\pi r$.

We are used to thinking that 360° makes a whole circle.



We are used to thinking that 360° makes a whole circle.

Can we extend our thinking to 2π makes a whole circle?



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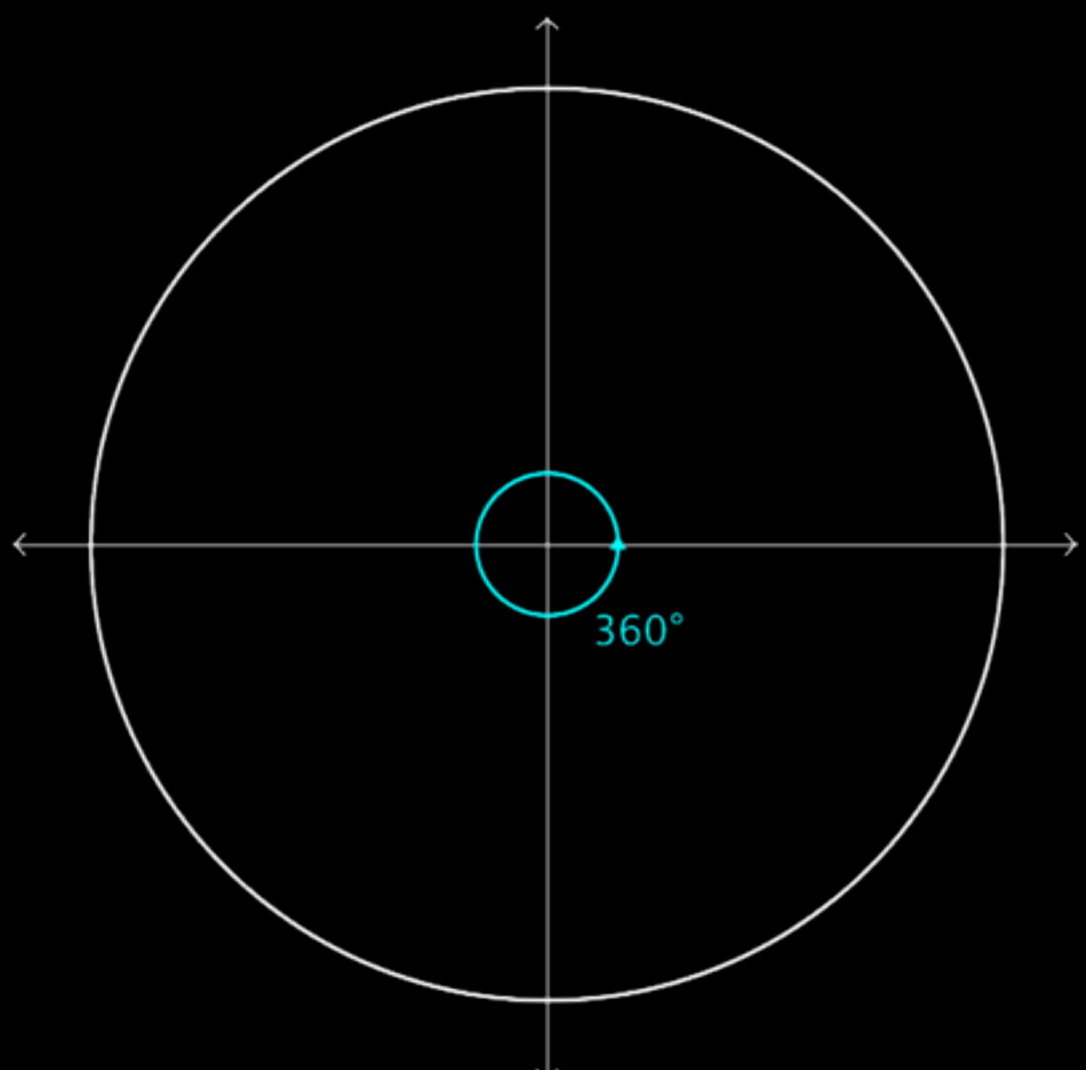

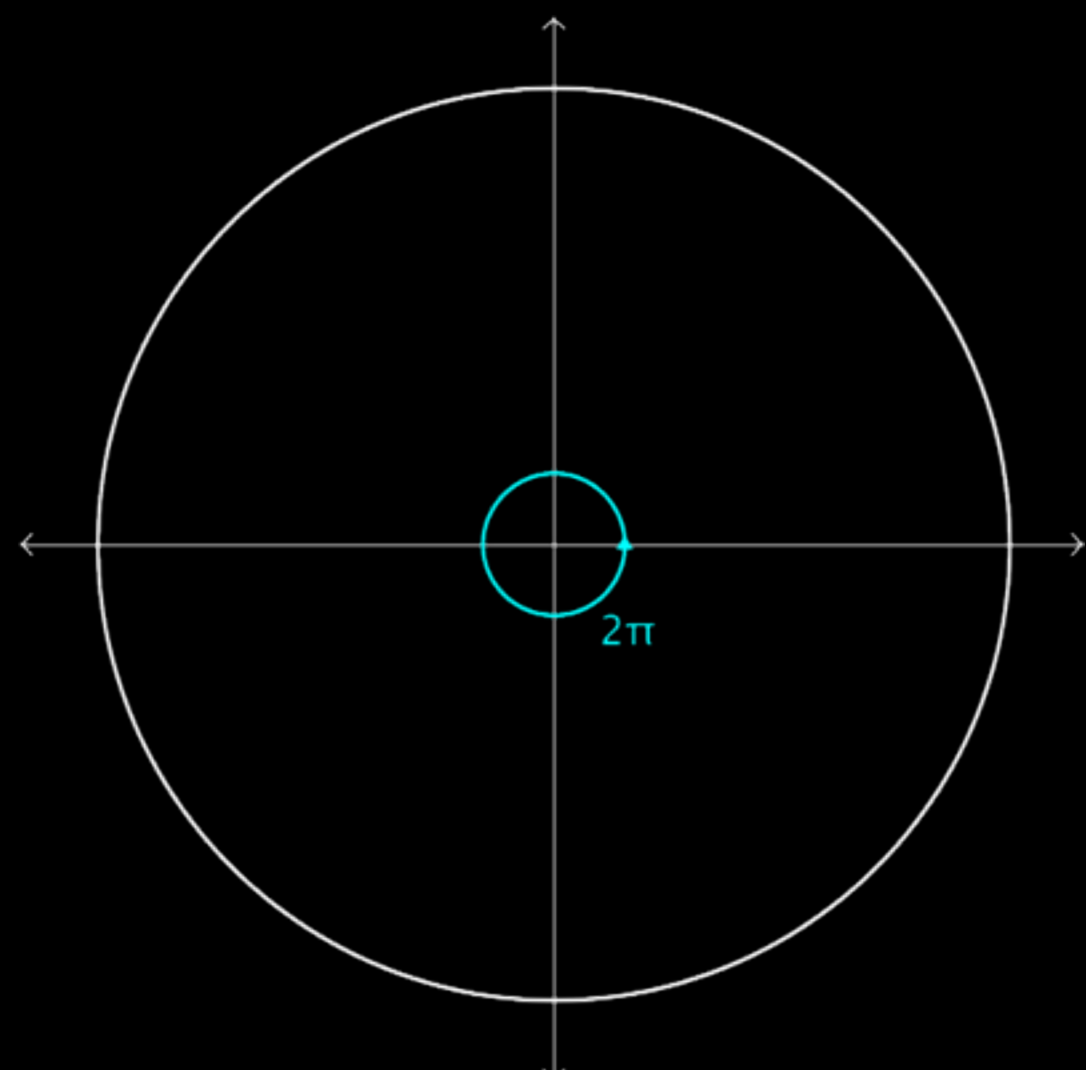

Define a New Angle Measure

CCSS.MATH.CONTENT 4.M.D.C.5A

An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the 2 rays intersect the circle. An angle that turns through 1/(360°) of a circle is called a "one-degree angle," and can be used to measure angles.

Define a New Angle Measure

CCSS.MATH.CONTENT 4.M.D.C.5A'

An angle is measured with reference to a circle with its center at the common endpoint of the rays, by considering the fraction of the circular arc between the points where the 2 rays intersect the circle. An angle that turns through 1/(2π) of a circle is called a "one-radian angle," and can be used to measure angles.

Define a New Angle Measure

An angle that turns through $1/(2\pi)$ of a circle is called a "one-radian angle," and can be used to measure angles.

Compare Angle Measures

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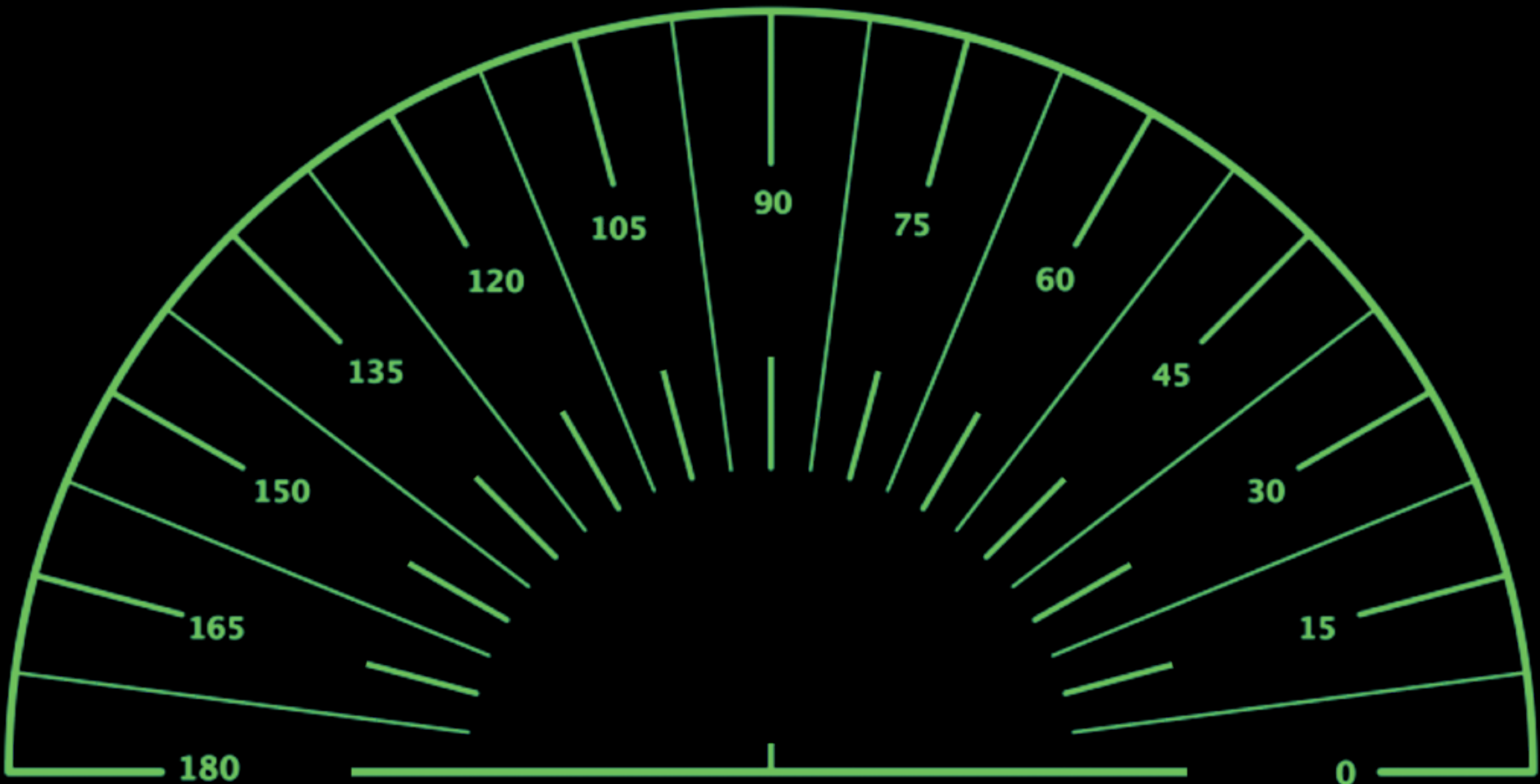
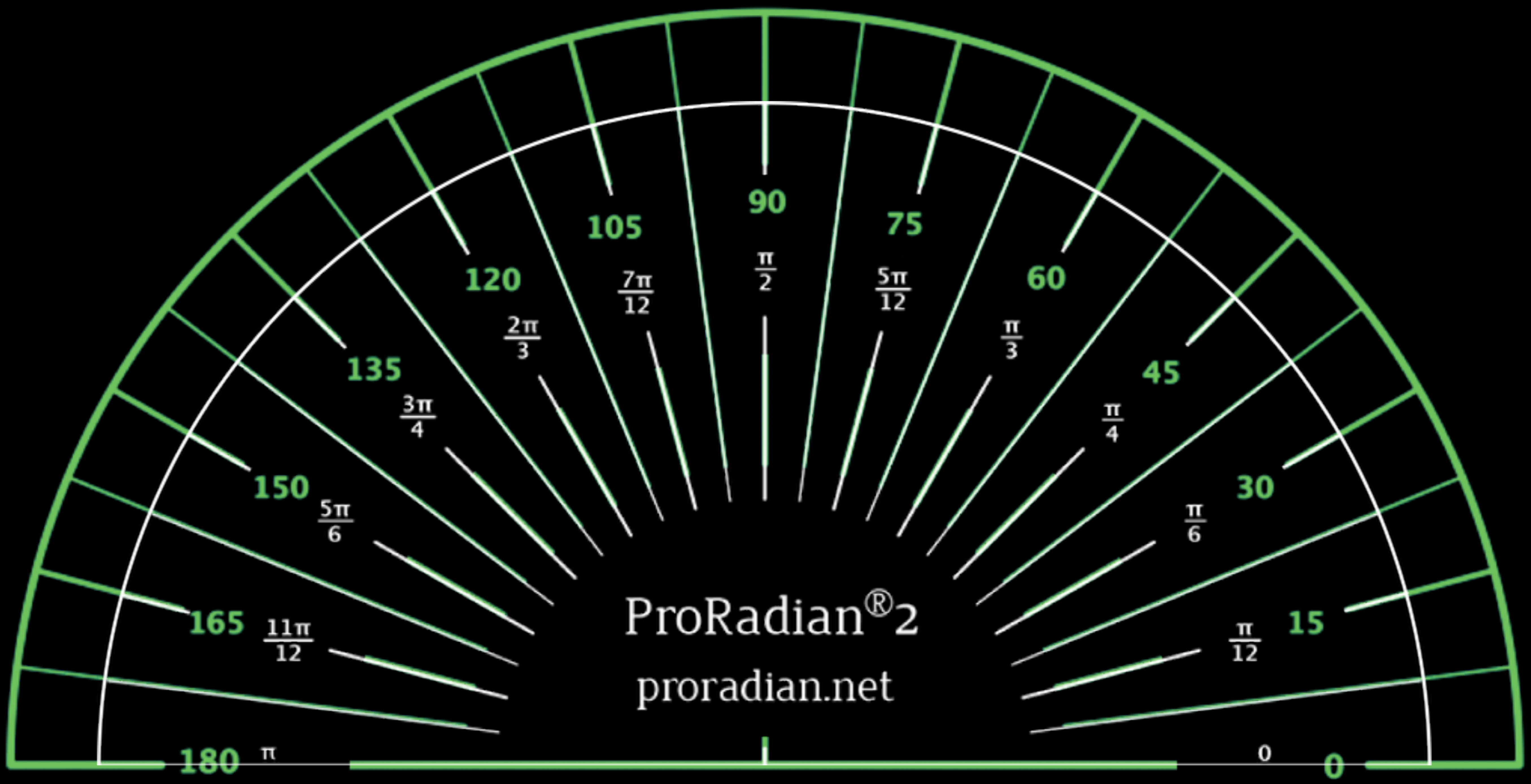
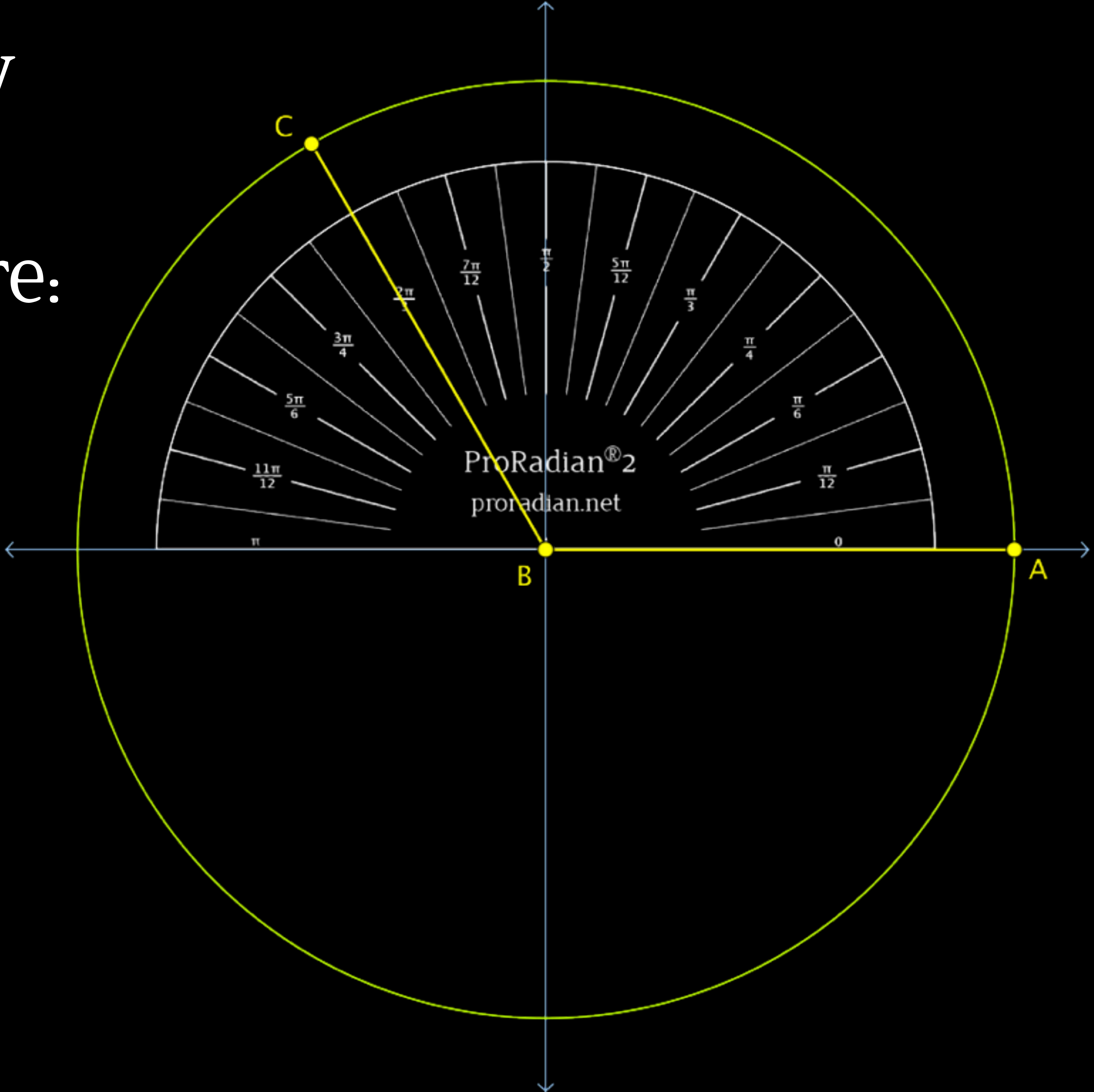


Compare Angle Measures



Identify radian measure:



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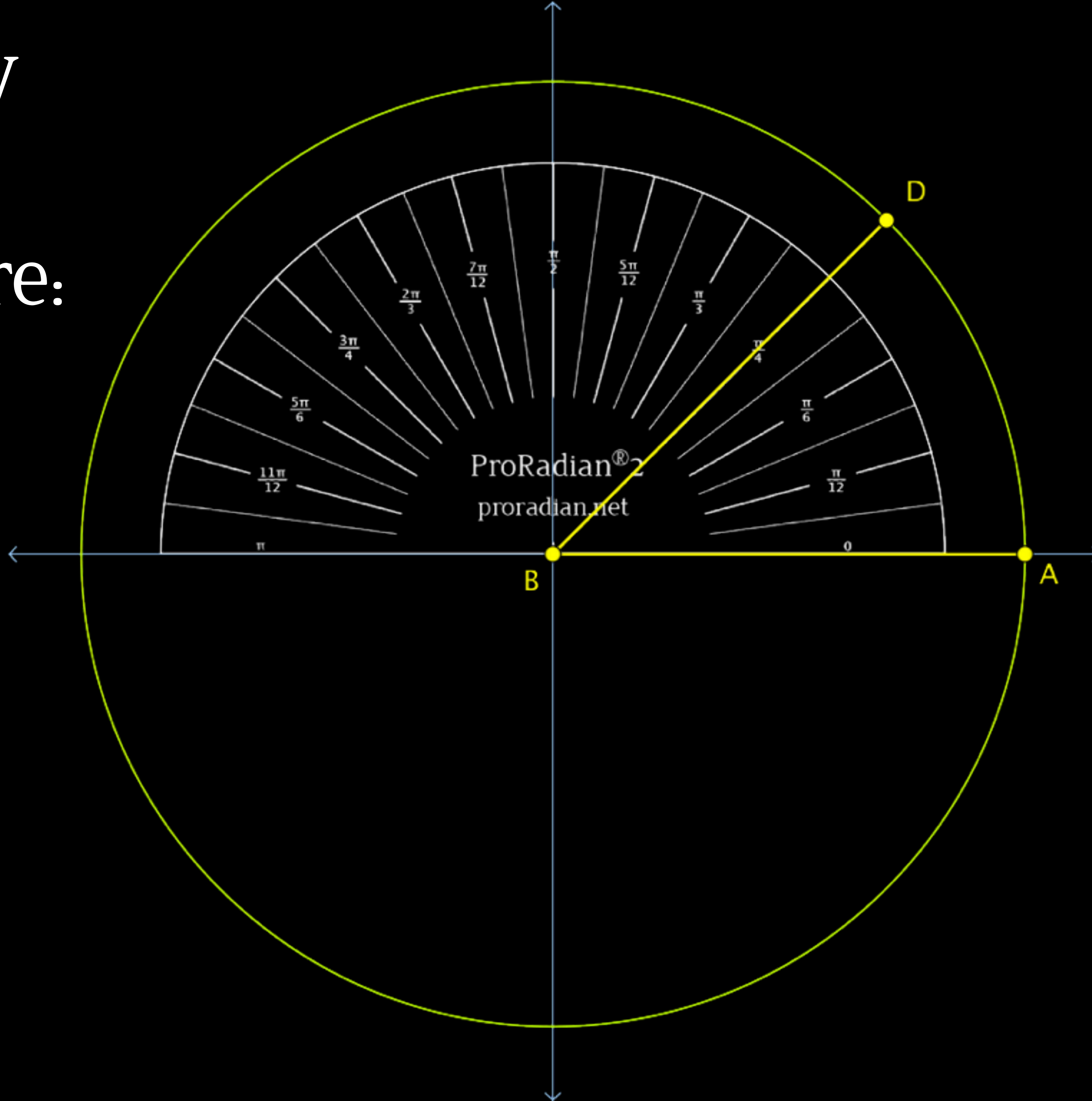

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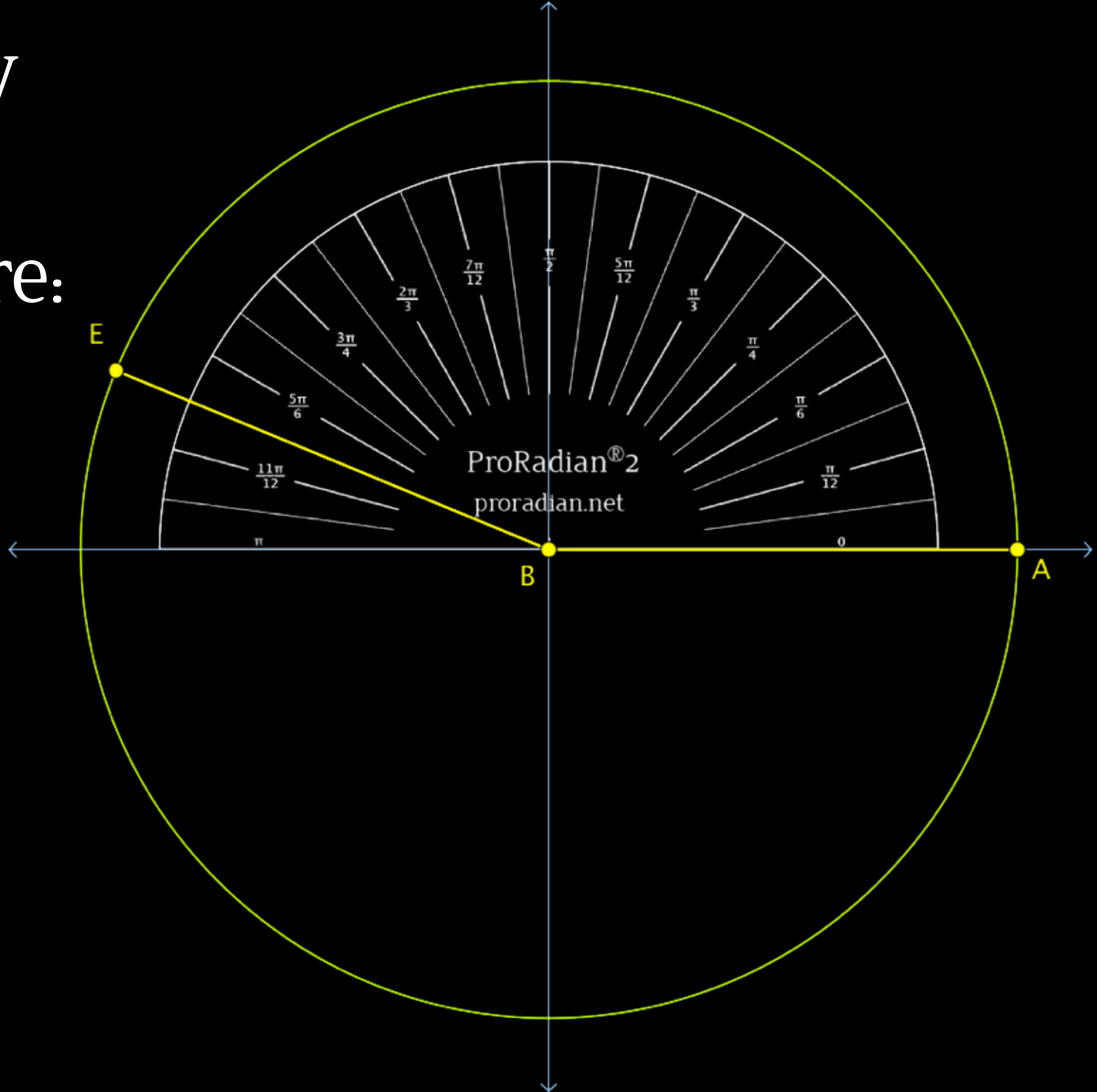

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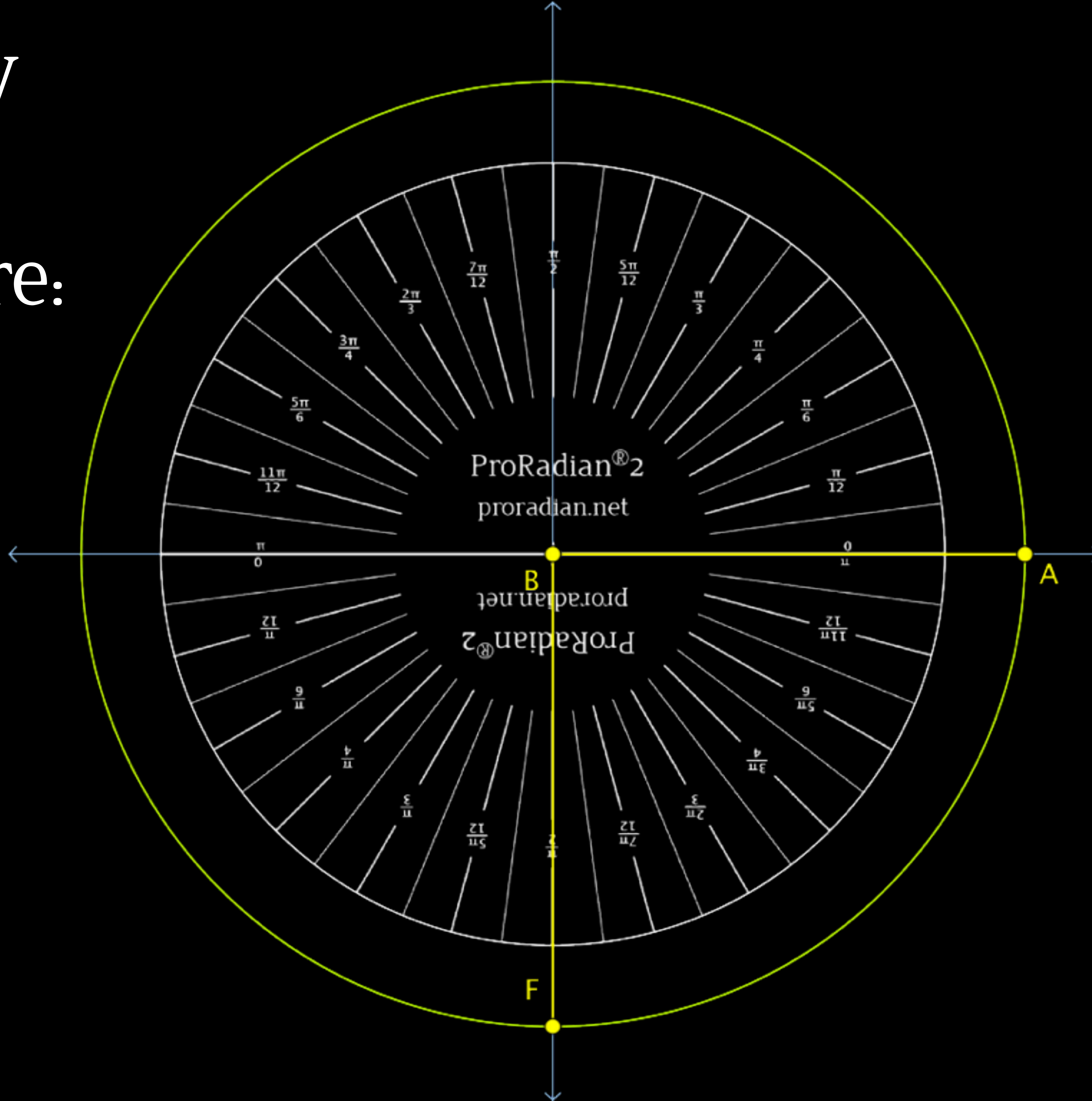


Apply Radian Measure I

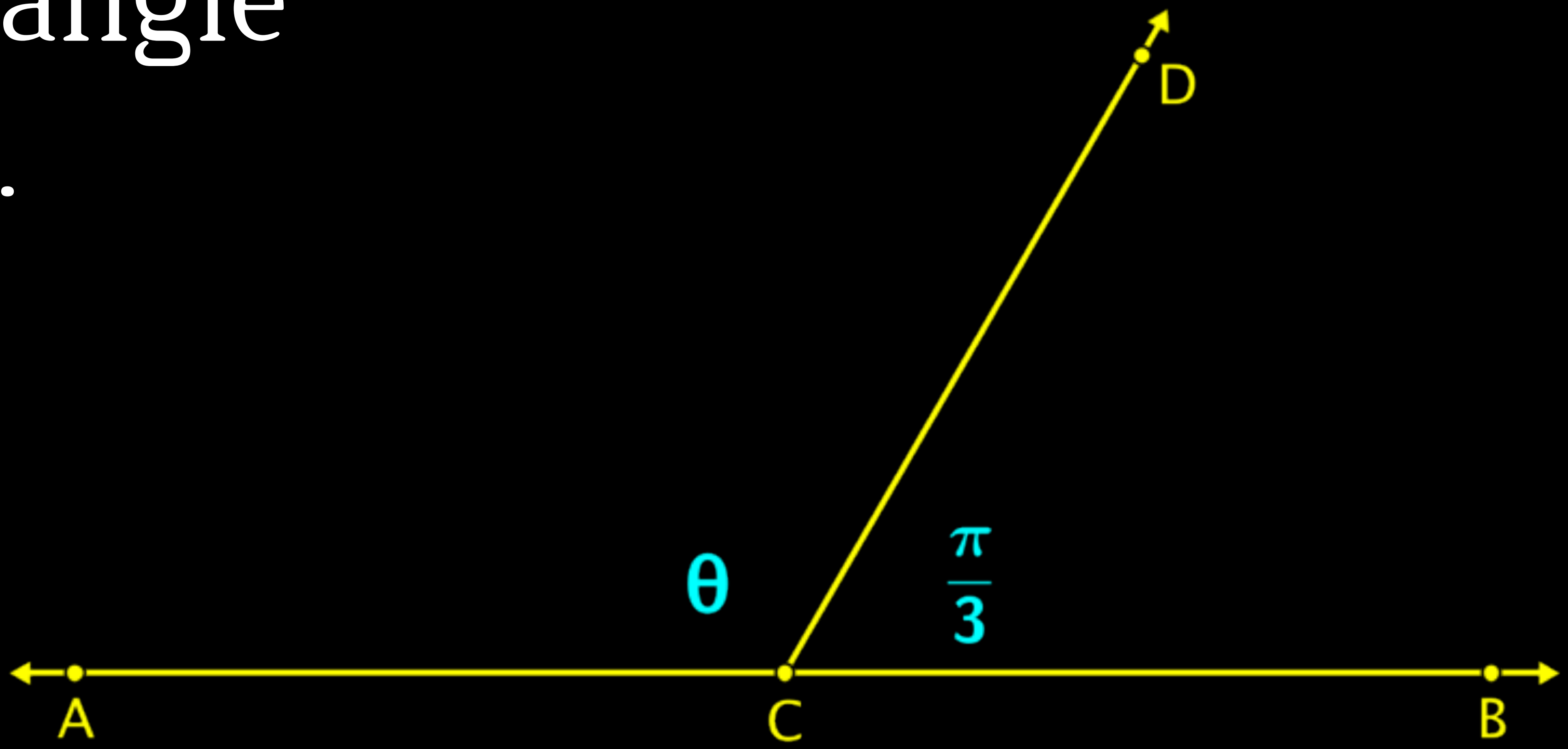
CCSS.MATH.CONTENT.7.G.B.5

Use facts about supplementary, complementary, vertical, and adjacent angles in a multi-step problem to write and solve simple equations for an unknown angle in a figure.

Apply Radian Measure I

Use facts about supplementary, complementary, vertical angles

Identify the missing angle measure.



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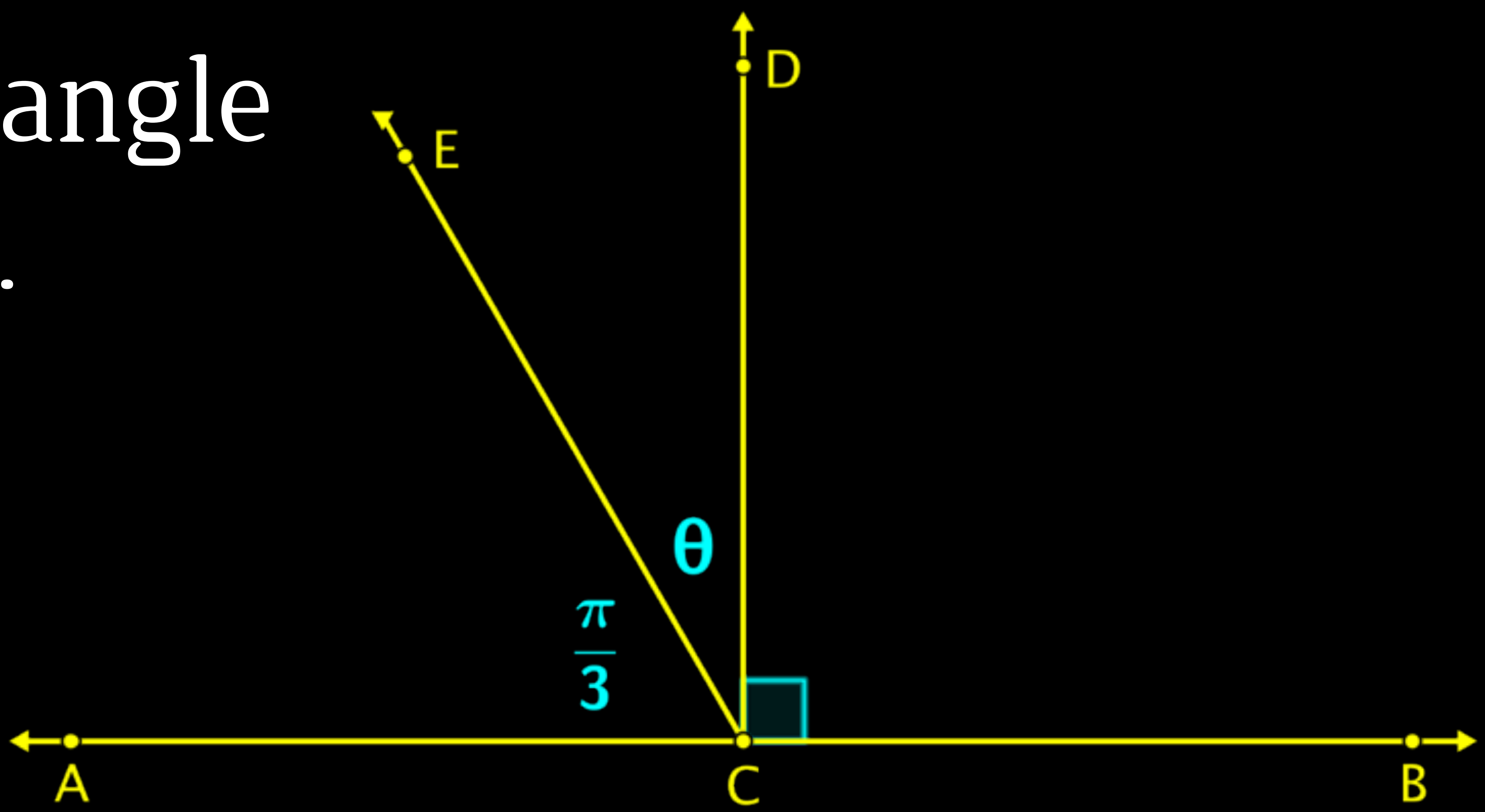

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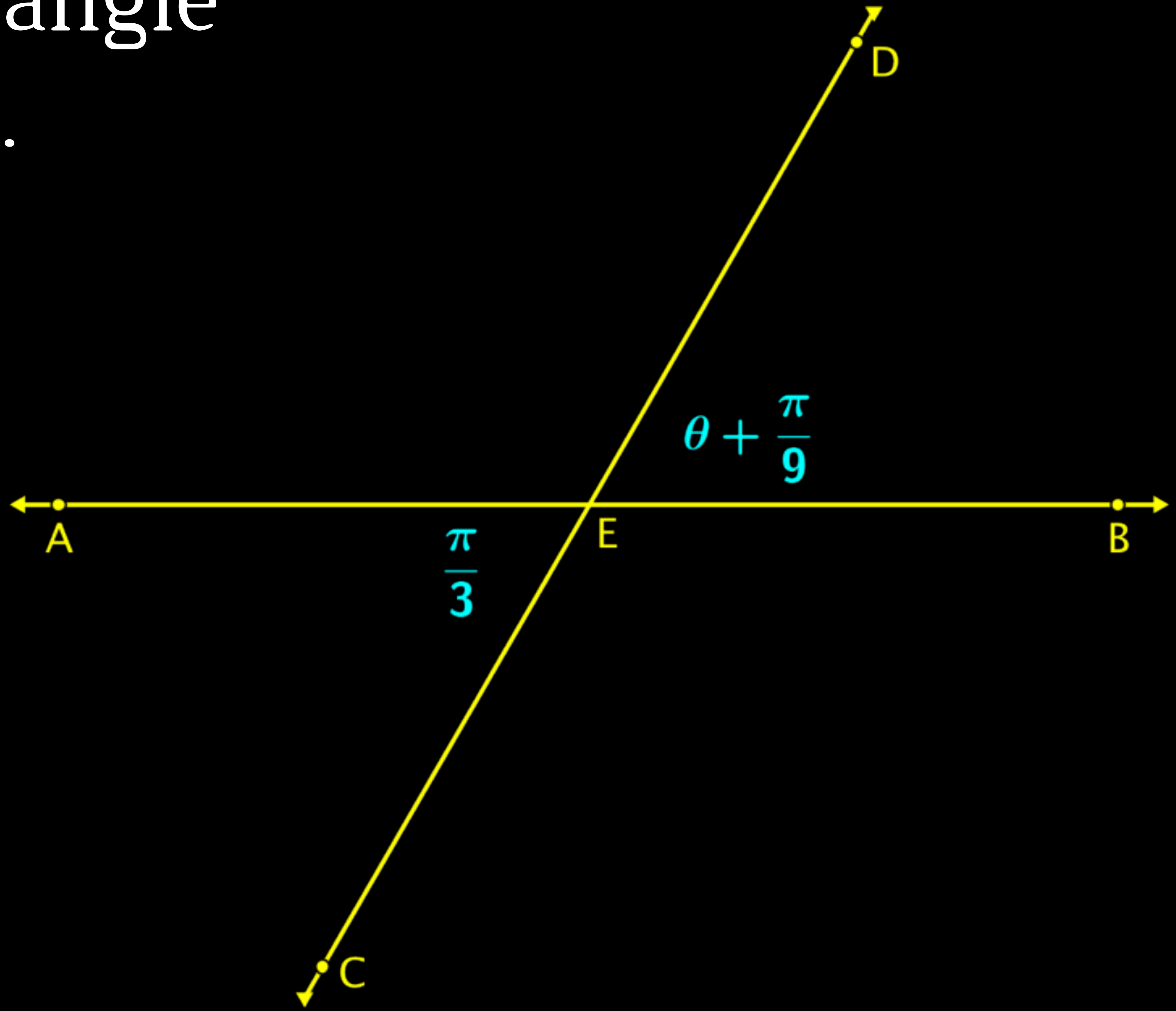

Handout Activity 1: Measure each angle and identify the angle pairs.

Handout Activity 1: Measure each angle and identify the angle pairs.

supplements:

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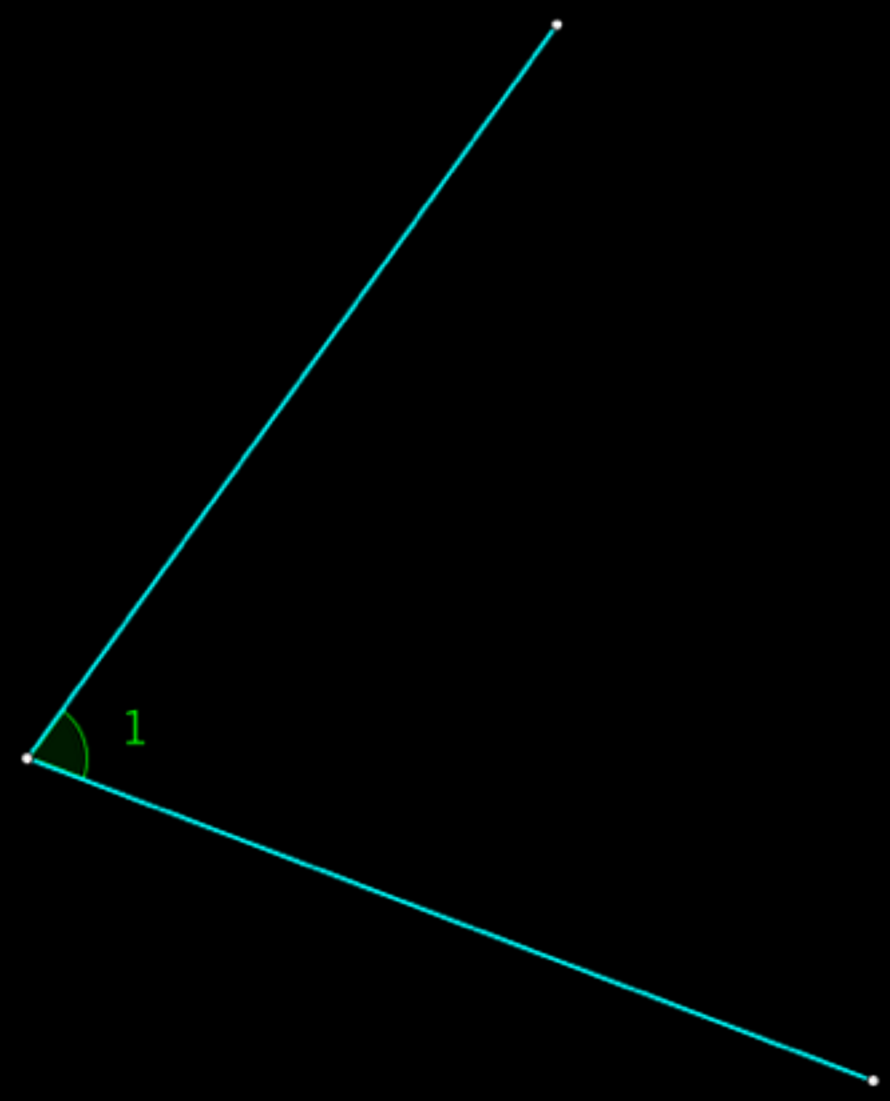

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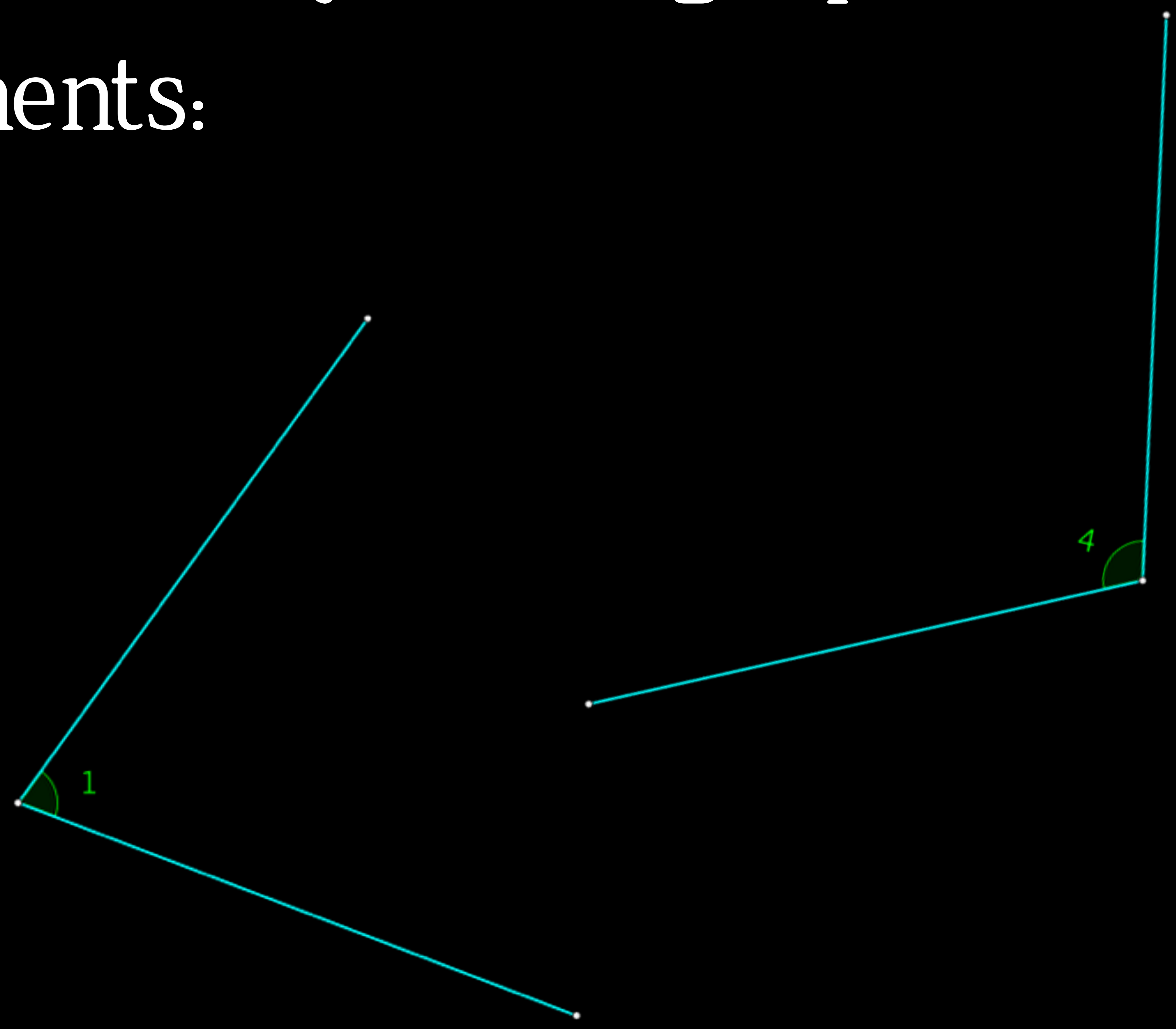

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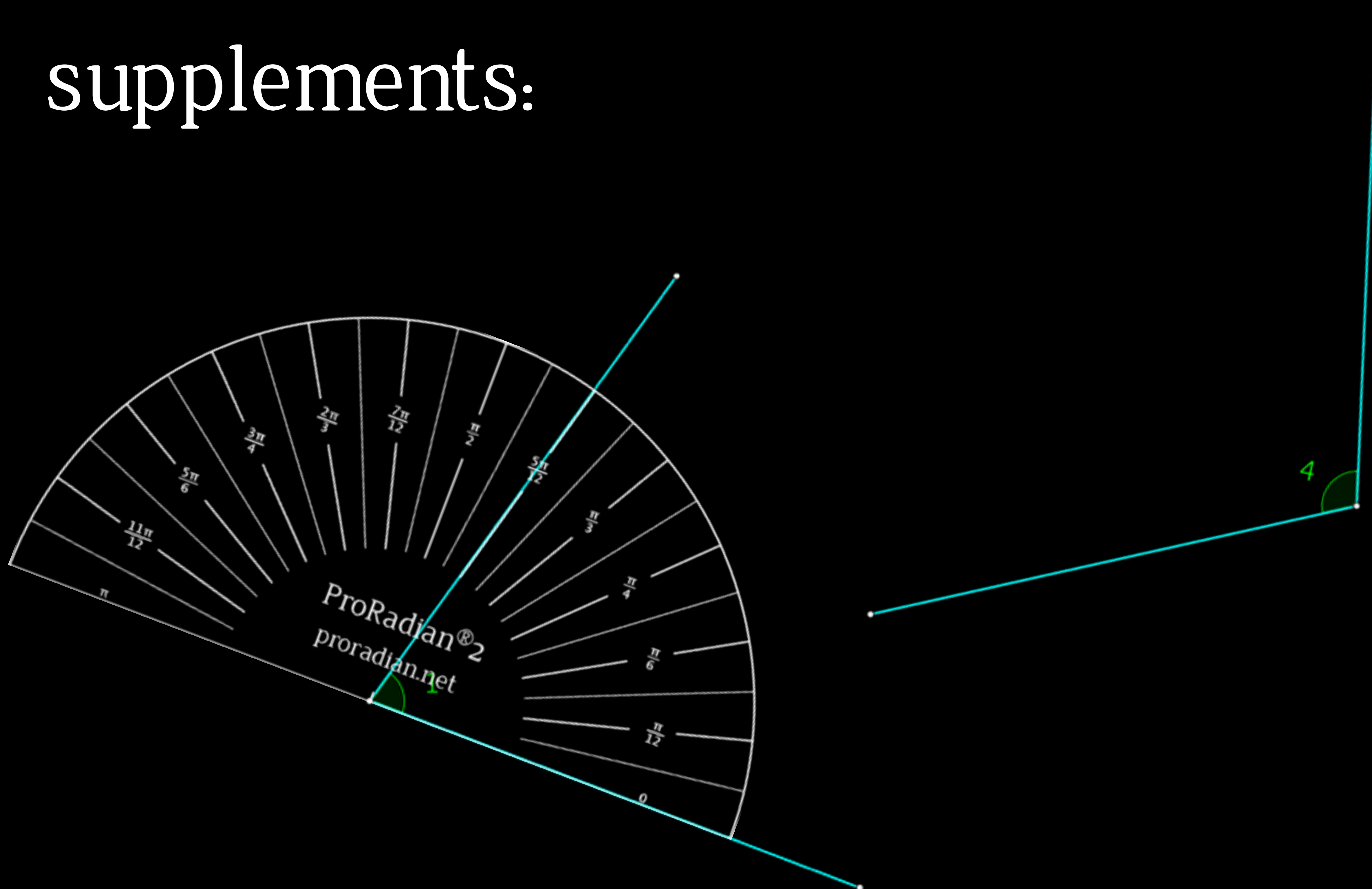

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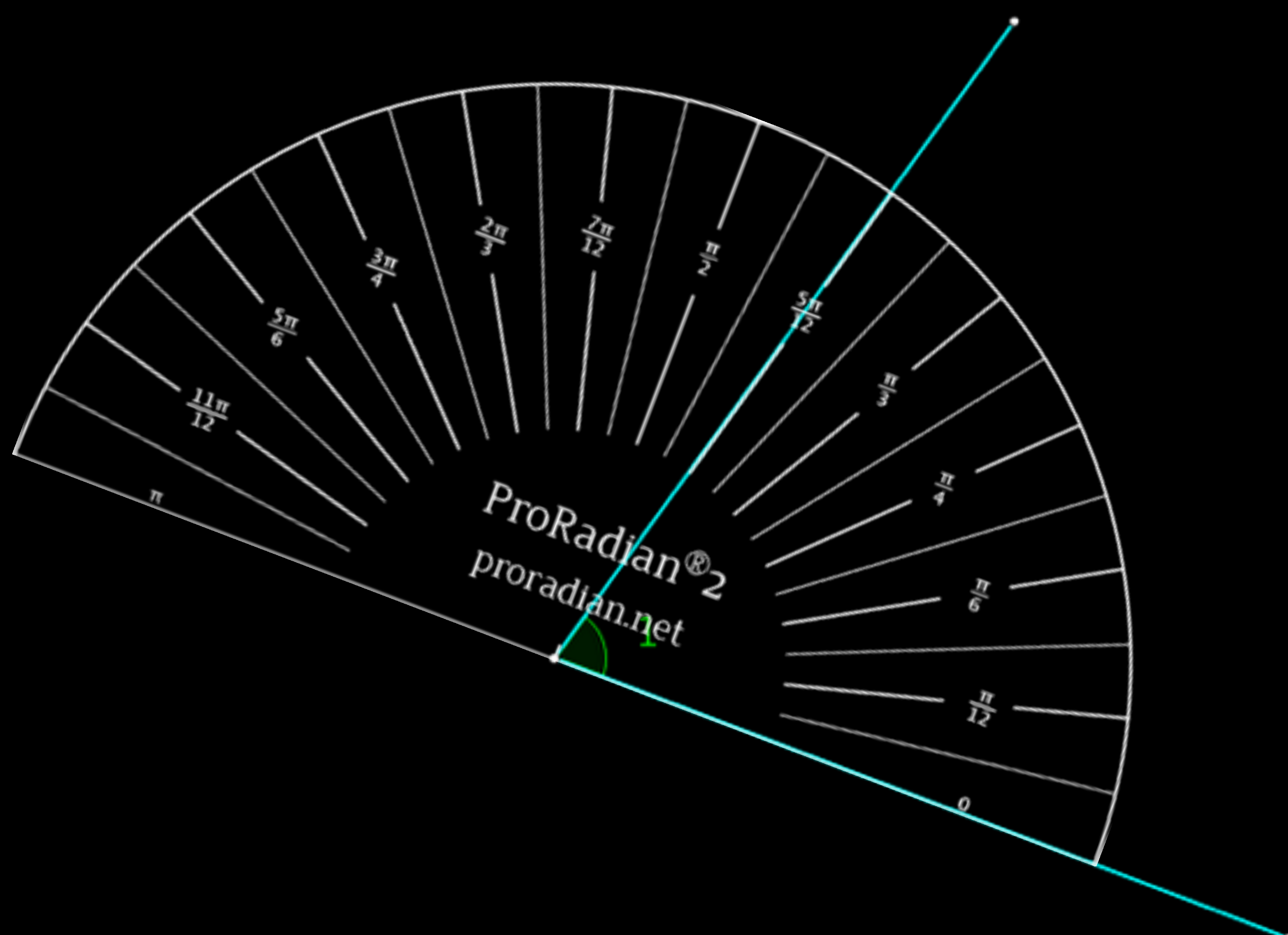

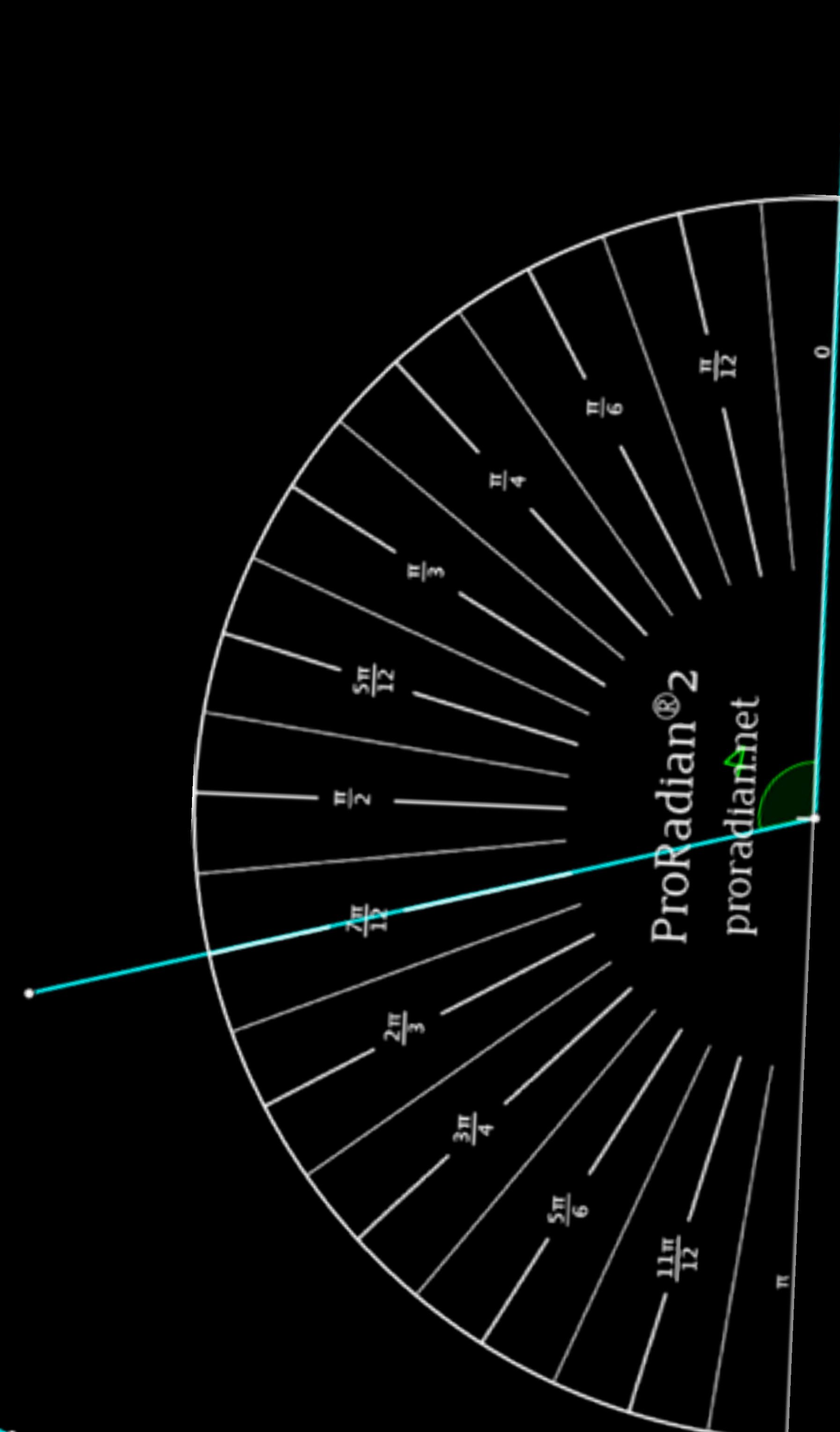

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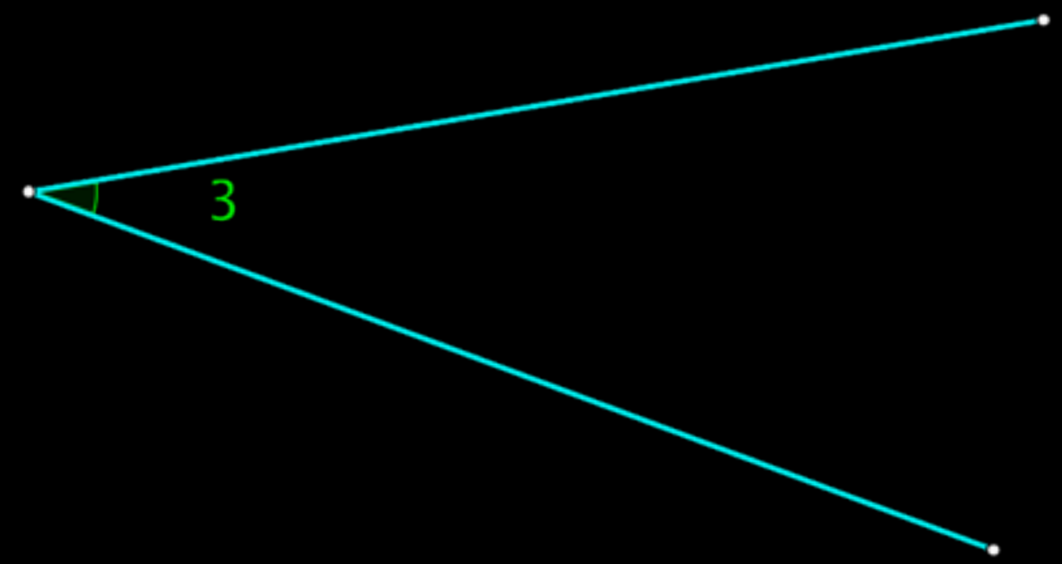

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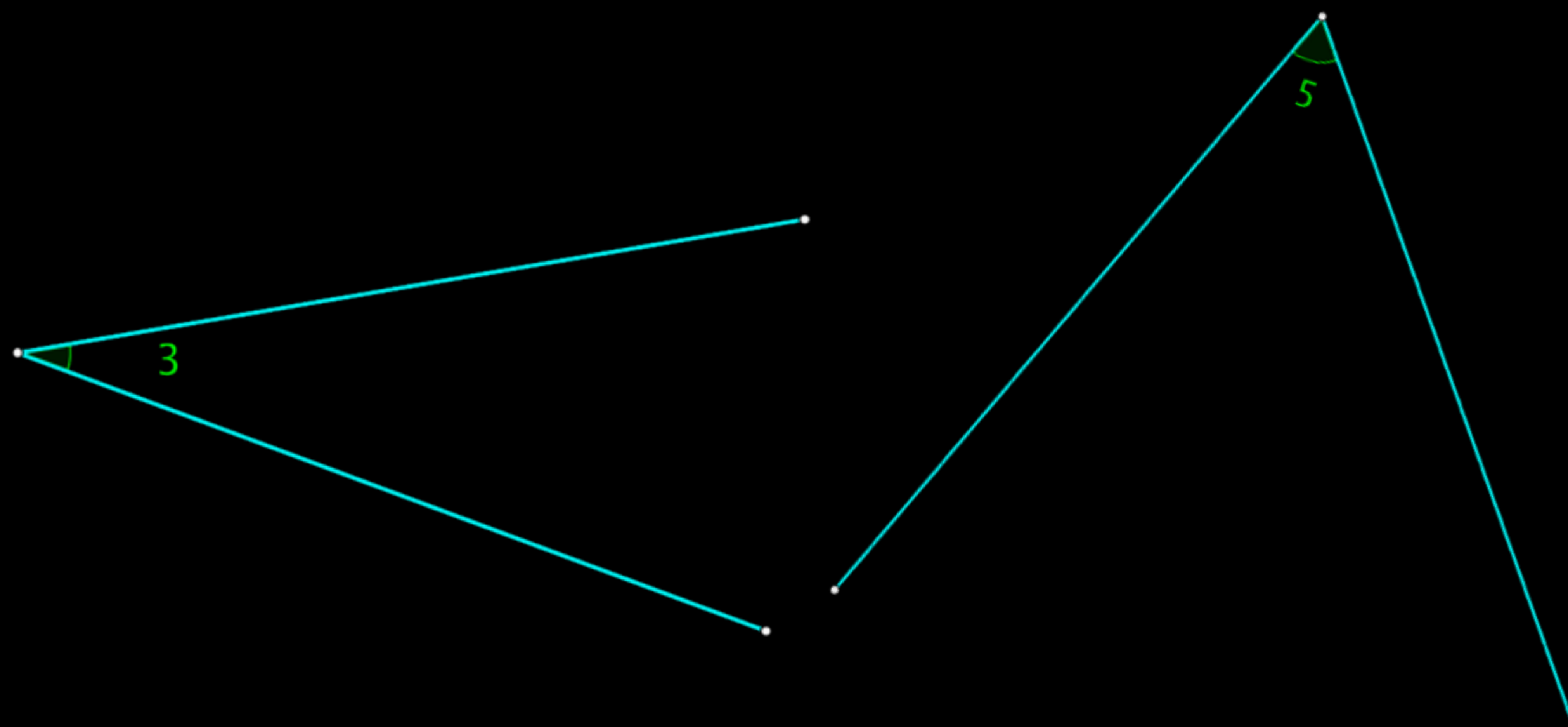

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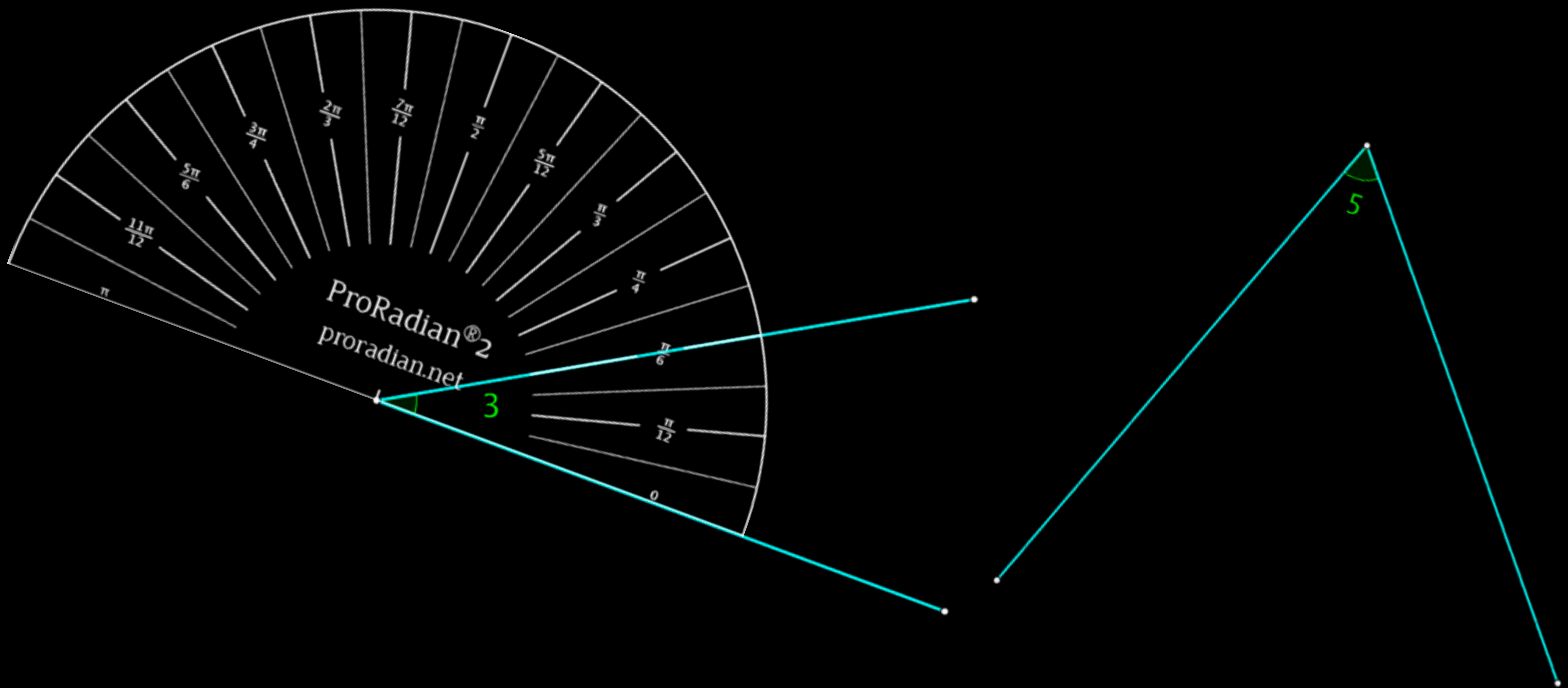

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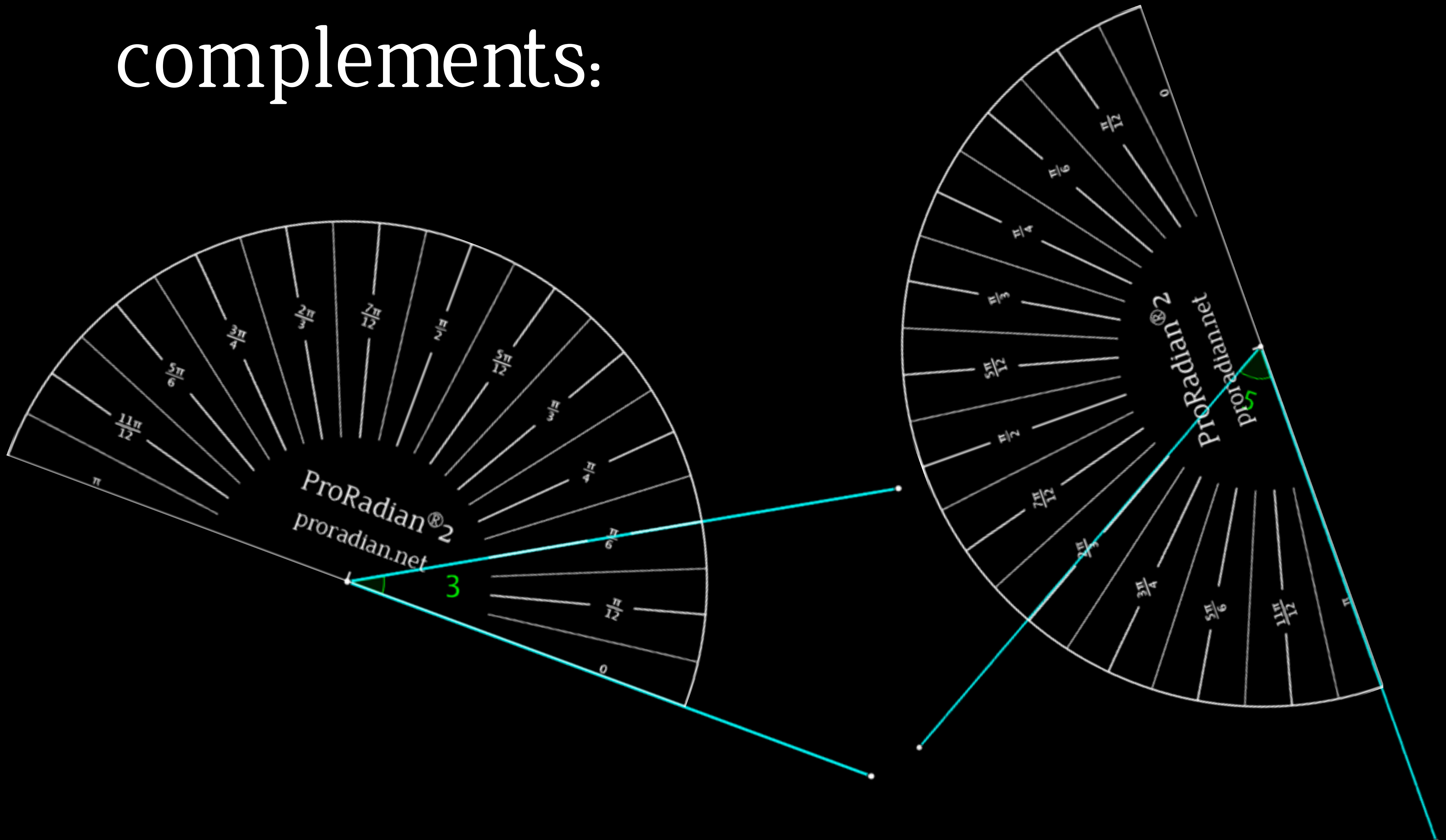


Apply Radian Measure II

CCSS.MATH.CONTENT.8.G.A.5

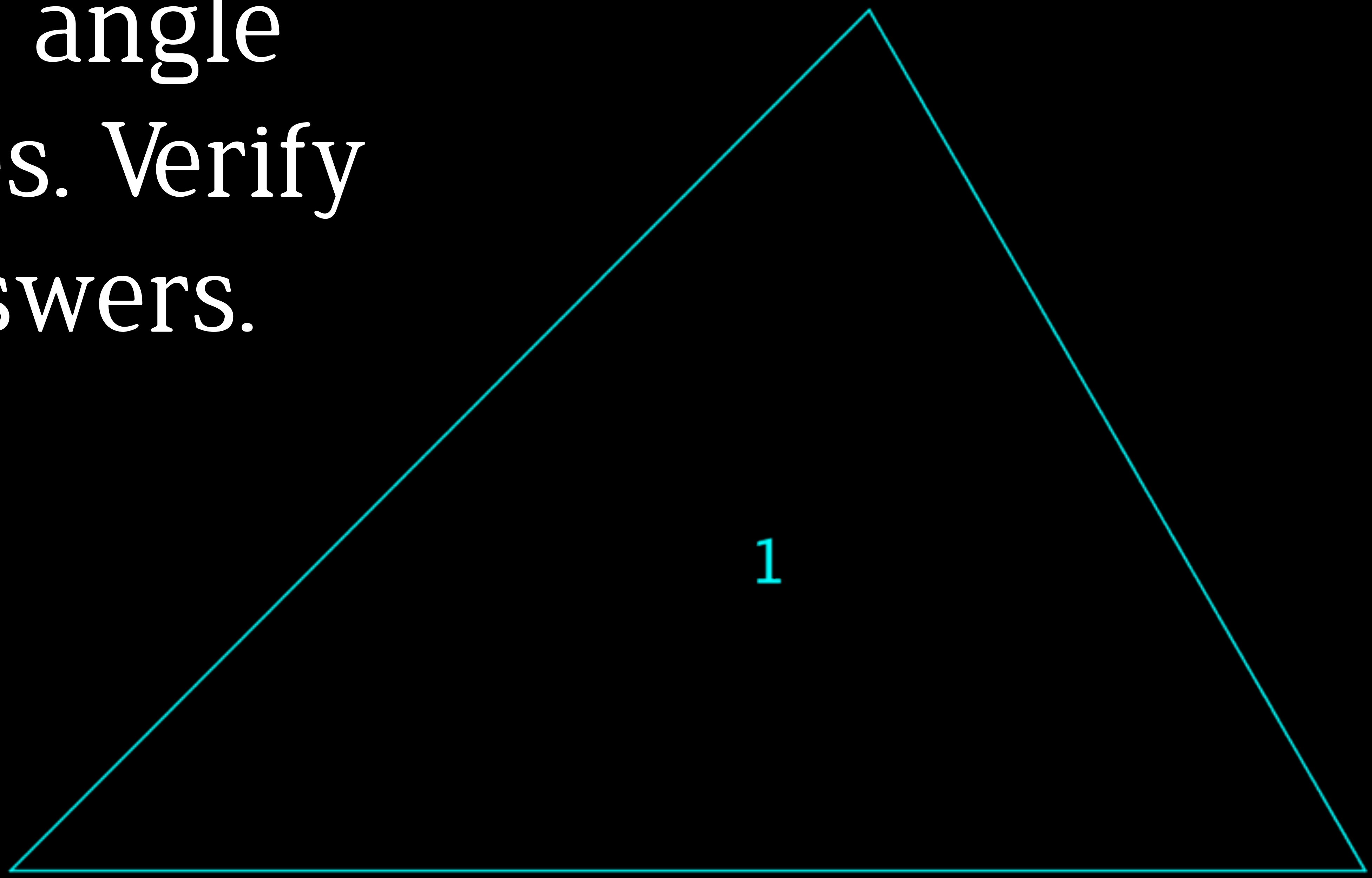
Use informal arguments to establish facts about the angle sum and exterior angle of triangles, about the angles created when parallel lines are cut by a transversal, and the angle-angle criterion for similarity of triangles.

Apply Radian Measure II

facts about the angle sum and exterior angle of triangles

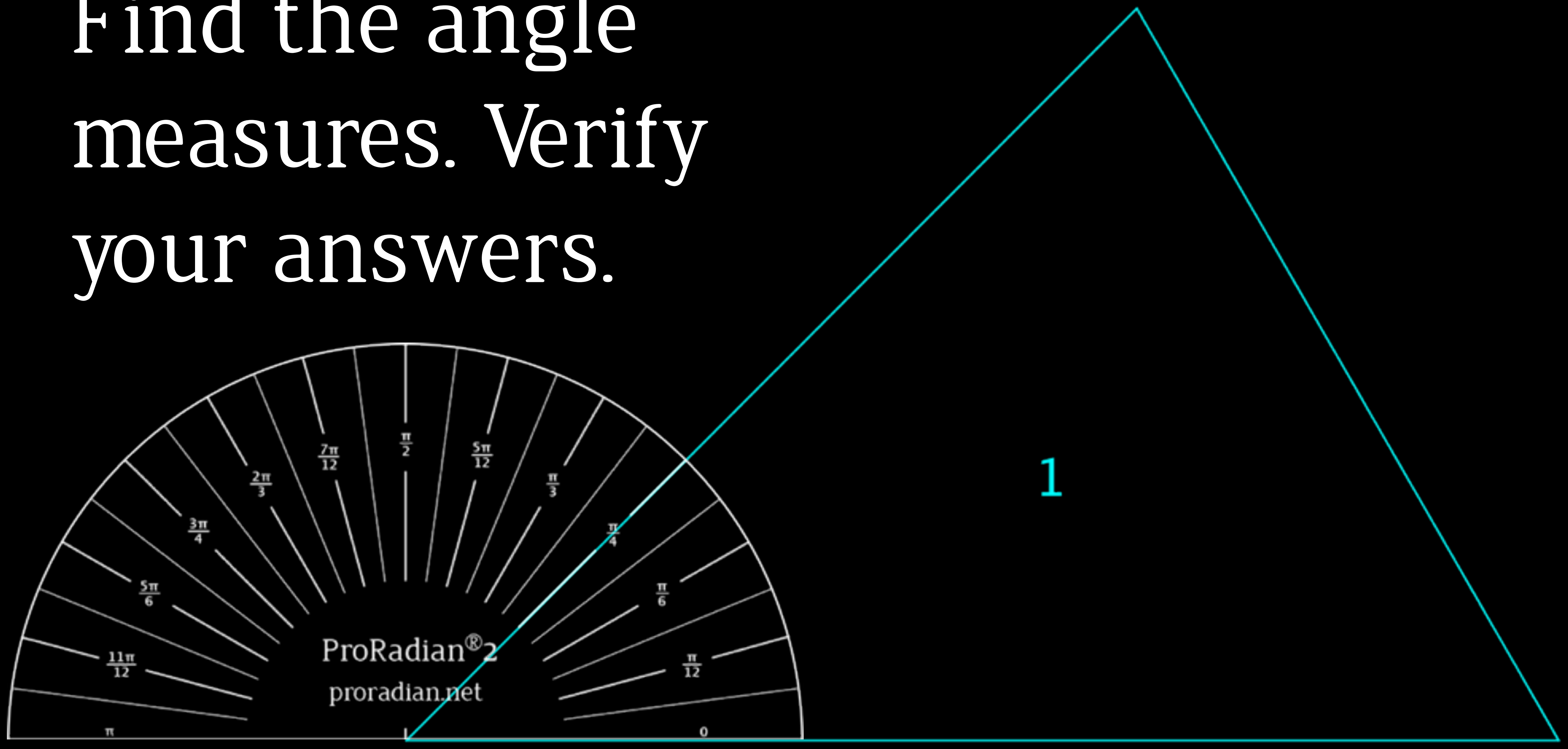
Handout Activity 2:

Find the angle measures. Verify your answers.



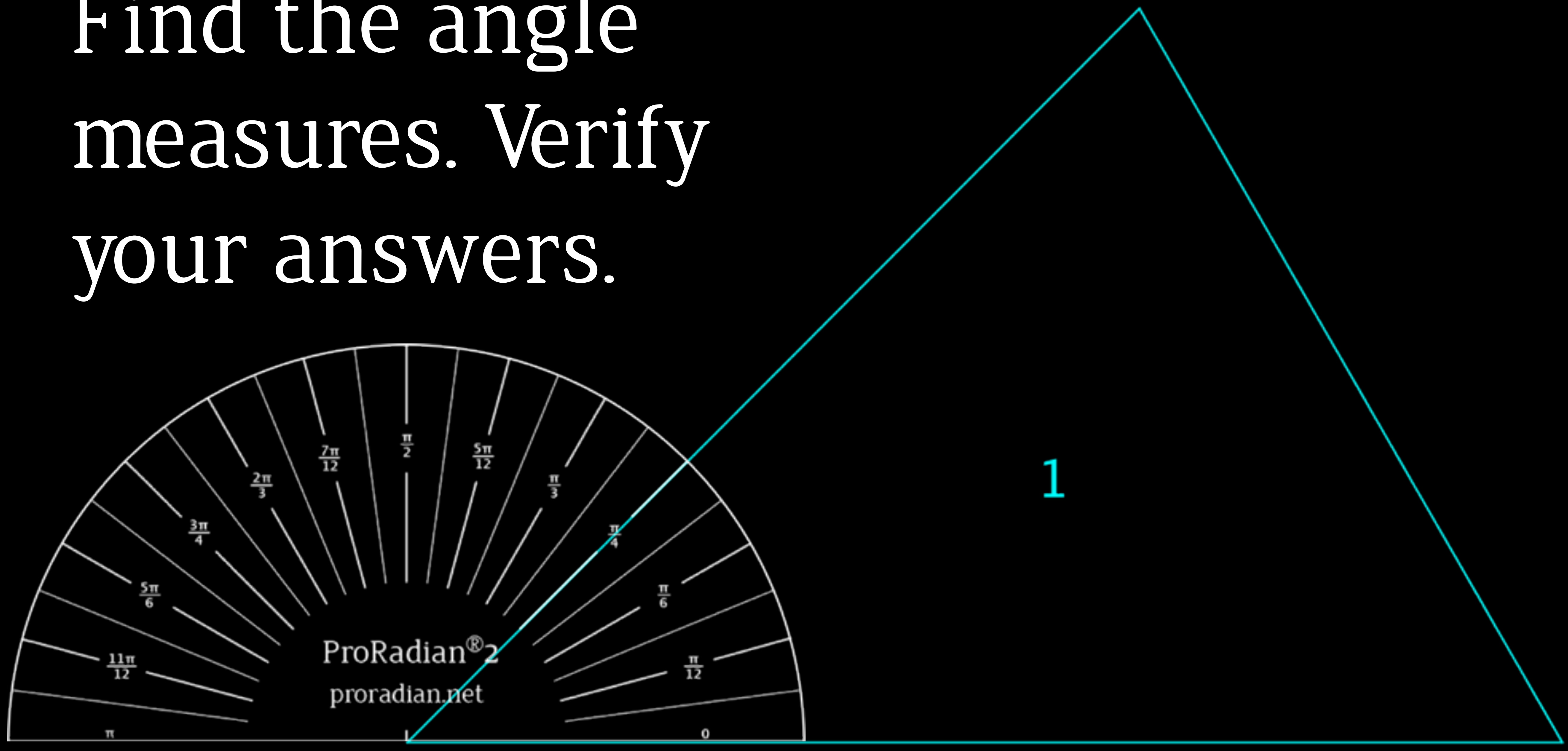
Handout Activity 2:

Find the angle measures. Verify your answers.



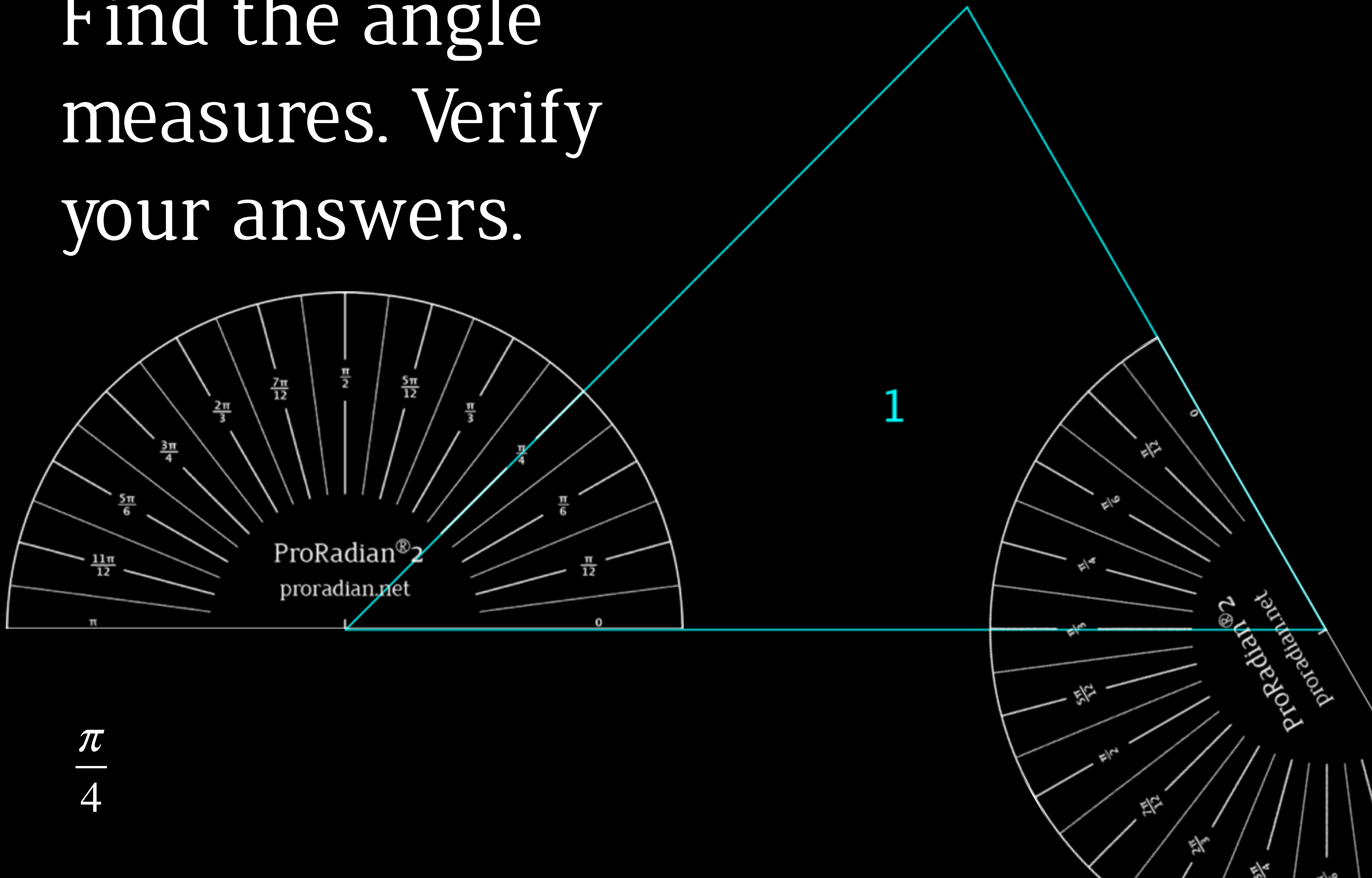
Handout Activity 2:

Find the angle measures. Verify your answers.



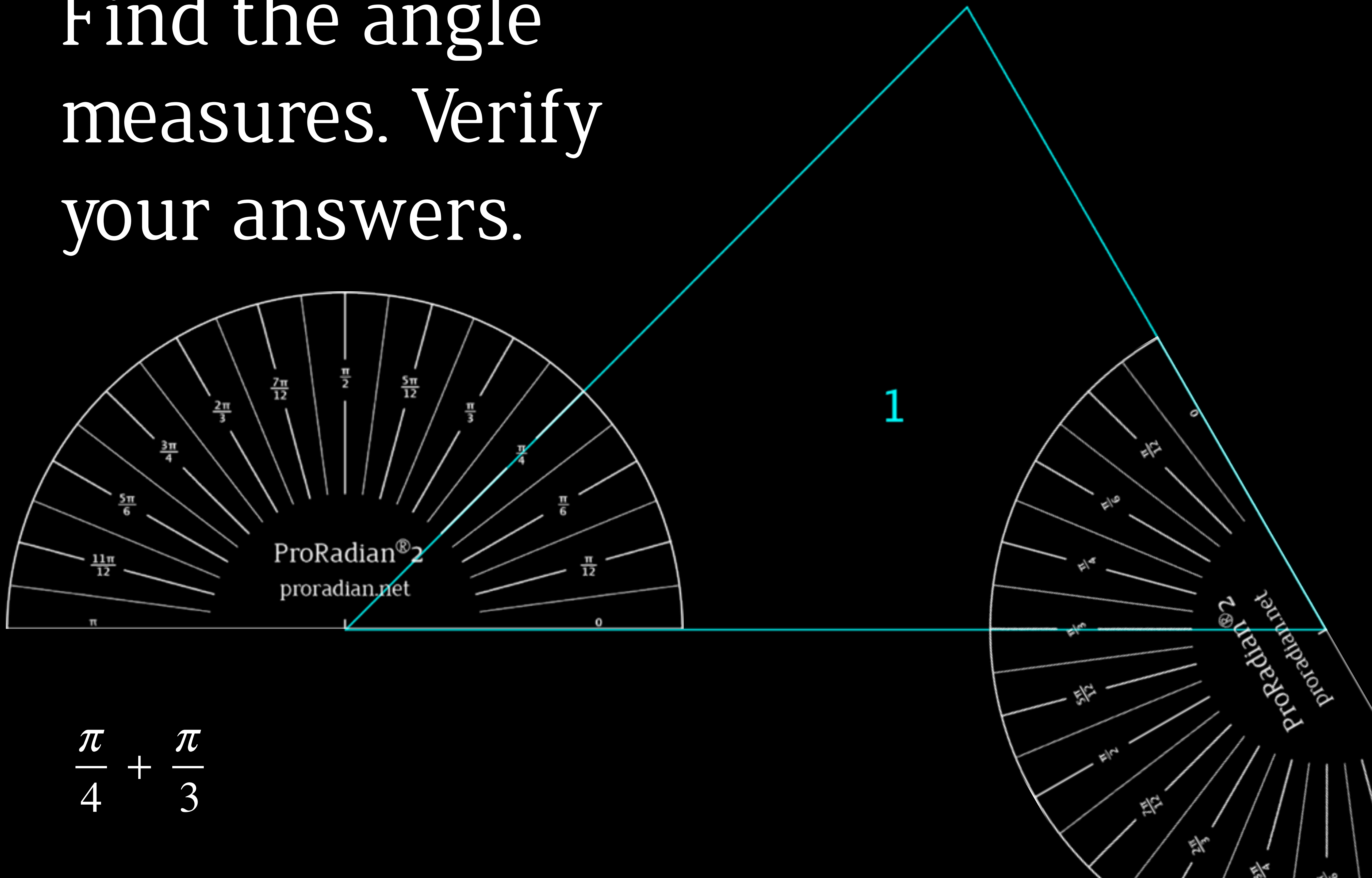
$\frac{\pi}{4}$

Handout Activity 2: Find the angle measures. Verify your answers.



$\frac{\pi}{4}$

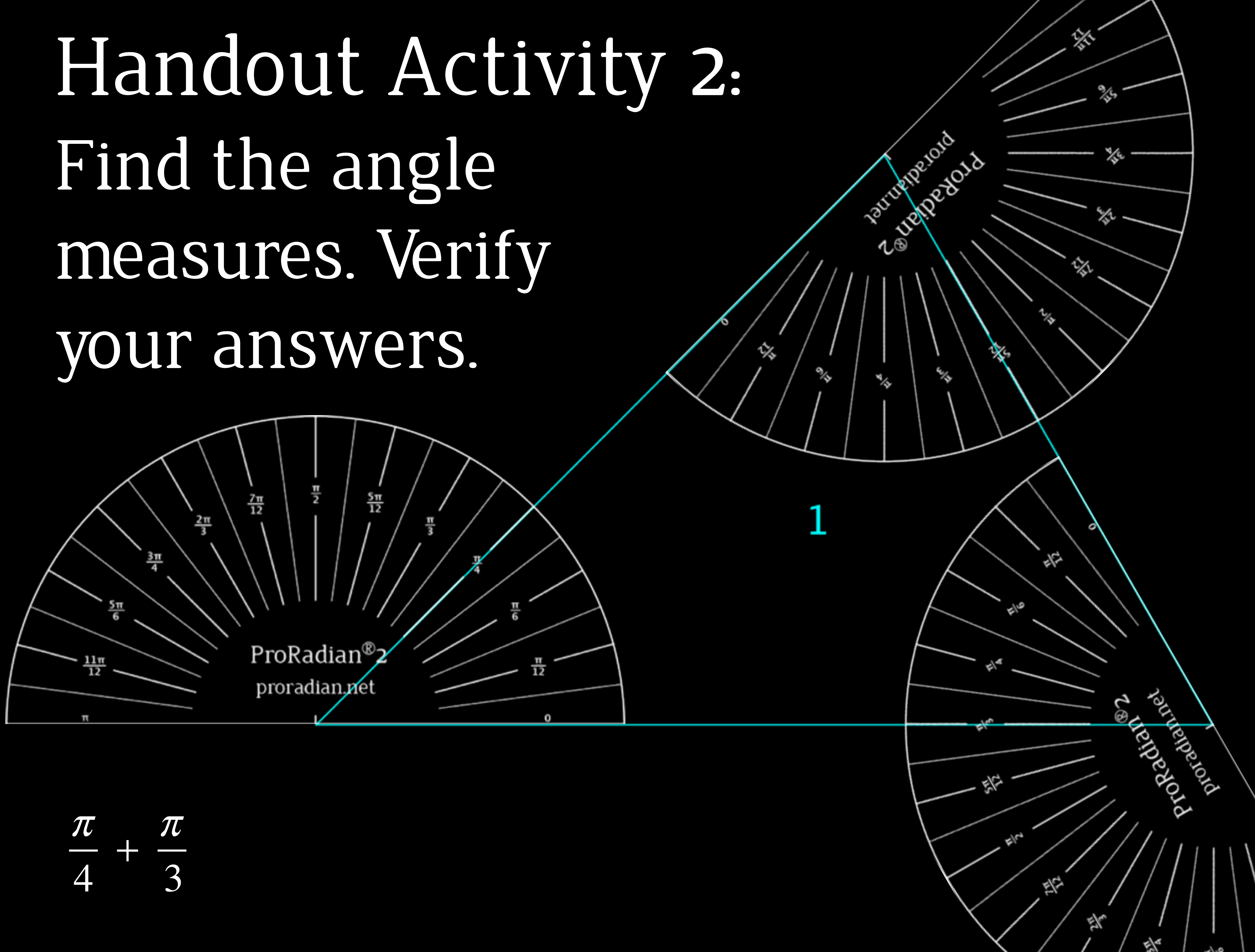
Handout Activity 2: Find the angle measures. Verify your answers.



$$\frac{\pi}{4} + \frac{\pi}{3}$$

Handout Activity 2:

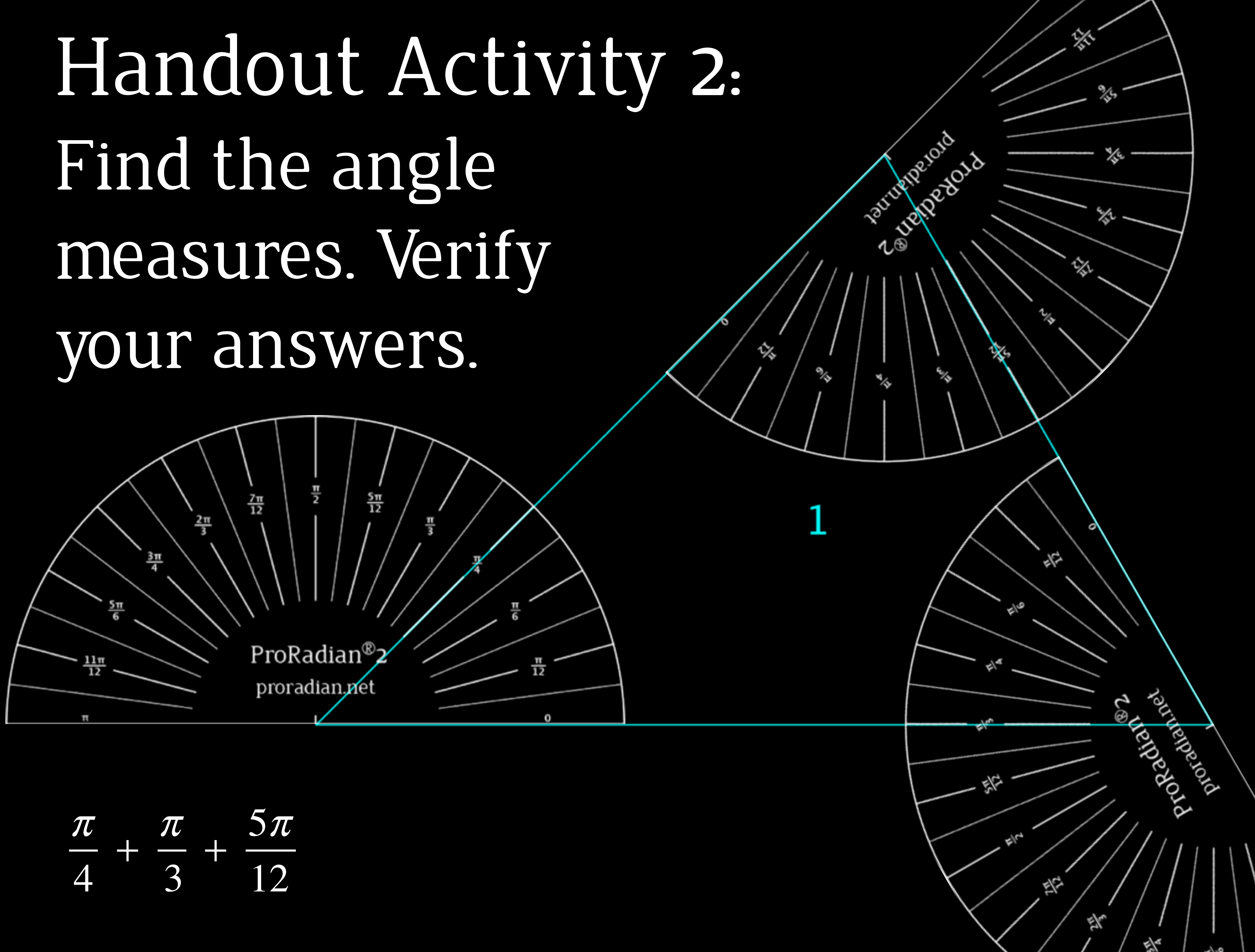
Find the angle measures. Verify your answers.



$$\frac{\pi}{4} + \frac{\pi}{3}$$

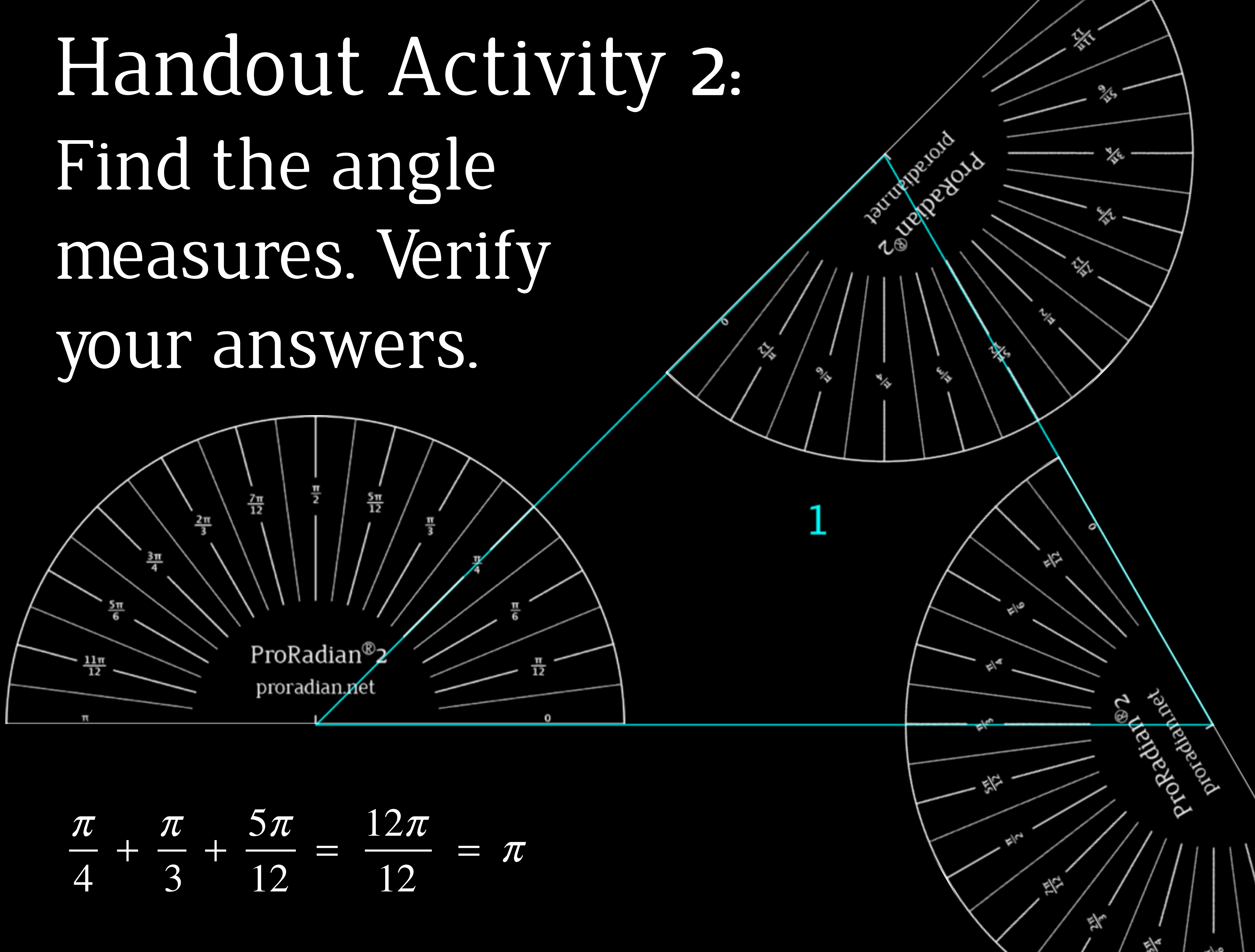
Handout Activity 2:

Find the angle measures. Verify your answers.



$$\frac{\pi}{4}+\frac{\pi}{3}+\frac{5\pi}{12}$$

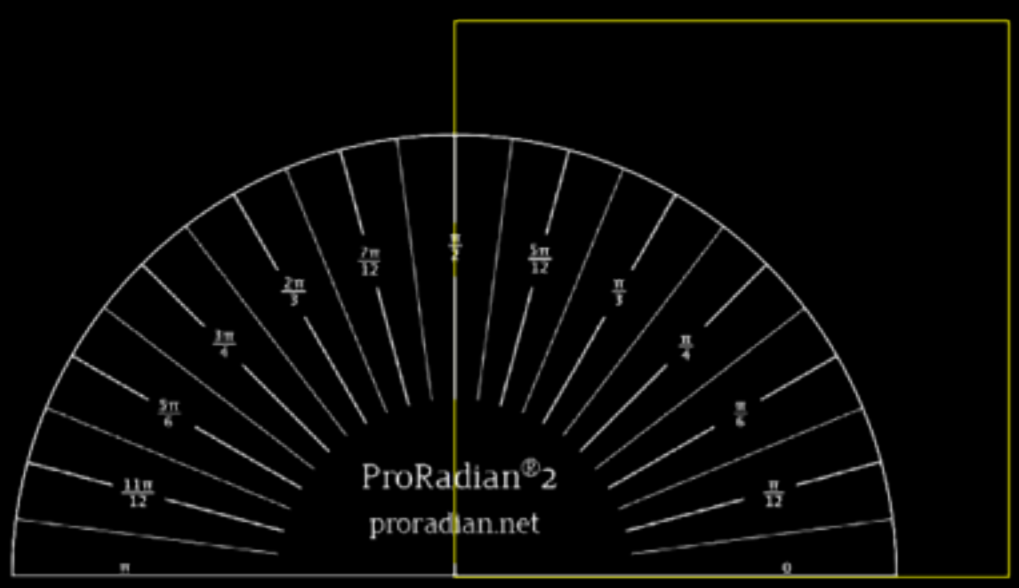
Handout Activity 2: Find the angle measures. Verify your answers.



$$\frac{\pi}{4} + \frac{\pi}{3} + \frac{5\pi}{12} = \frac{12\pi}{12} = \pi$$

Handout Activity 3: Polygons!

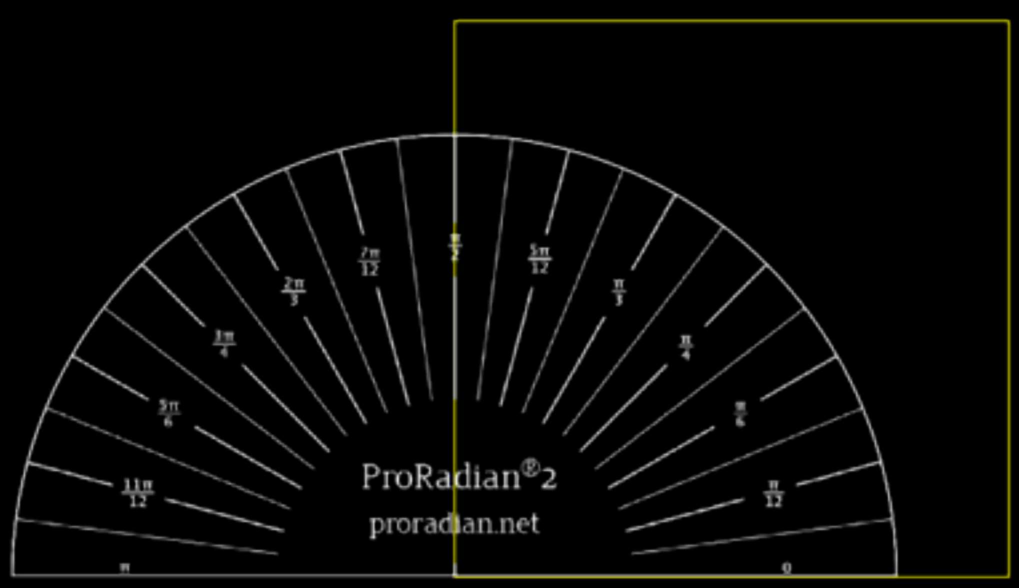
Measure the angles in some regular polygons. Complete the table.



Handout Activity 3: Polygons!

Measure the angles in some regular polygons. Complete the table.

Regular Polygon	Measure of 1 Interior Angle	Sum of Interior Angles



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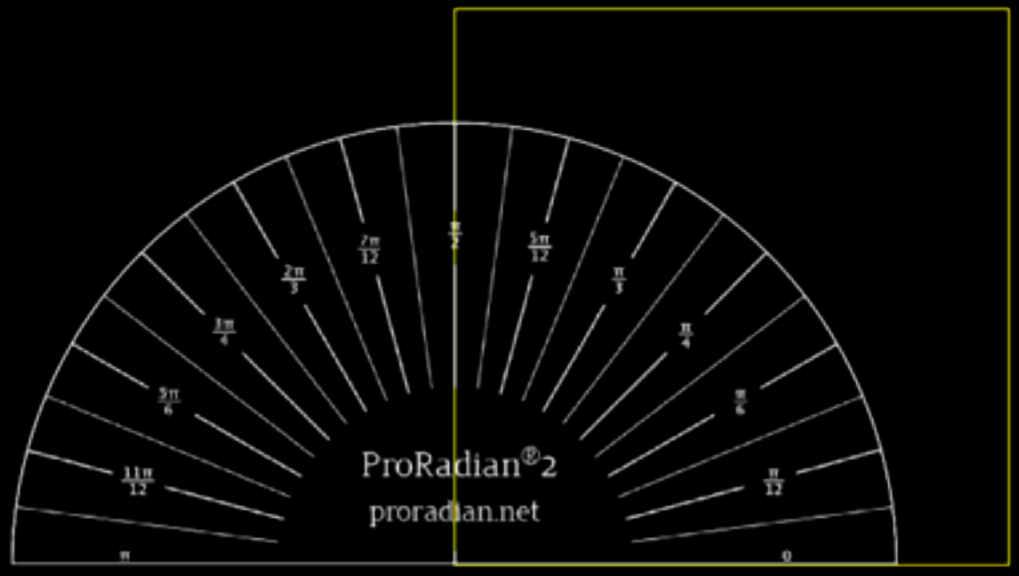
Measure the angles in some regular polygons. Complete the table.

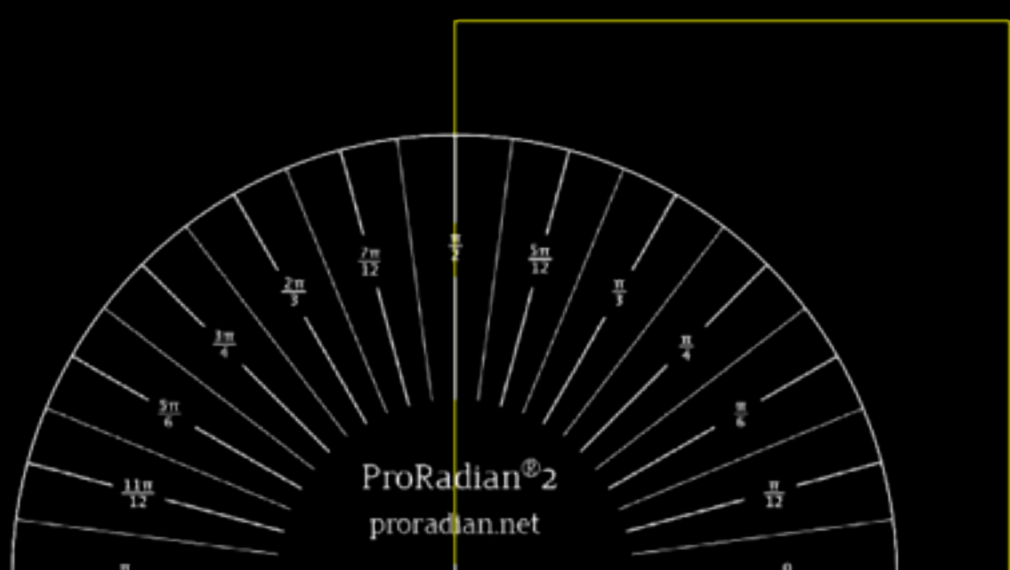


Regular Polygon	Measure of 1 Interior Angle	Sum of Interior Angles
Triangle	$1\pi/3$	π

Handout Activity 3: Polygons!

Measure the angles in some regular polygons. Complete the table.

Regular Polygon	Measure of 1 Interior Angle	Sum of Interior Angles
Triangle	$1\pi/3$	π
Square	$2\pi/4$	2π



Handout Activity 3: Polygons!

Regular Polygon	Measure of 1 Interior Angle	Sum of Interior Angles
Triangle	$1\pi/3$	π
Square	$2\pi/4$	2π

Handout Activity 3: Polygons!

Use patterns to fill-in those you can't measure.

Regular Polygon	Measure of 1 Interior Angle	Sum of Interior Angles
Triangle	$1\pi/3$	π
Square	$2\pi/4$	2π

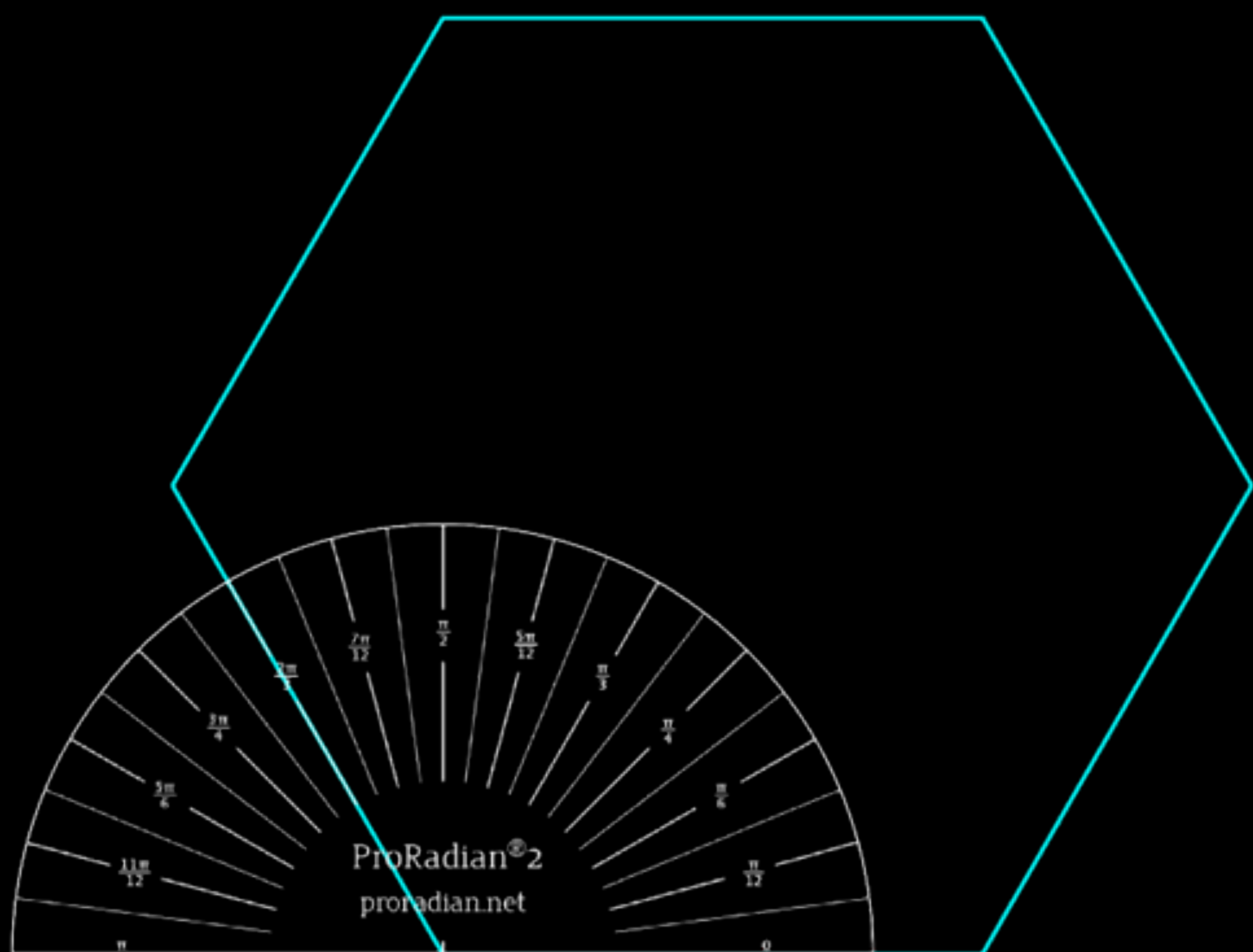
Handout Activity 3: Polygons!

Use patterns to fill-in those you can't measure.

Regular Polygon	Measure of 1 Interior Angle	Sum of Interior Angles
Triangle	$1\pi/3$	π
Square	$2\pi/4$	2π
Pentagon	$3\pi/5$	3π

Handout Activity 3: Polygons!

Use patterns to fill-in those you can't measure.



Regular Polygon	Measure of 1 Interior Angle	Sum of Interior Angles
Triangle	$1\pi/3$	π
Square	$2\pi/4$	2π
Pentagon	$3\pi/5$	3π
Hexagon	$4\pi/6$	4π

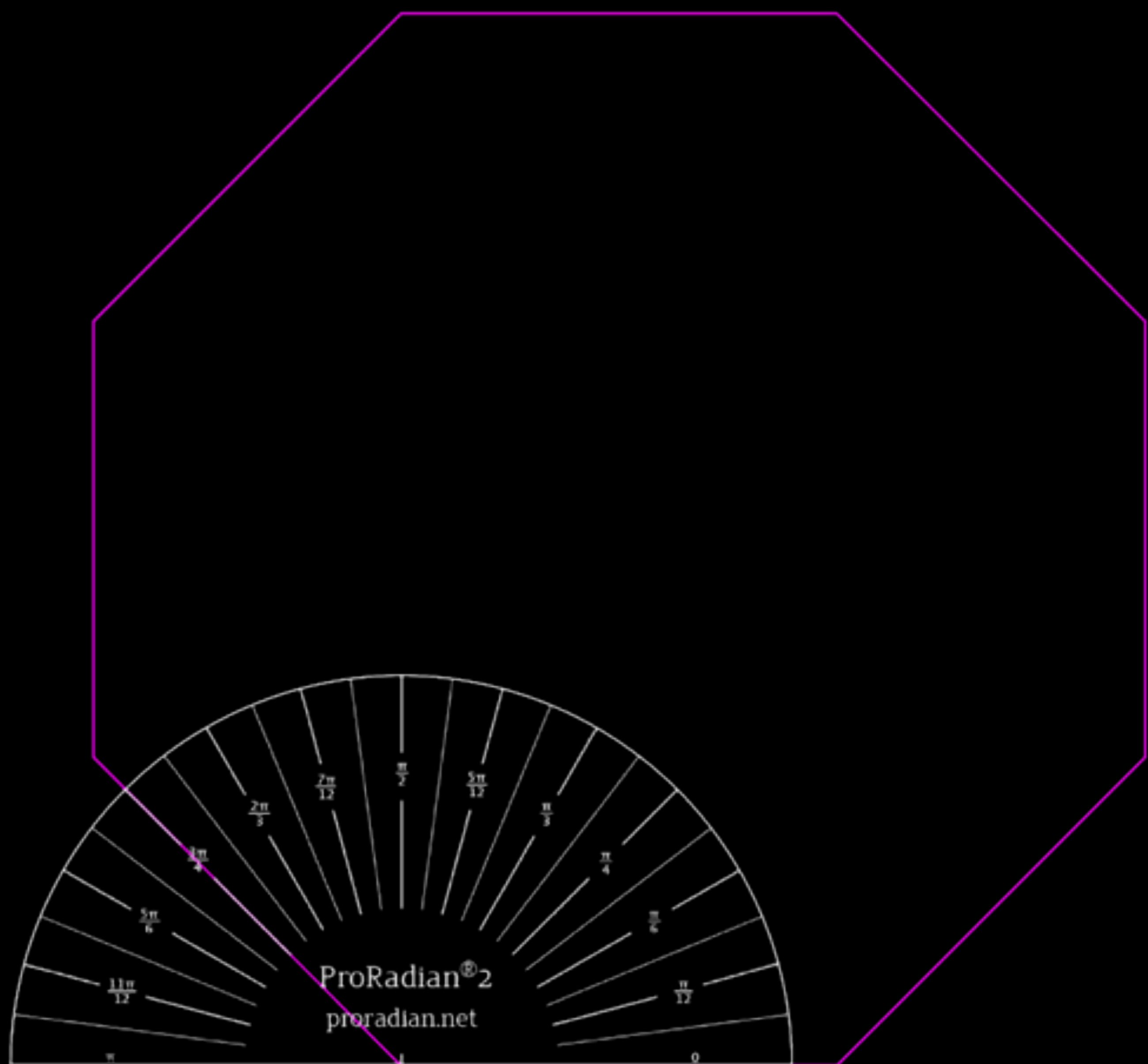
Handout Activity 3: Polygons!

Use patterns to fill-in those you can't measure.

Regular Polygon	Measure of 1 Interior Angle	Sum of Interior Angles
Triangle	$1\pi/3$	π
Square	$2\pi/4$	2π
Pentagon	$3\pi/5$	3π
Hexagon	$4\pi/6$	4π
Septagon	$5\pi/7$	5π

Handout Activity 3: Polygons!

Use patterns to fill-in those you can't measure.



Regular Polygon	Measure of 1 Interior Angle	Sum of Interior Angles
Triangle	1π/3	π
Square	2π/4	2π
Pentagon	3π/5	3π
Hexagon	4π/6	4π
Septagon	5π/7	5π
Octagon	6π/8	6π

Handout Activity 3: Polygons!

Use patterns to fill-in those you can't measure.

Regular Polygon	Measure of 1 Interior Angle	Sum of Interior Angles
Triangle	$1\pi/3$	π
Square	$2\pi/4$	2π
Pentagon	$3\pi/5$	3π
Hexagon	$4\pi/6$	4π
Septagon	$5\pi/7$	5π
Octagon	$6\pi/8$	6π
Nonagon	$7\pi/9$	7π

Handout Activity 3: Polygons!

Use patterns to fill-in those you can't measure.

Regular Polygon	Measure of 1 Interior Angle	Sum of Interior Angles
Triangle	$1\pi/3$	π
Square	$2\pi/4$	2π
Pentagon	$3\pi/5$	3π
Hexagon	$4\pi/6$	4π
Septagon	$5\pi/7$	5π
Octagon	$6\pi/8$	6π
Nonagon	$7\pi/9$	7π
Decagon	$8\pi/10$	8π

Handout Activity 3: Polygons!

Regular Polygon	Measure of 1 Interior Angle	Sum of Interior Angles
Triangle	$1\pi/3$	π
Square	$2\pi/4$	2π
Pentagon	$3\pi/5$	3π
Hexagon	$4\pi/6$	4π
Septagon	$5\pi/7$	5π
Octagon	$6\pi/8$	6π
Nonagon	$7\pi/9$	7π
Decagon	$8\pi/10$	8π

Handout Activity 3: Polygons!

He would be pleased.

Regular Polygon	Measure of 1 Interior Angle	Sum of Interior Angles
Triangle	$1\pi/3$	π
Square	$2\pi/4$	2π
Pentagon	$3\pi/5$	3π
Hexagon	$4\pi/6$	4π
Septagon	$5\pi/7$	5π
Octagon	$6\pi/8$	6π
Nonagon	$7\pi/9$	7π
Decagon	$8\pi/10$	8π
n-gon	$(n-2)\pi/n$	$(n-2)\pi$

High School Activities

Radian Measure & Arc Length

Radian Measure & Arc Length

CCSS.MATH.CONTENT. HSF-TF.A.1

Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

Radian Measure & Arc Length

CCSS.MATH.CONTENT. HSF-TF.A.1

Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

So, the angle measure is the same as the length of the arc?

Radian Measure & Arc Length

CCSS.MATH.CONTENT.HSF-TF.A.1

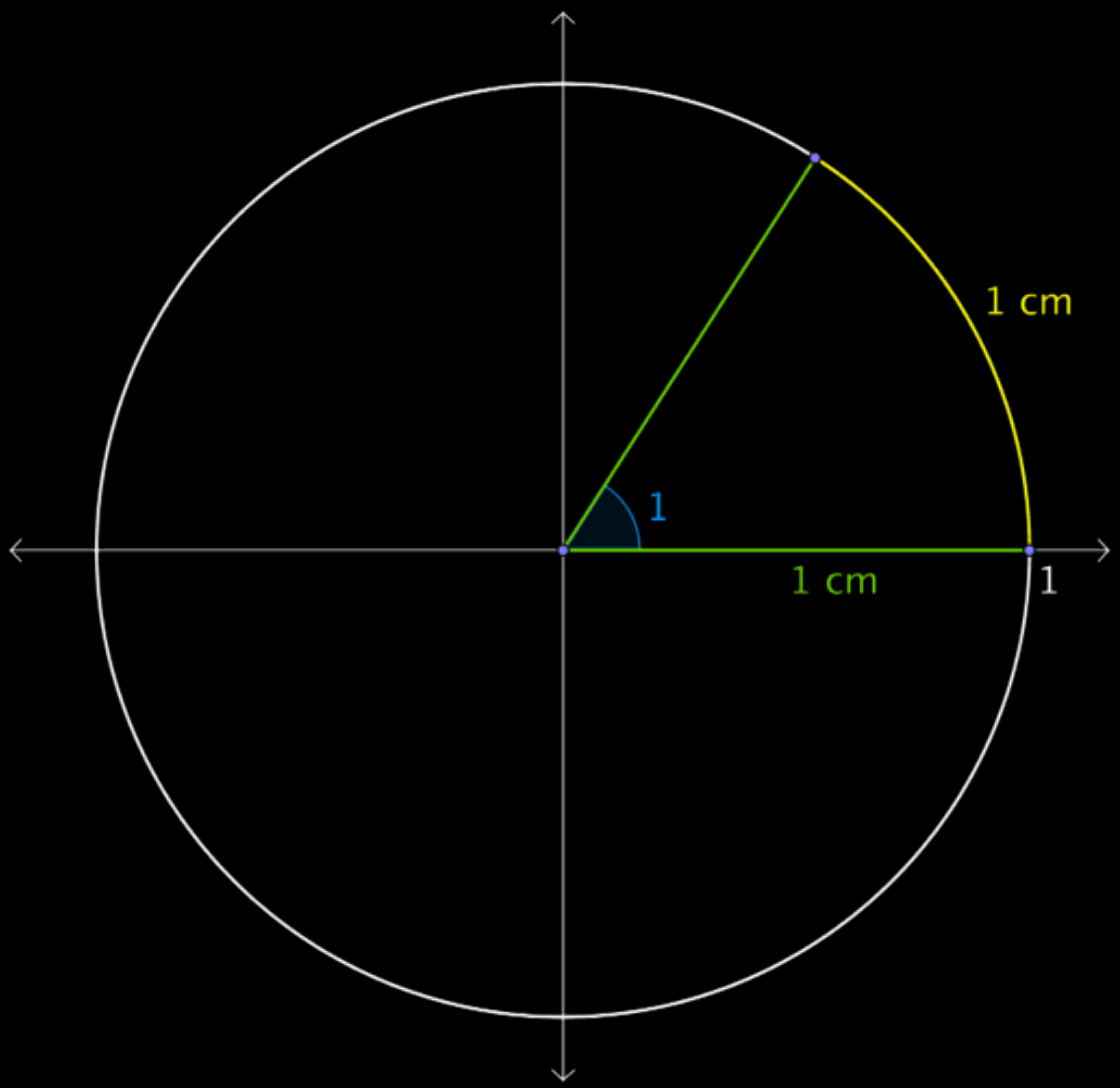
Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.

So, the angle measure is the same as the length of the arc?

REALLY?

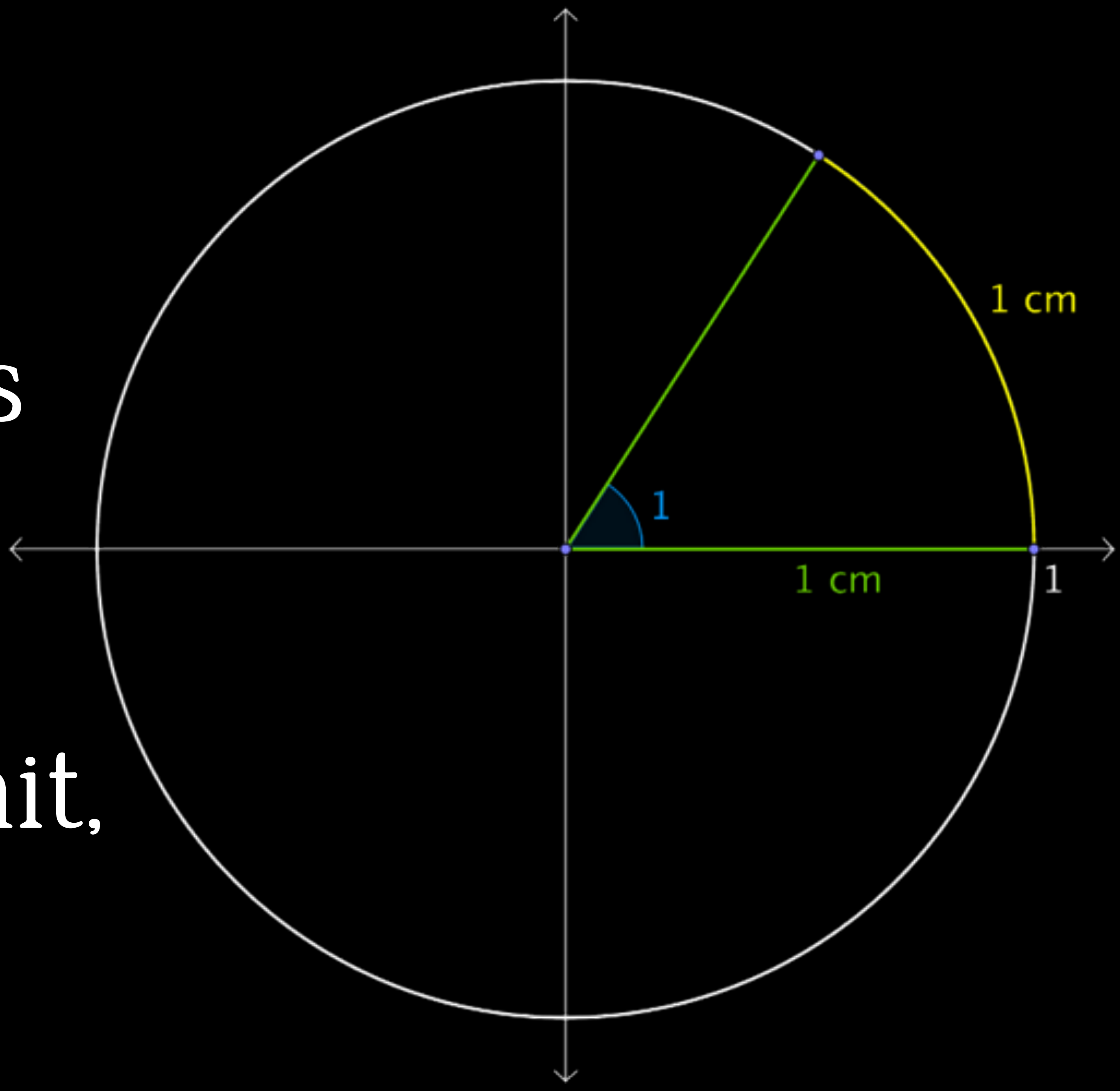
What about the units?

What about the units?



What about the units?

Yes, the arc and the angle have the same numeric value, but it is incorrect to say that they are the same. Length has a linear unit, and radian angle measure is unit-less, because it is a ratio.



Ok, I'll overlook that…

This is what the standard means.

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If the radius is 1, the radian measure of the central angle is numerically equal to the length of the intercepted arc.

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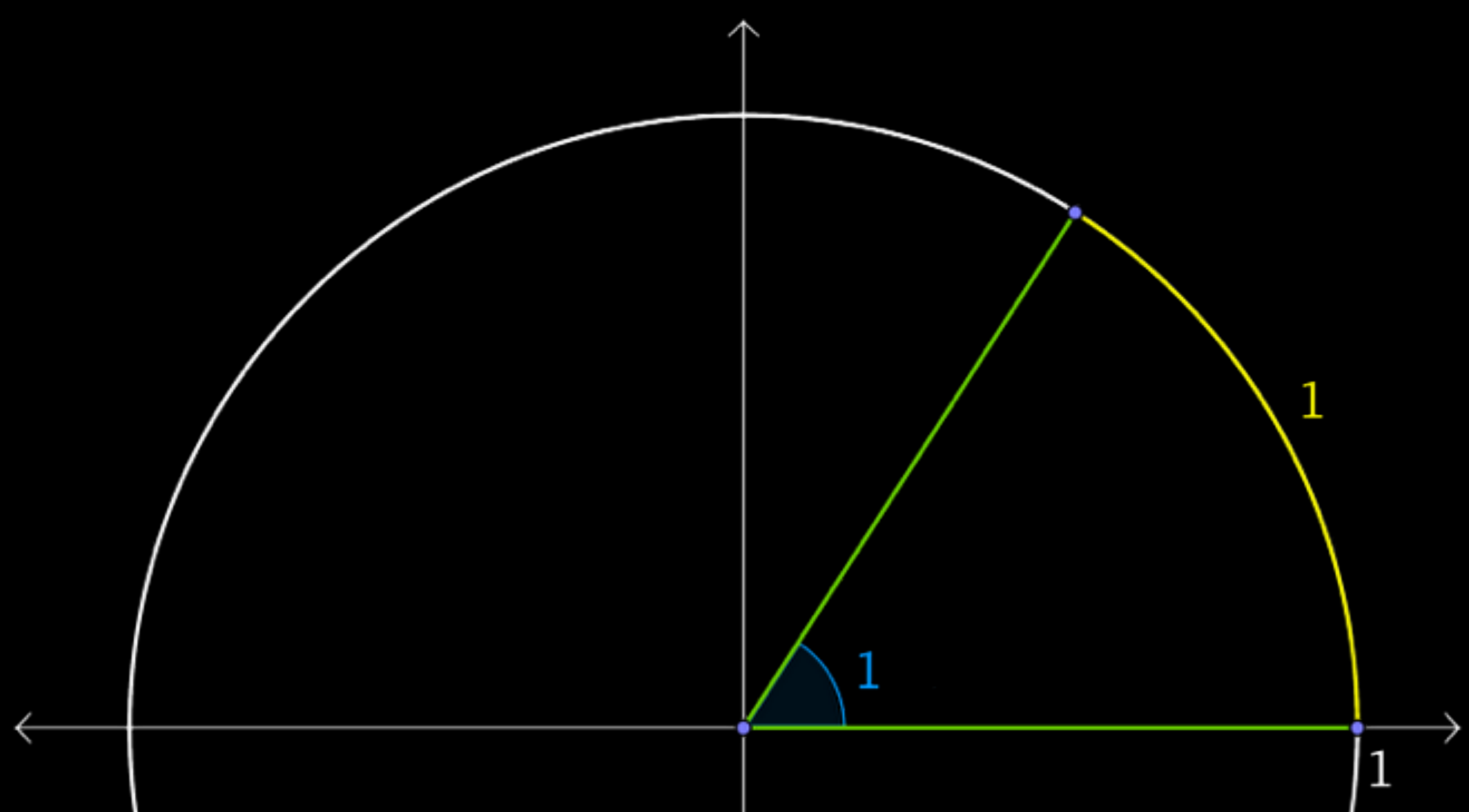
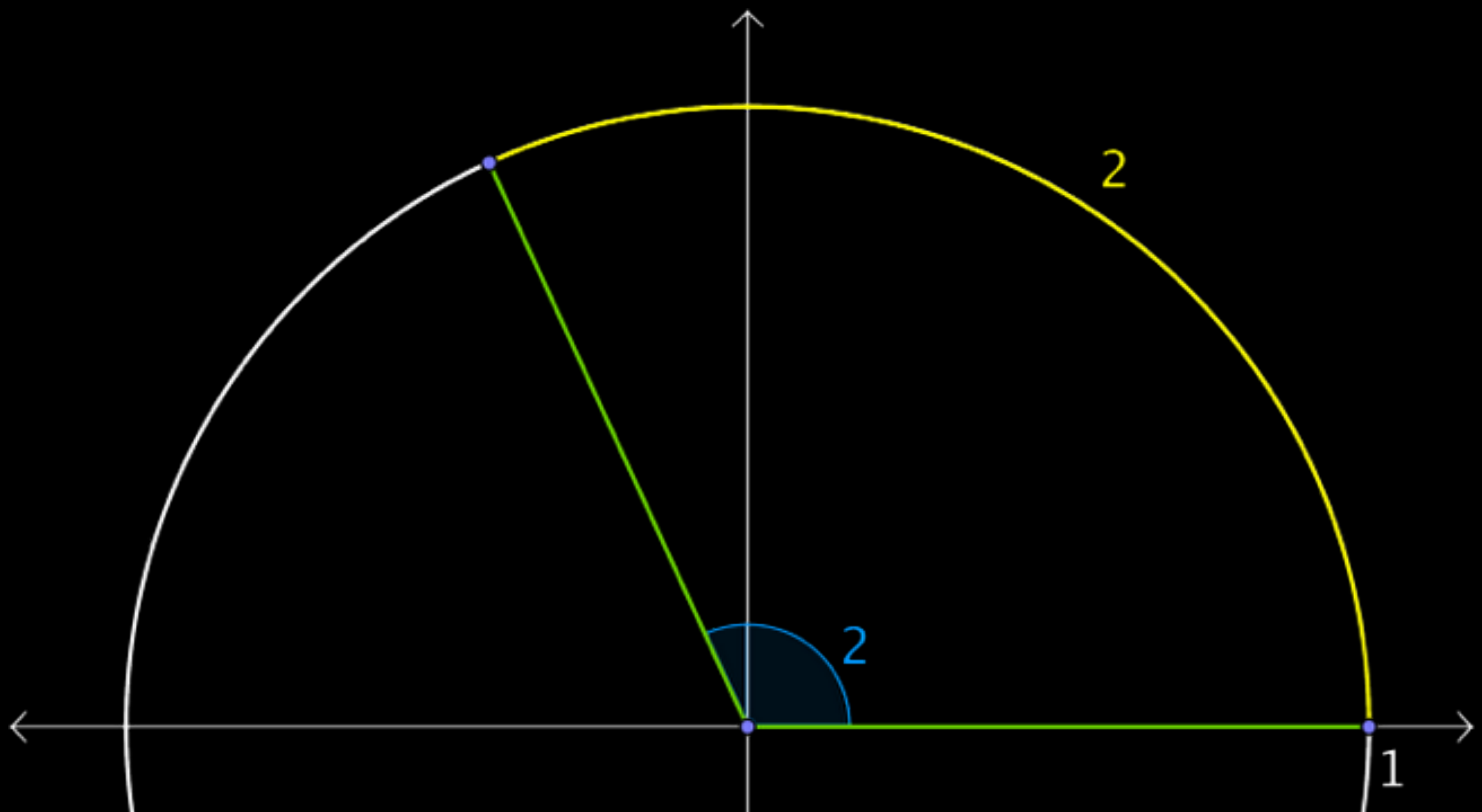
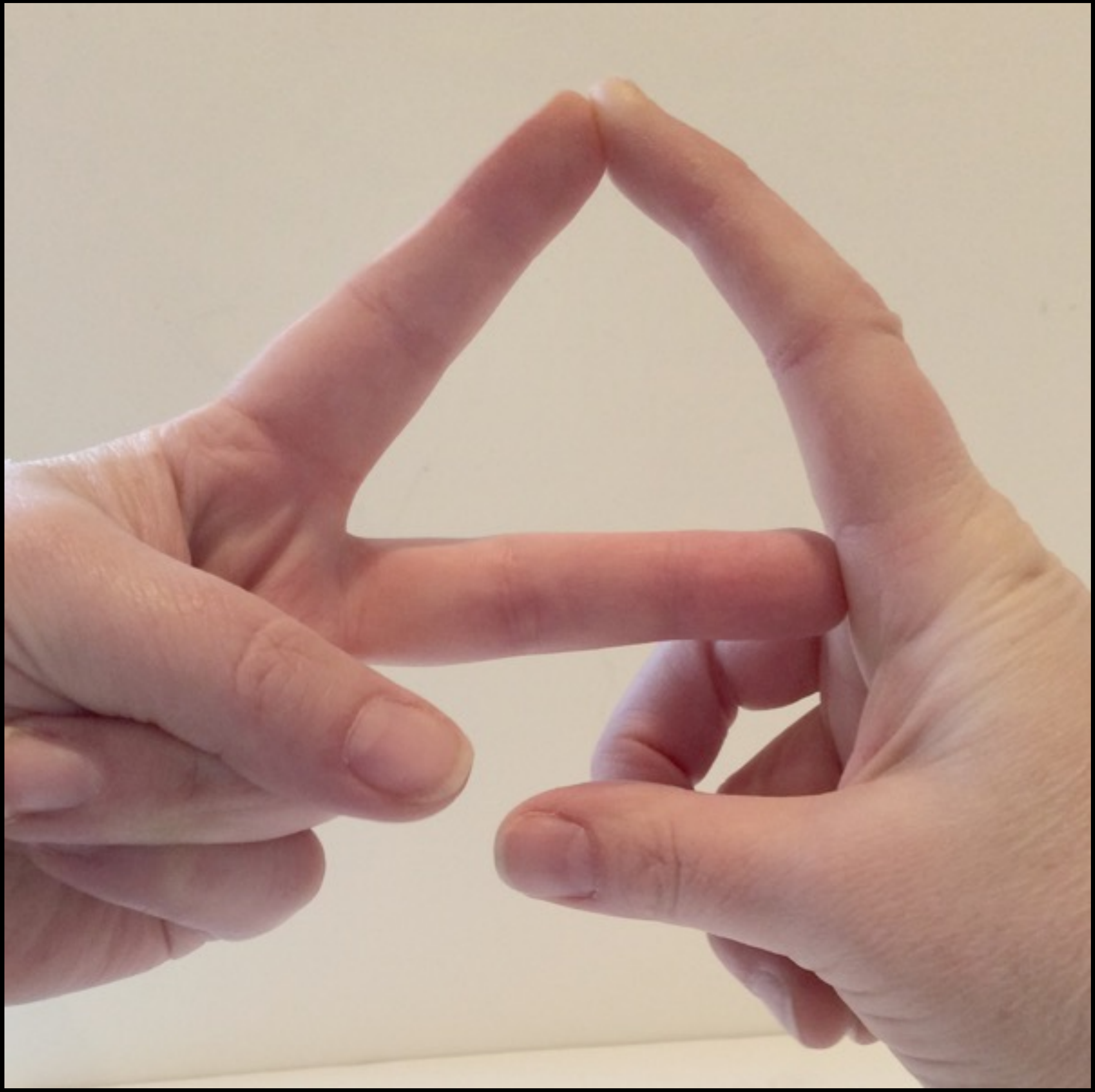


This is what the standard means.

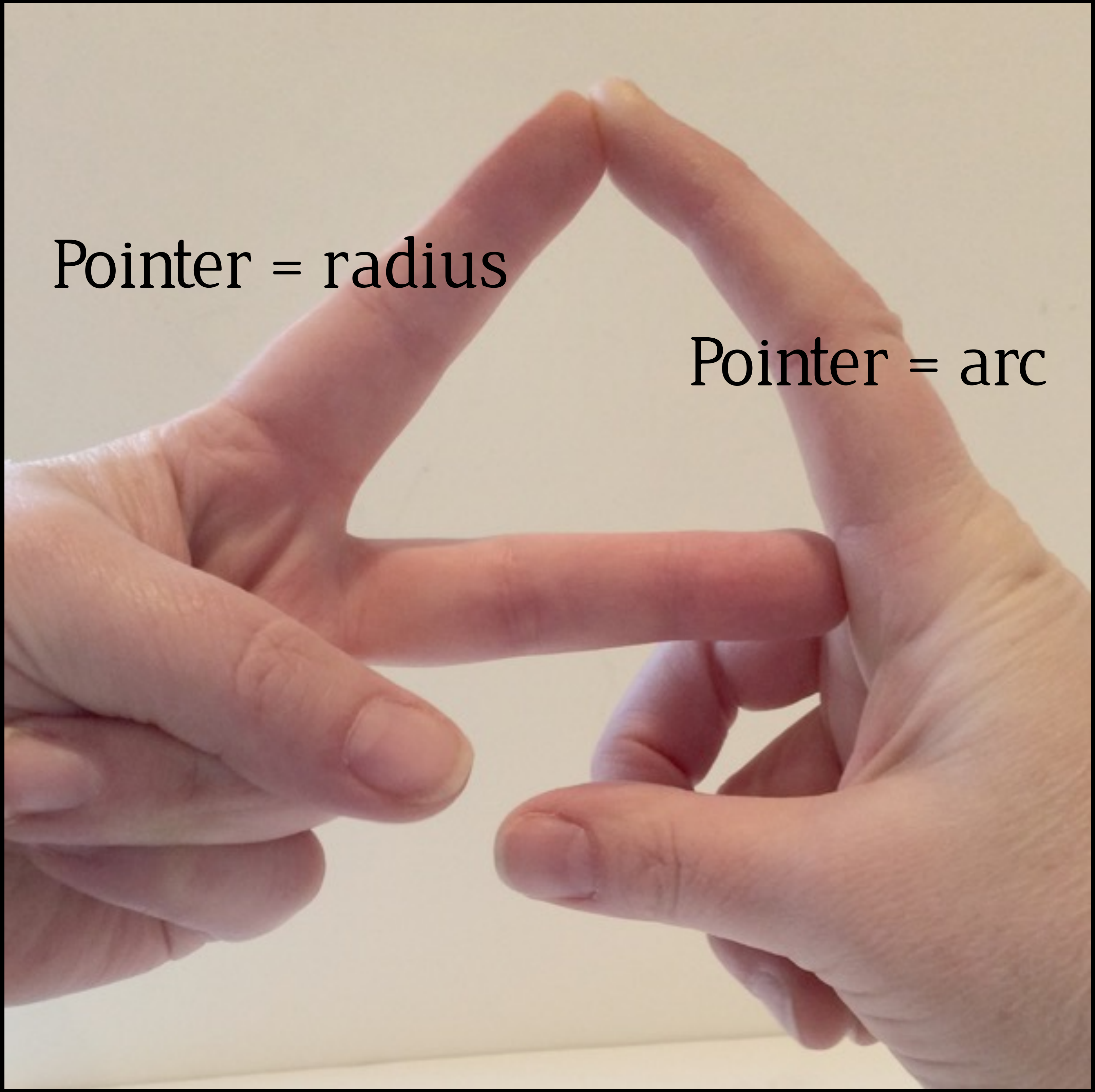
If the radius is 1, the radian measure of the central angle is numerically equal to the length of the intercepted arc.



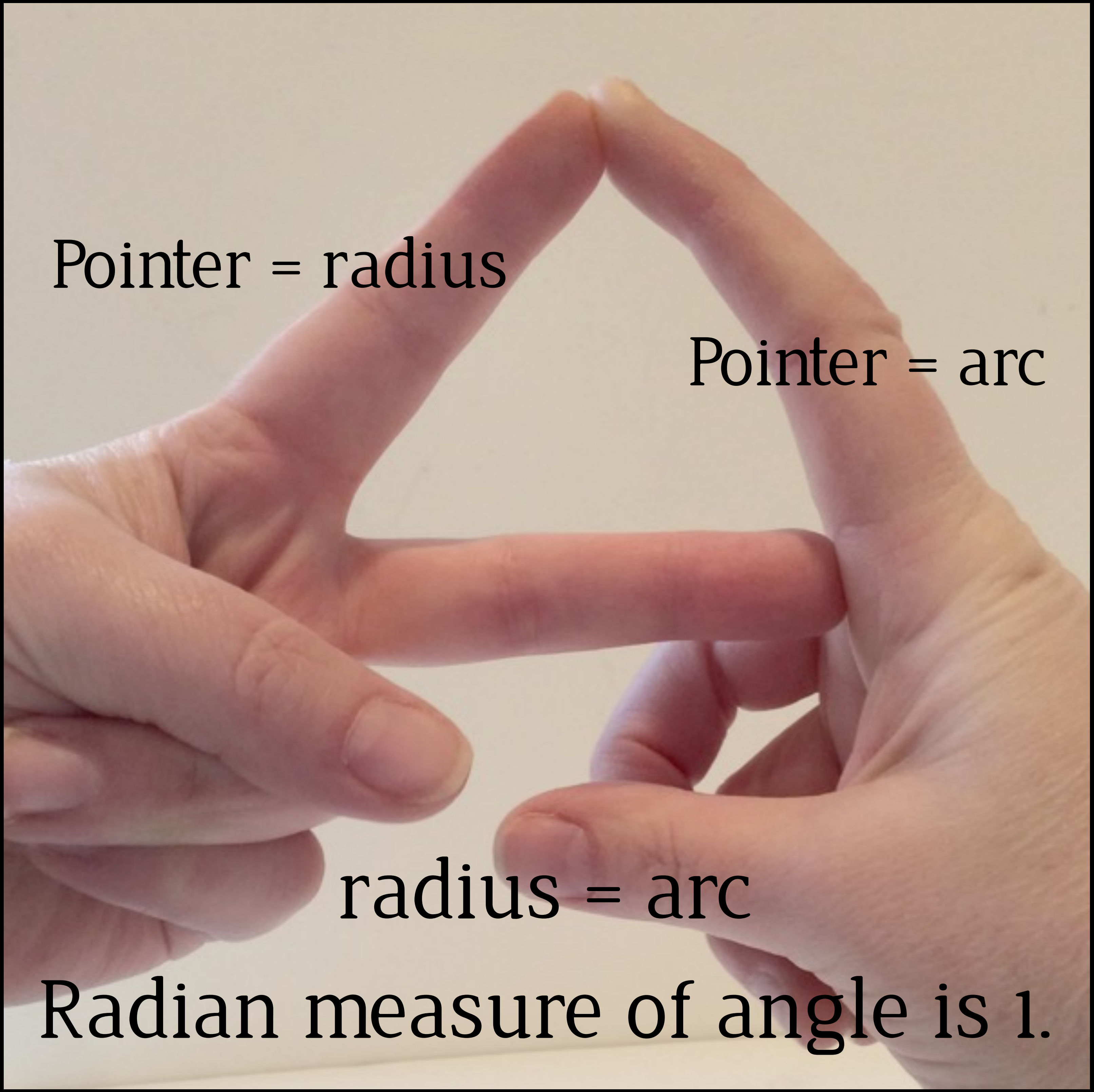
Kid-Friendly Demo:



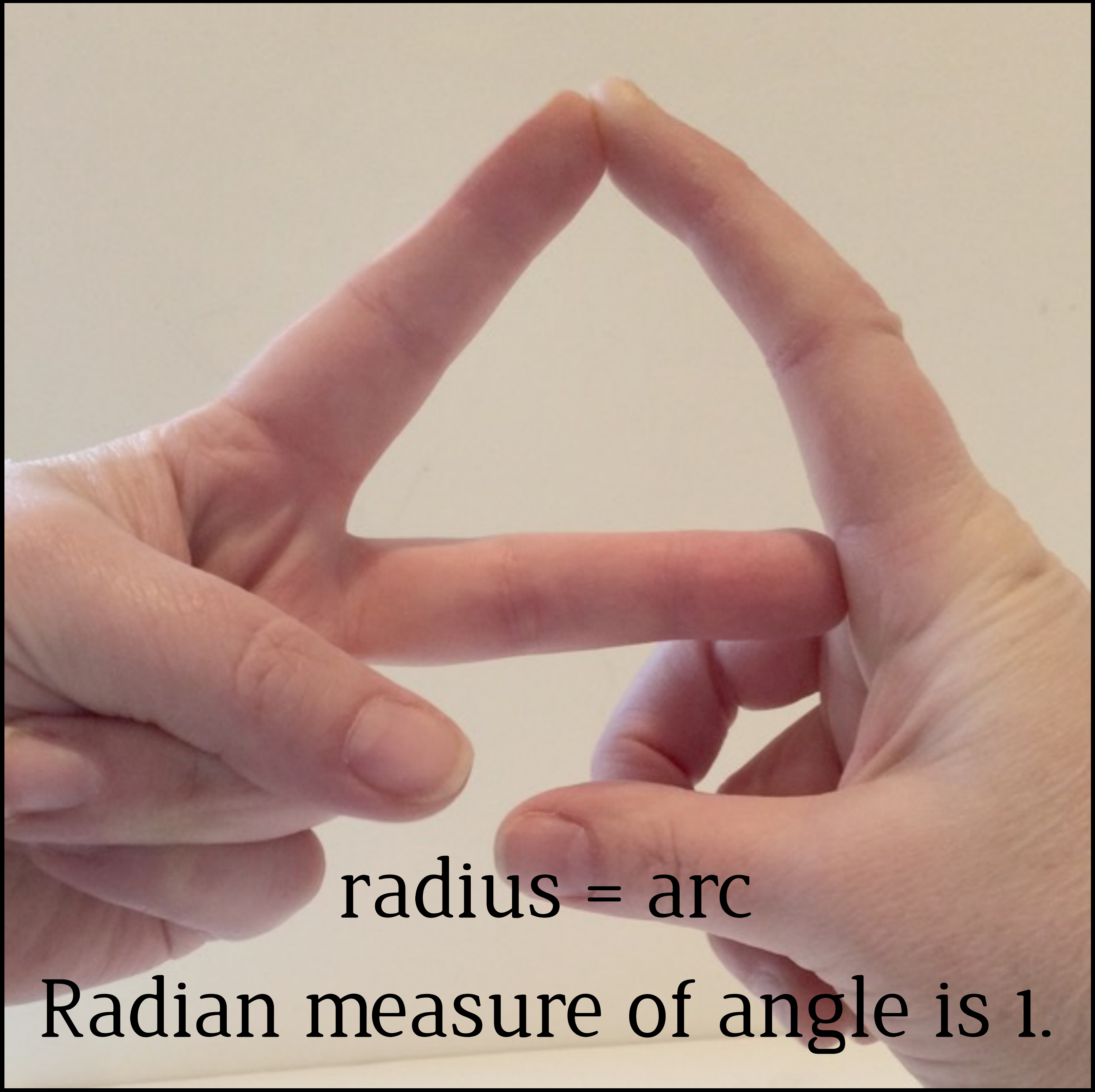
Kid-Friendly Demo:



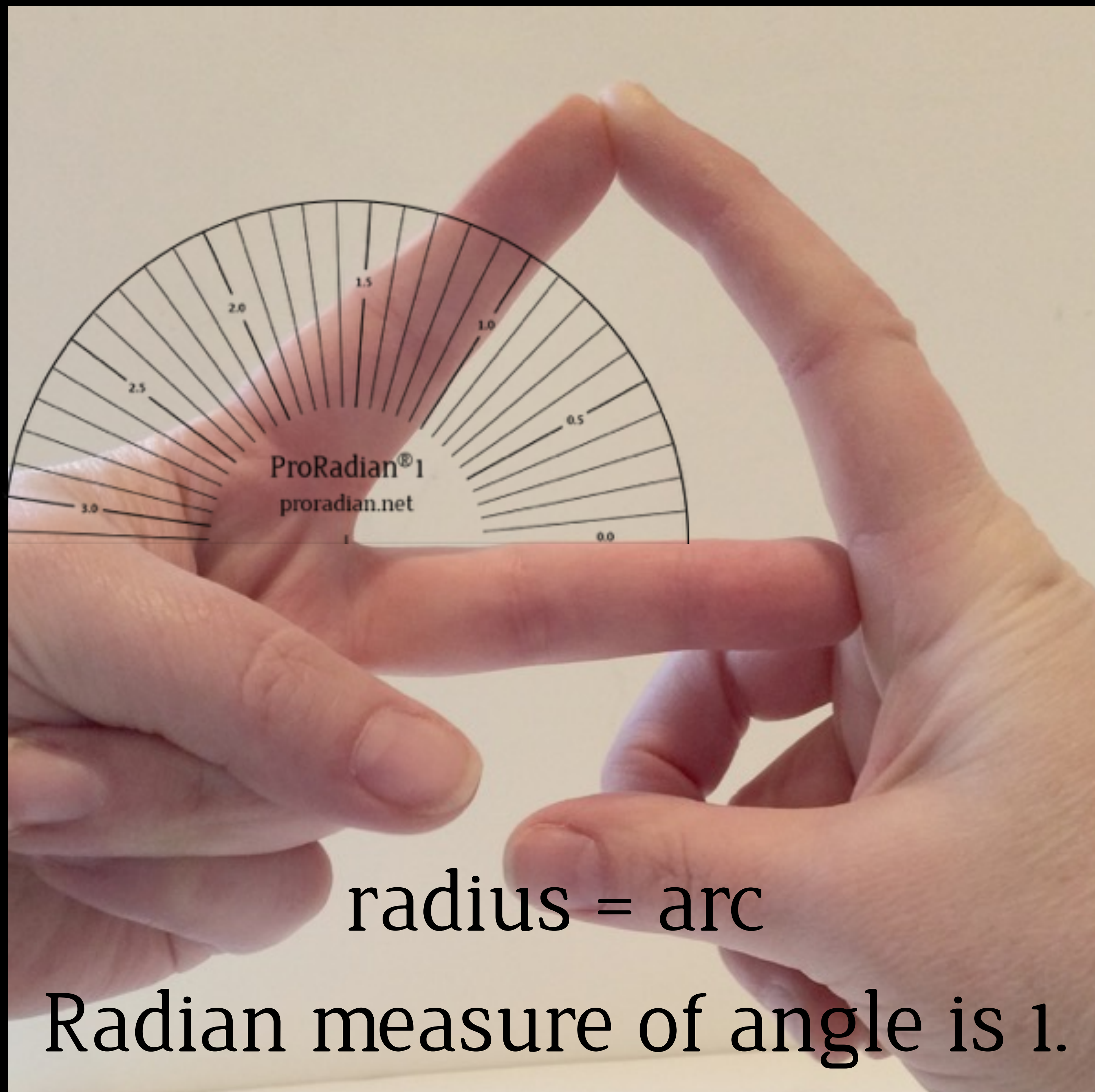
Kid-Friendly Demo:



Kid-Friendly Demo:



Kid-Friendly Demo:



Plain Paper Activity 2

Make a Number Line

Make a Number Line

Take a plain piece of paper and fold it in half length-wise. Put it down with the fold facing you.

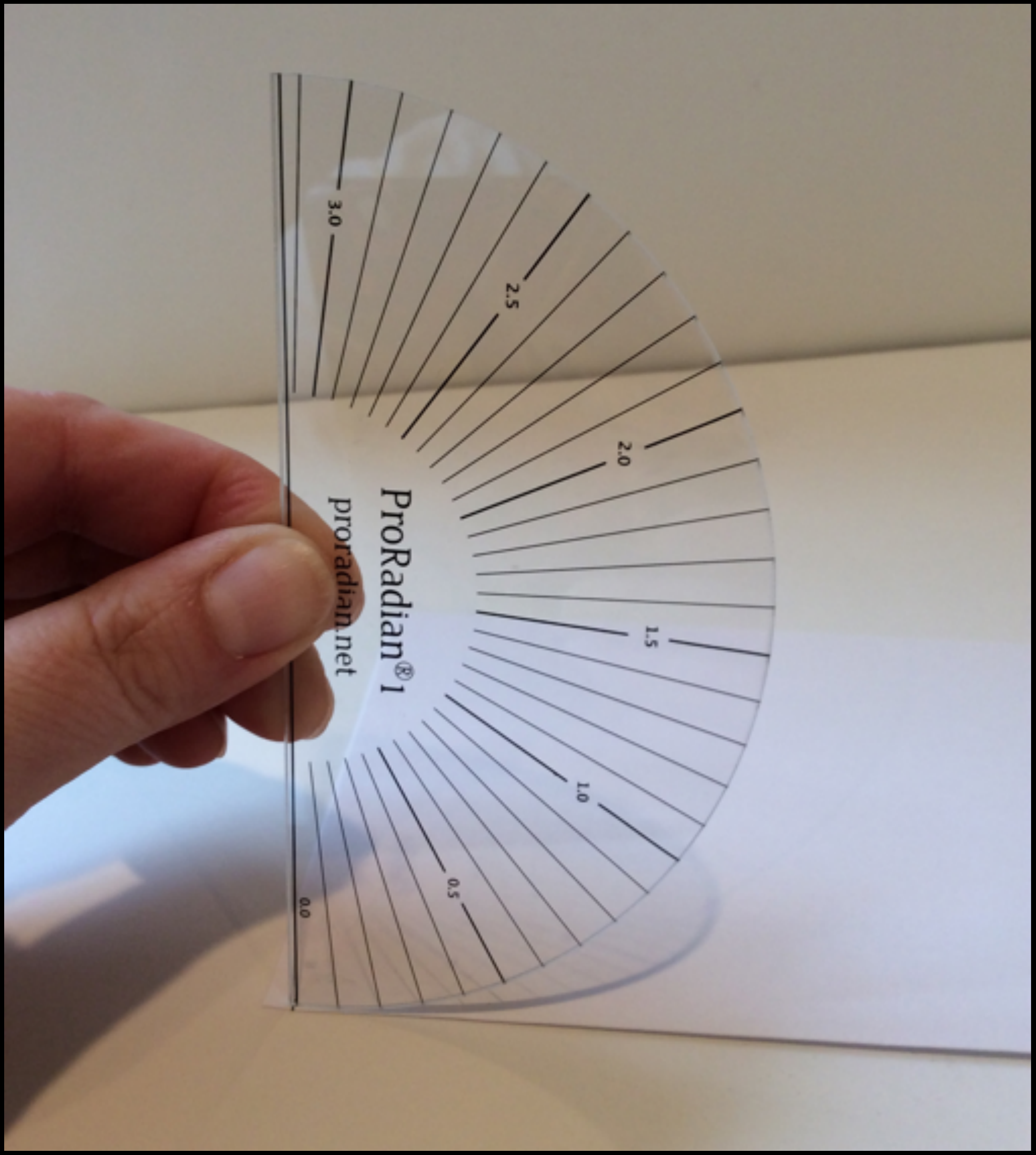
Make a Number Line

Take a plain piece of paper and fold it in half length-wise. Put it down with the fold facing you.

Make a small mark on the folded edge, near the left open edge.

Make a Number Line

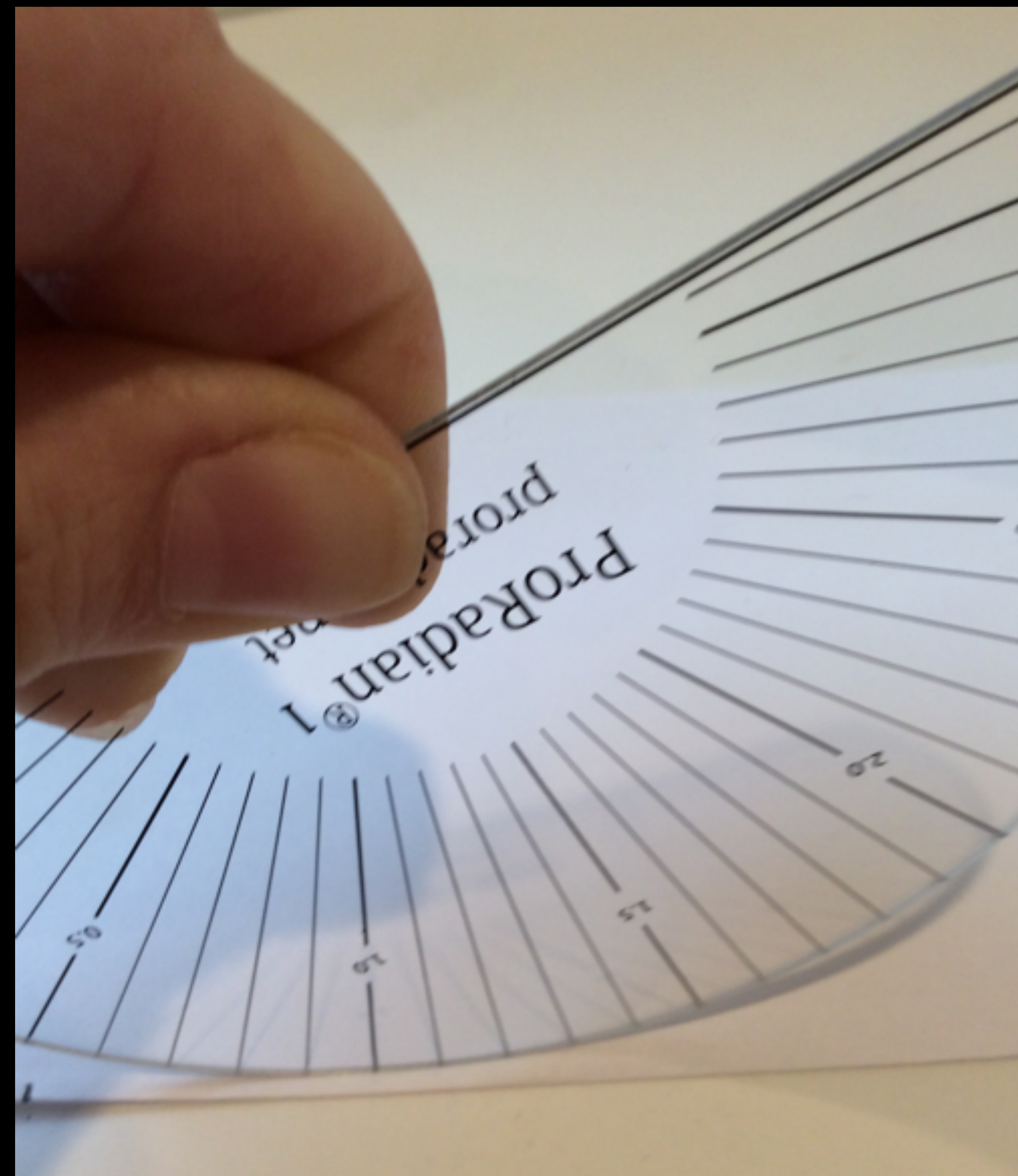
Hold ProRadian1 perpendicular to the table, with the 0.0 on the mark you made.



Make a Number Line

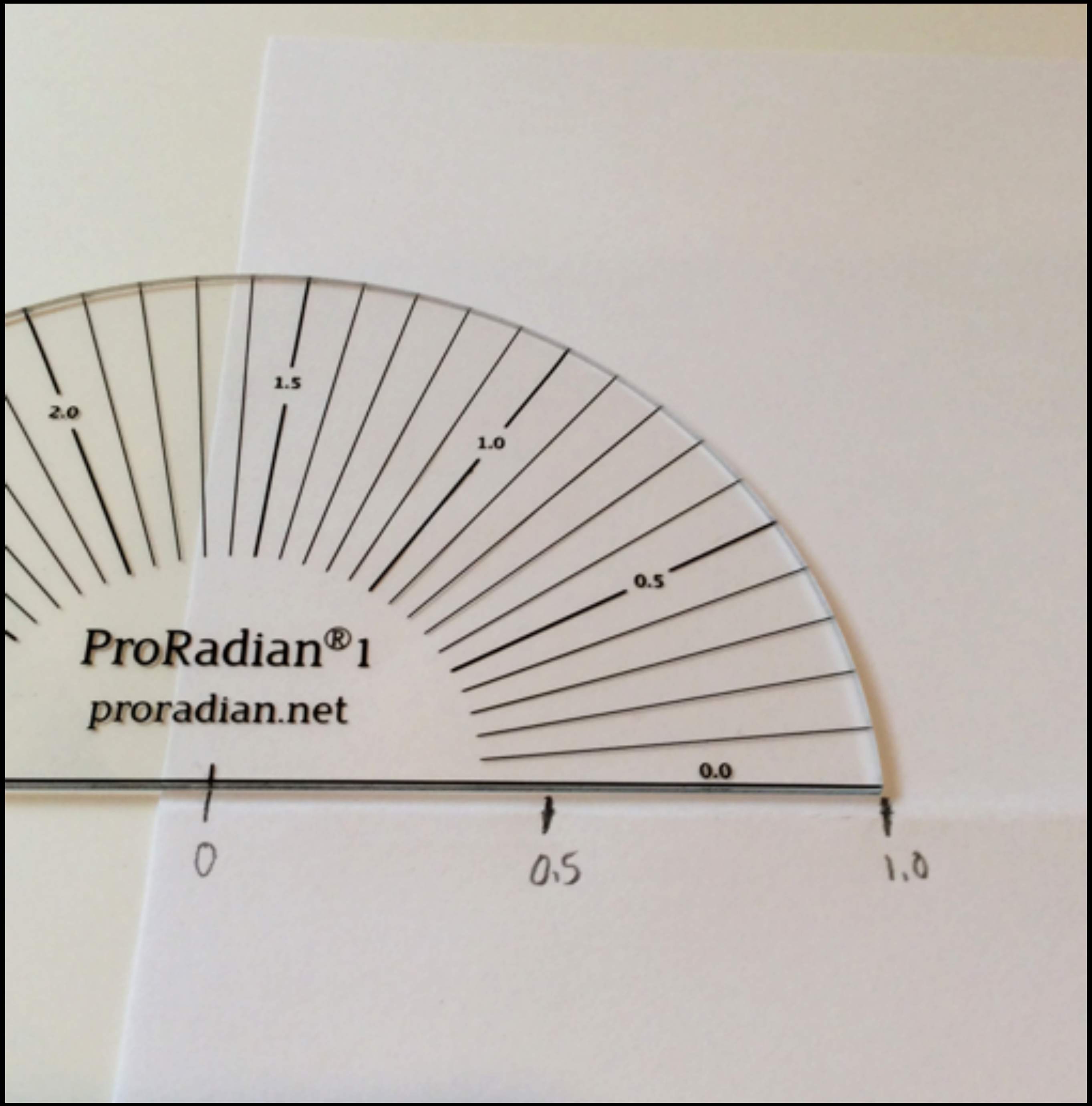
Hold ProRadian1 perpendicular to the table, with the 0.0 on the mark you made.

Roll ProRadian1 along the fold, marking every half radian.



Make a Number Line

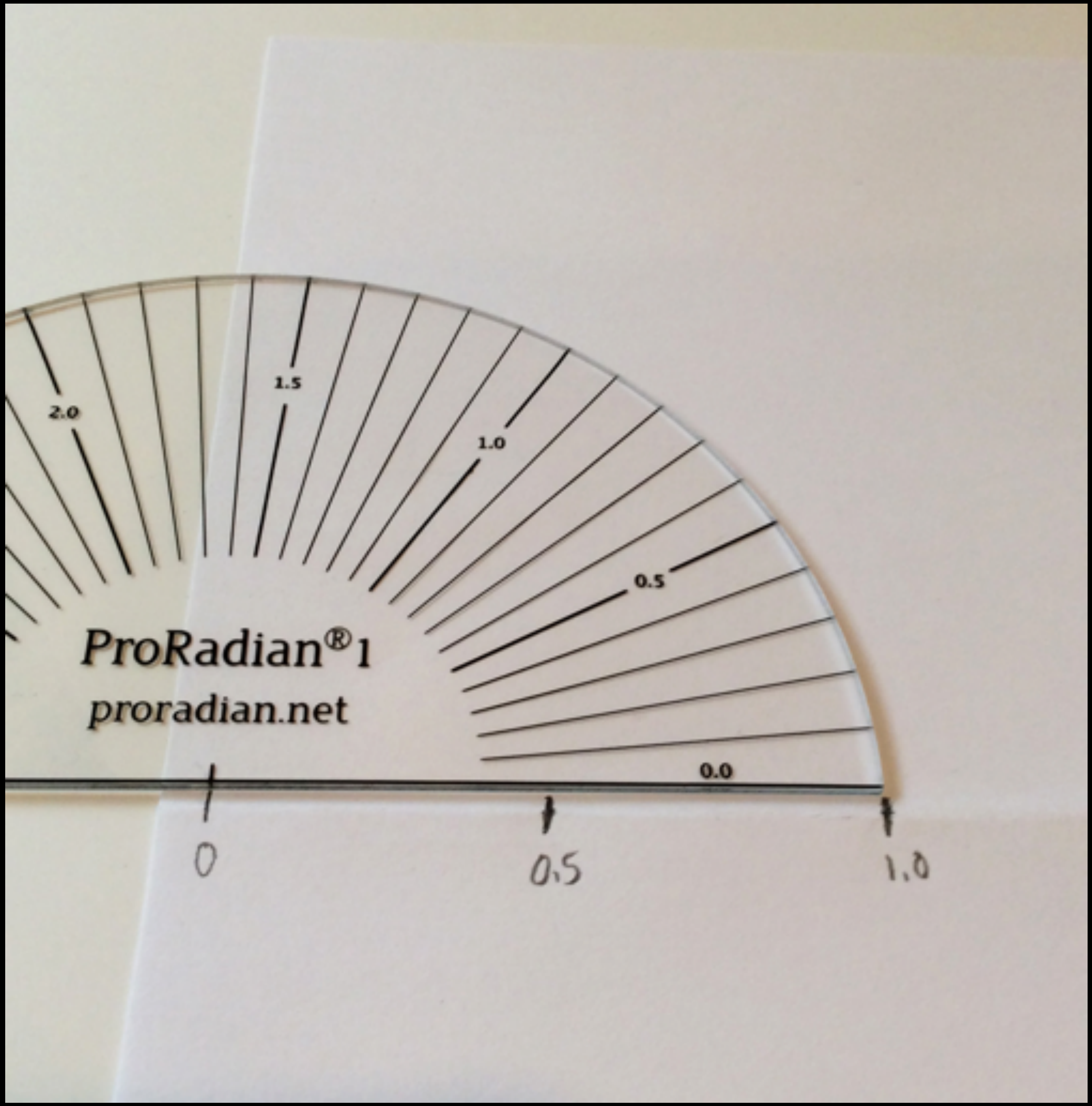
Unfold the paper and consider the number line you've created. What's special about it?



Make a Number Line

Unfold the paper and consider the number line you've created. What's special about it?

Now roll ProRadian2 along the same fold, marking every π/6. What's special about this number line?



Radian Measure & Proportion

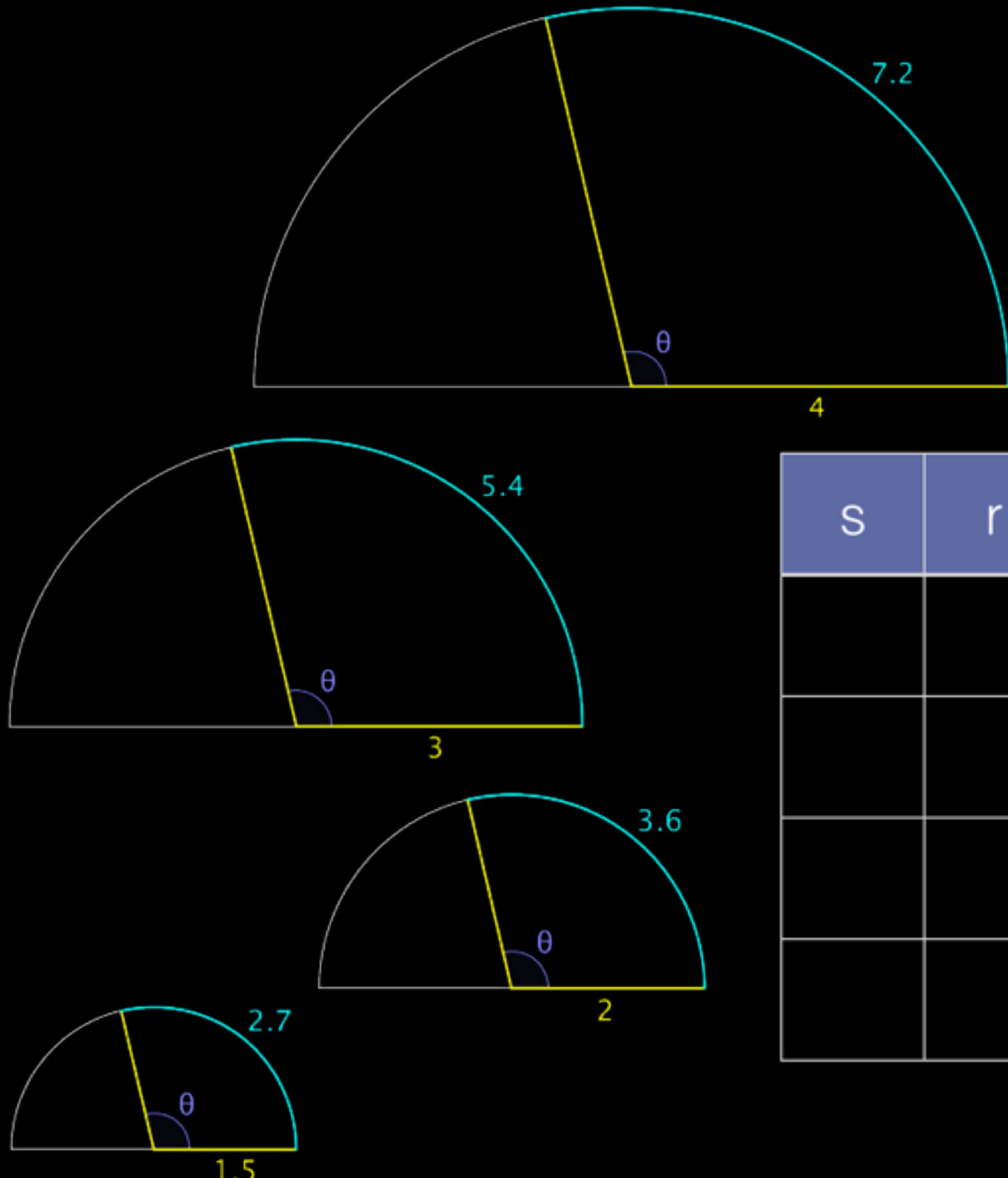
CCSS.MATH.CONTENT.HSG-C.B.5

Derive using similarity the fact that the length of the arc intercepted by an angle is proportional to the radius, and define the radian measure of the angle as the constant of proportionality; derive the formula for the area of a sector.

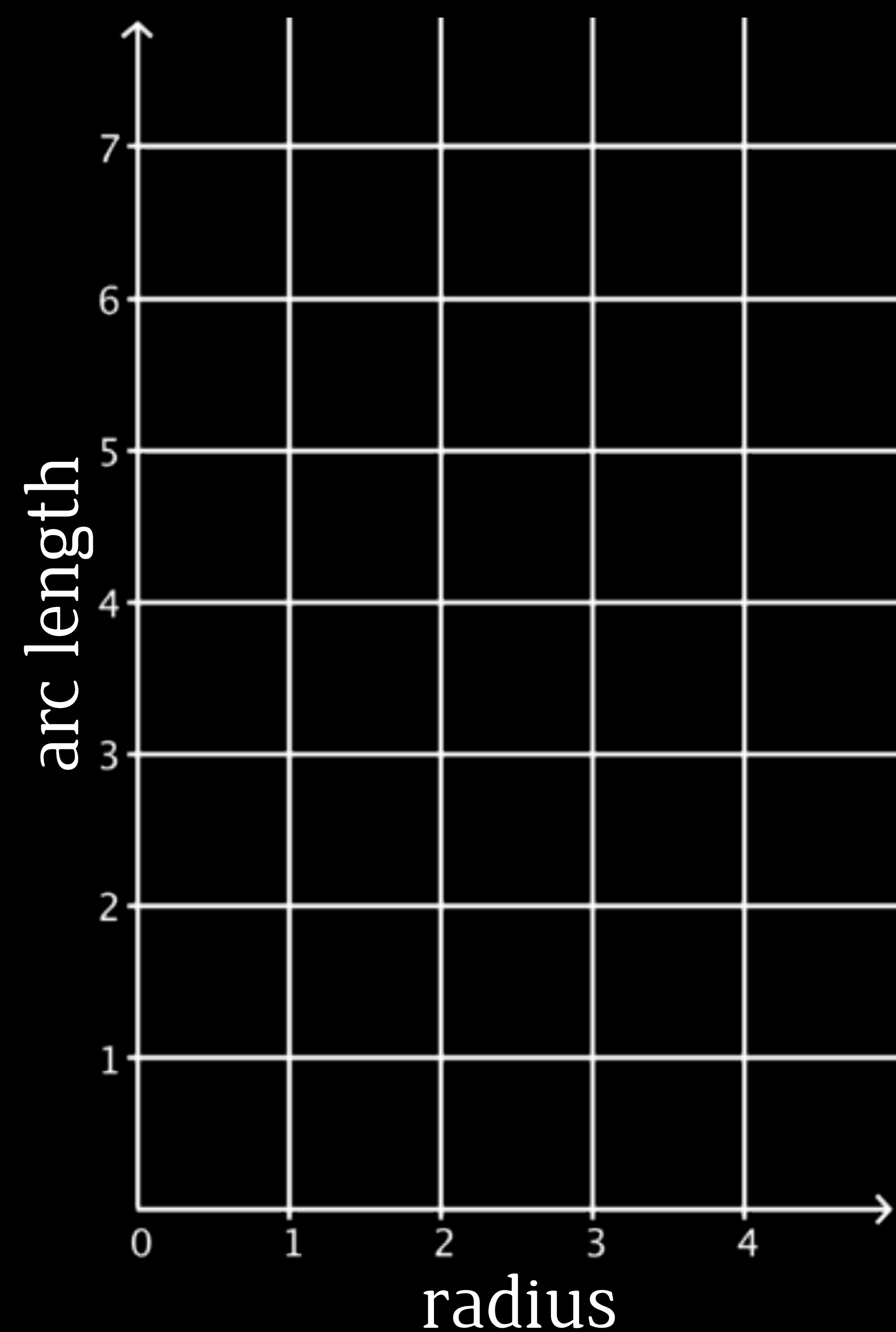
Radian Measure & Proportion

the length of the arc intercepted by an angle is proportional to the radius

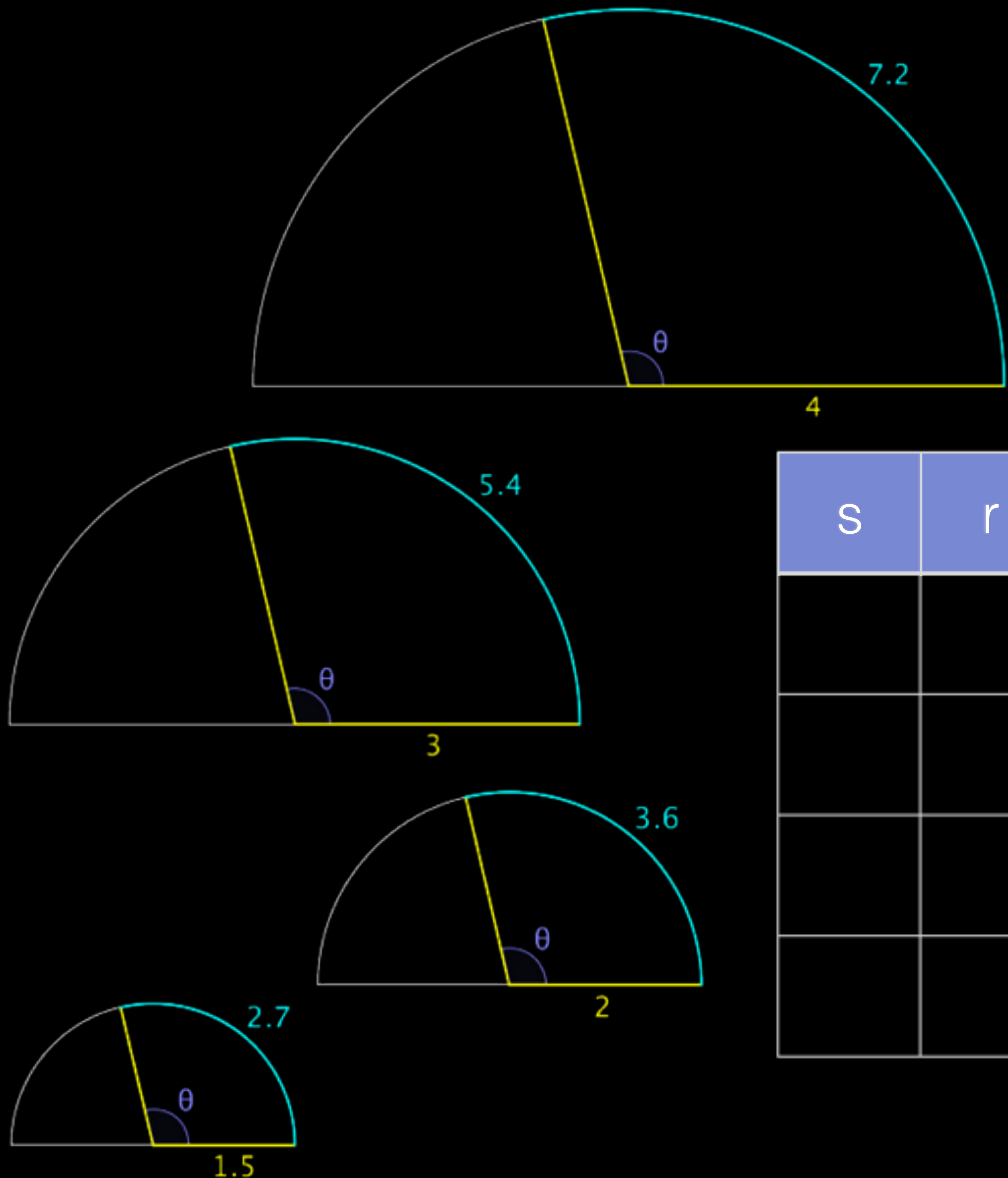
Handout Activity 5



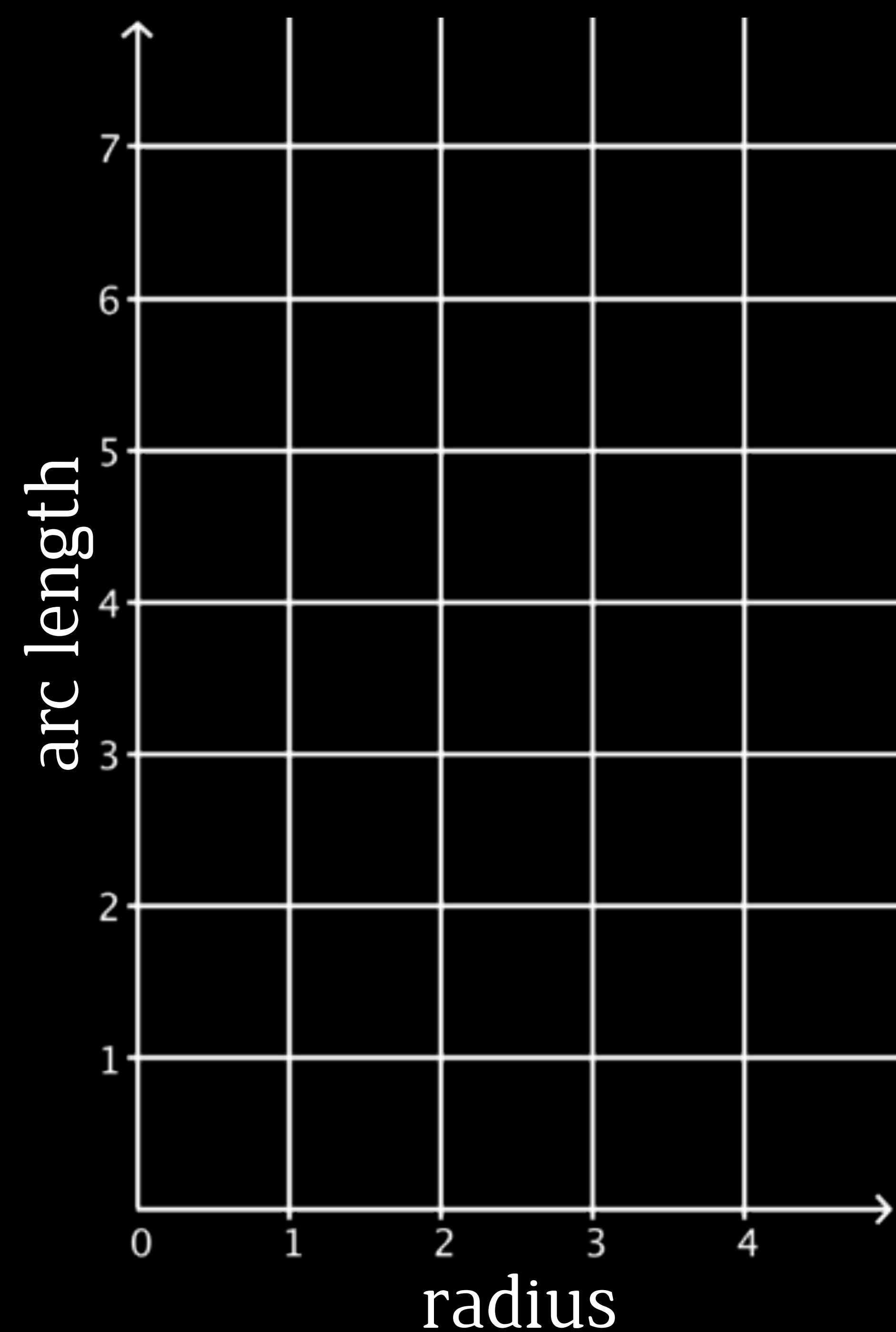
s	r	s/r



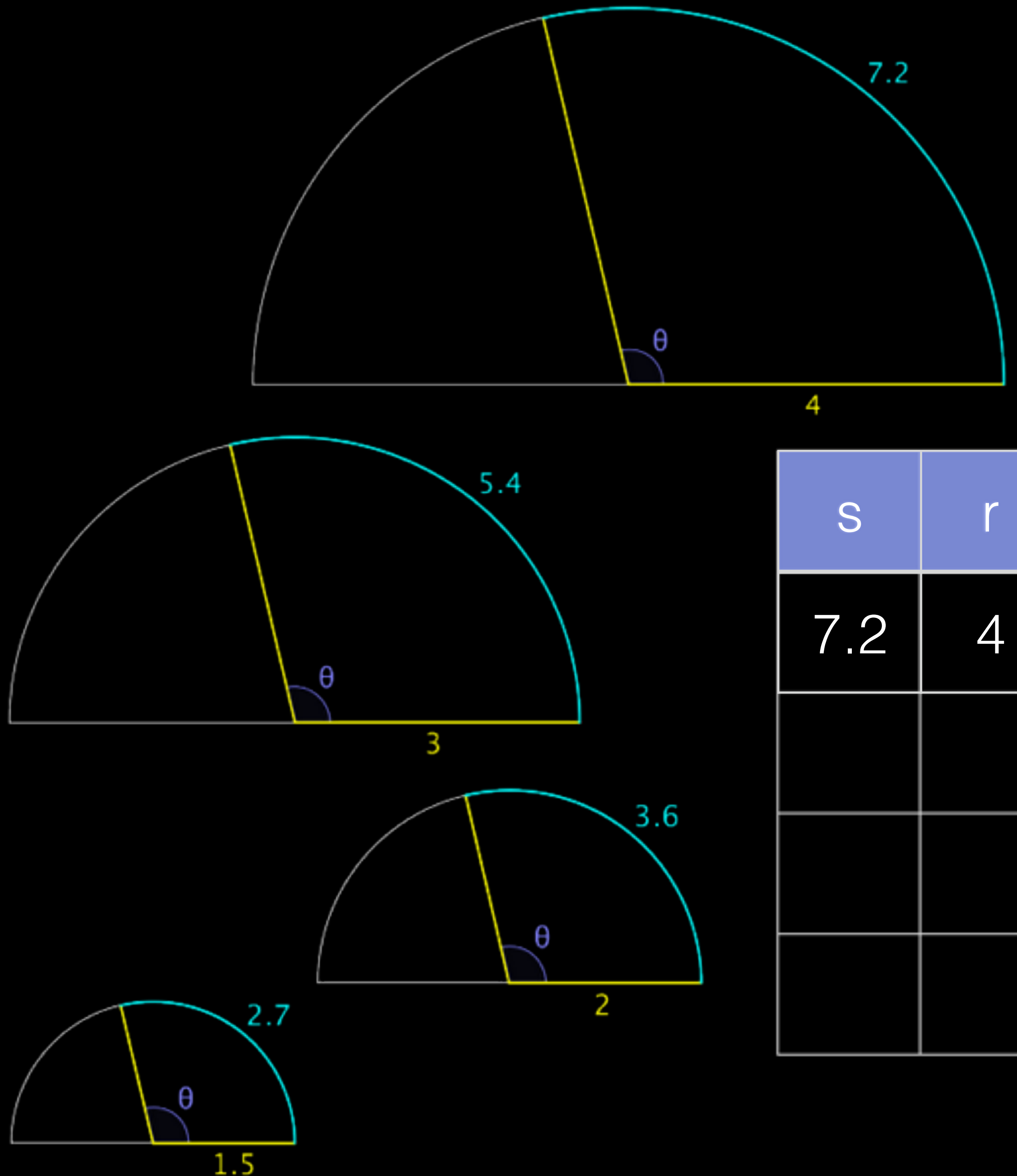
Use the diagrams to complete the table & graph.



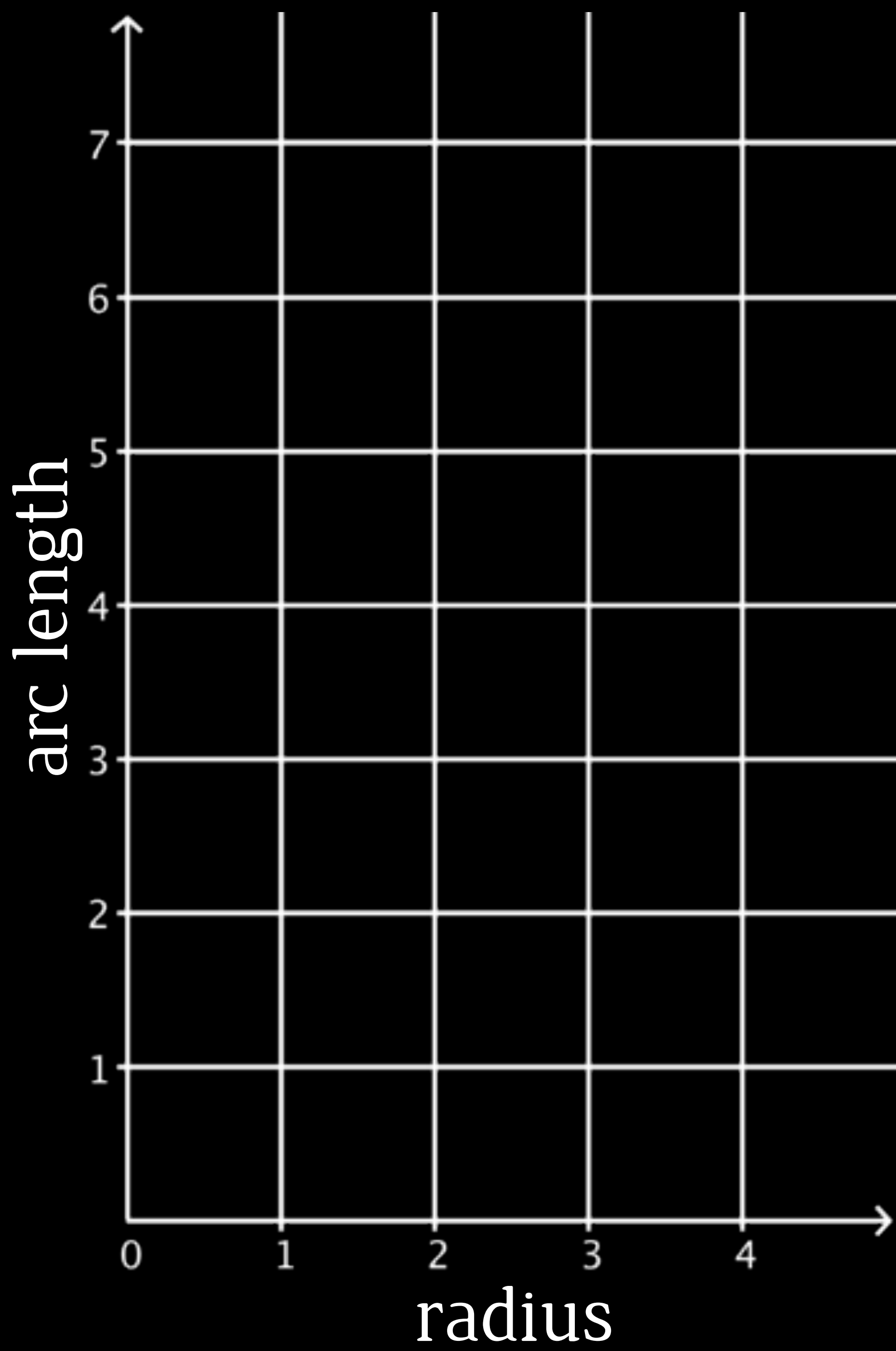
s	r	s/r



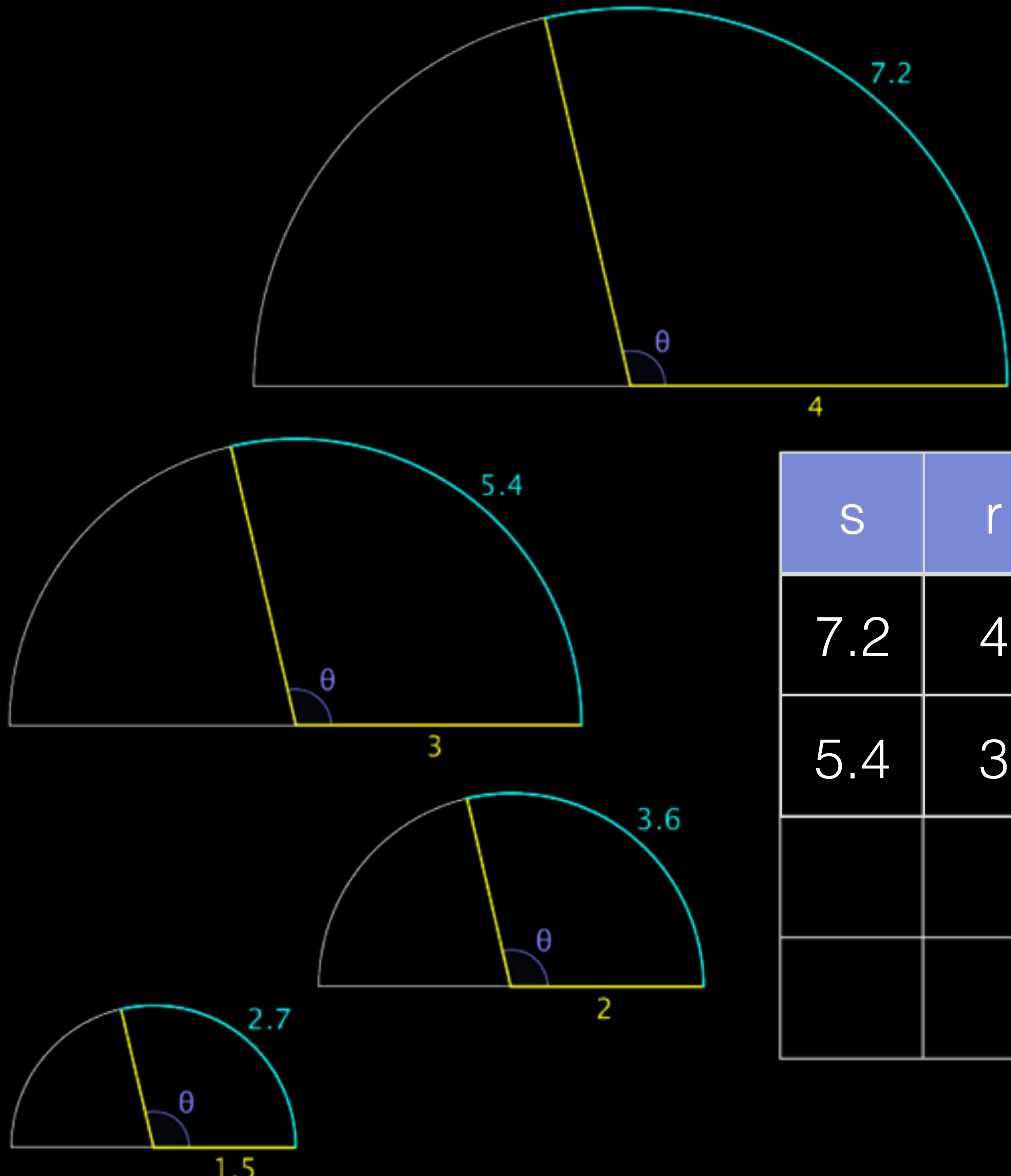
Use the diagrams to complete the table & graph.



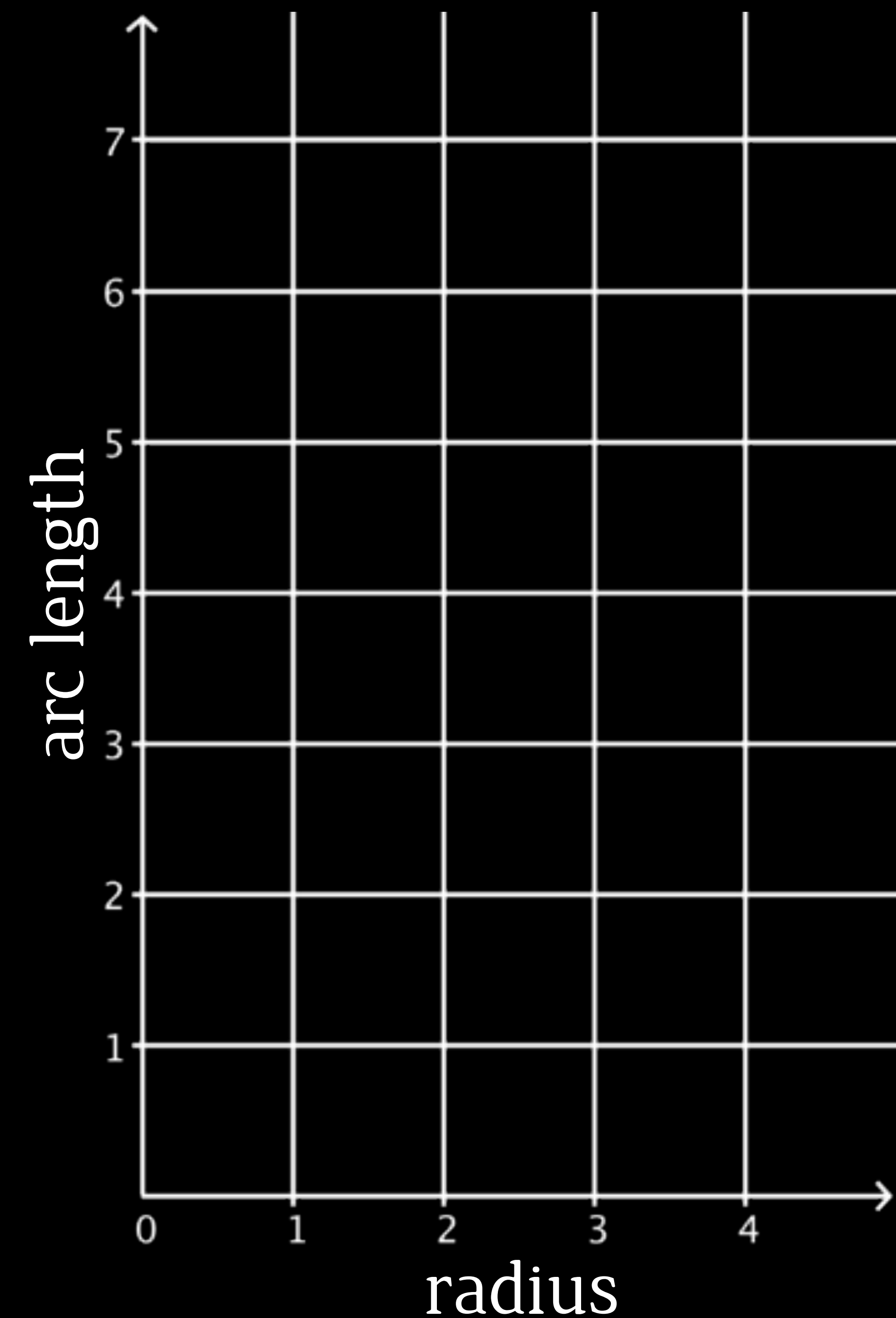
s	r	s/r
7.2	4	1.8



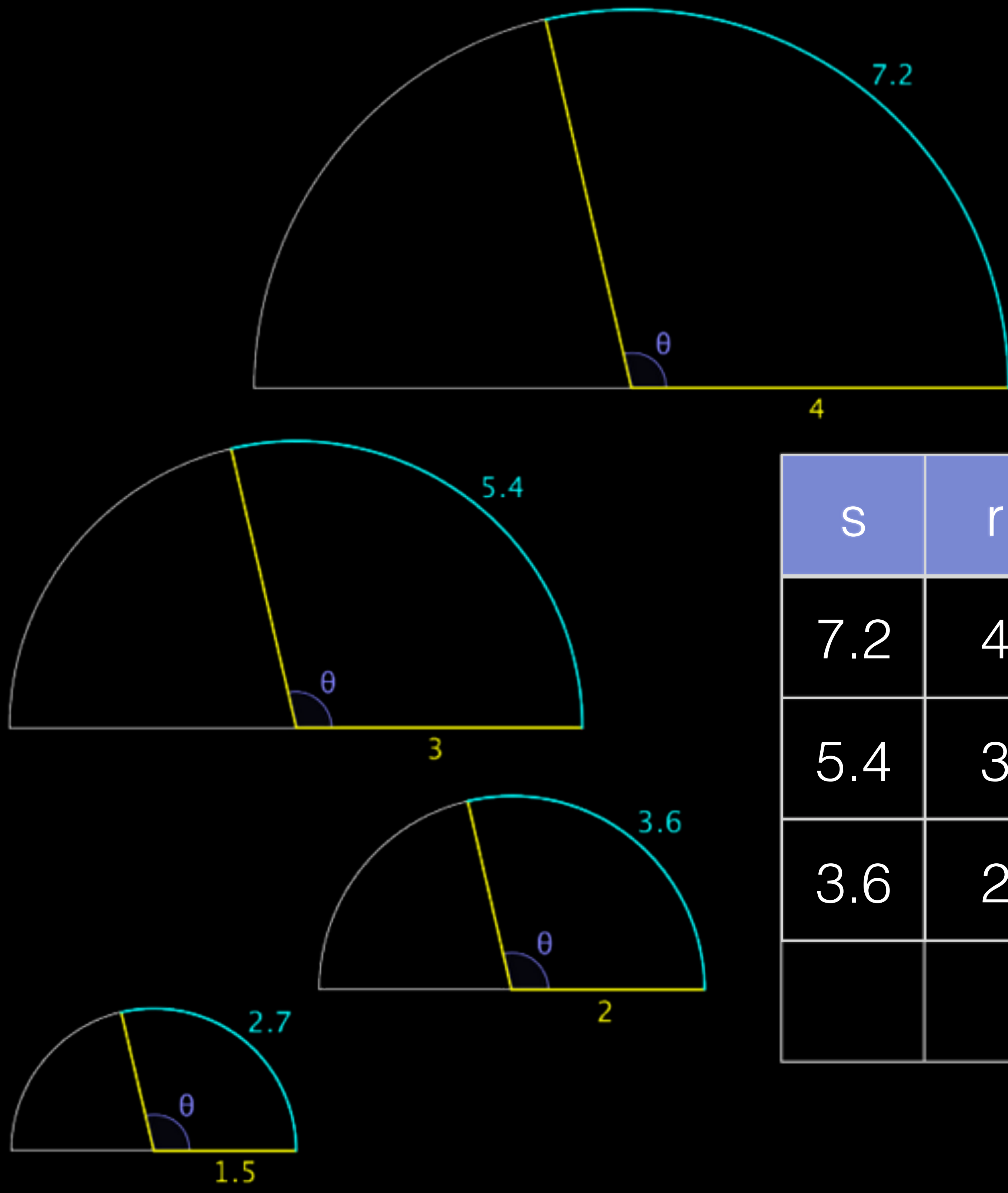
Use the diagrams to complete the table & graph.



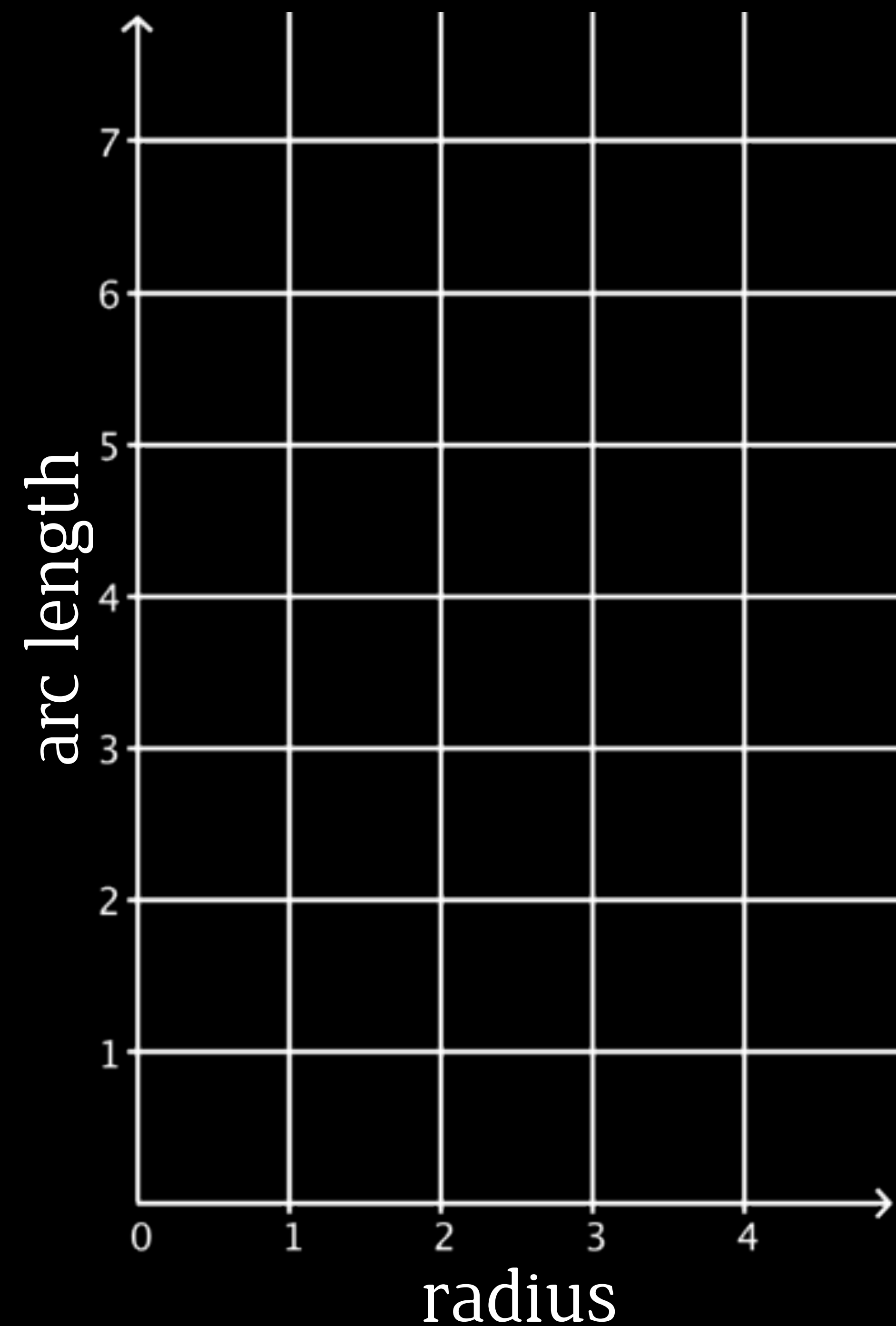
s	r	s/r
7.2	4	1.8
5.4	3	1.8



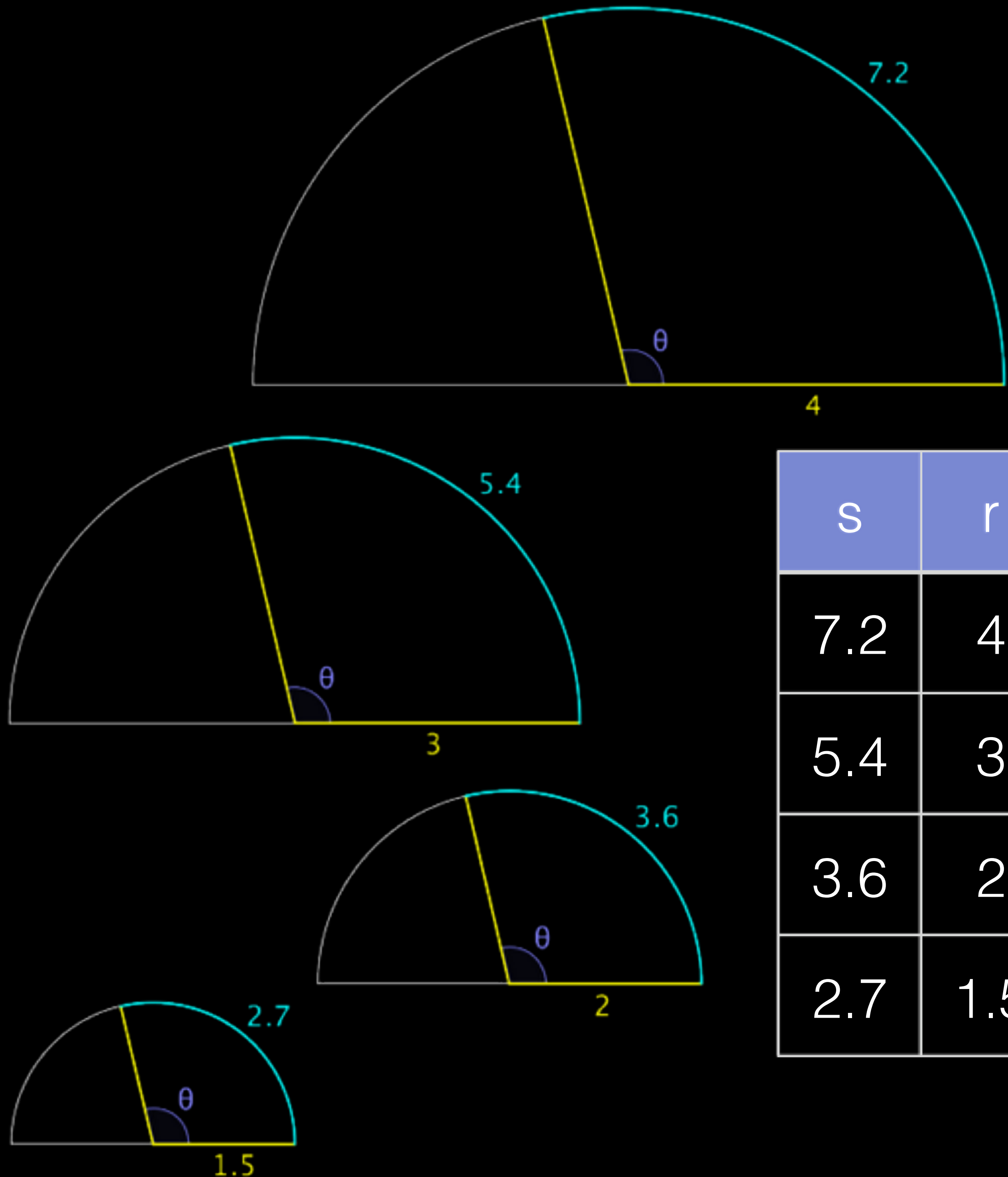
Use the diagrams to complete the table & graph.



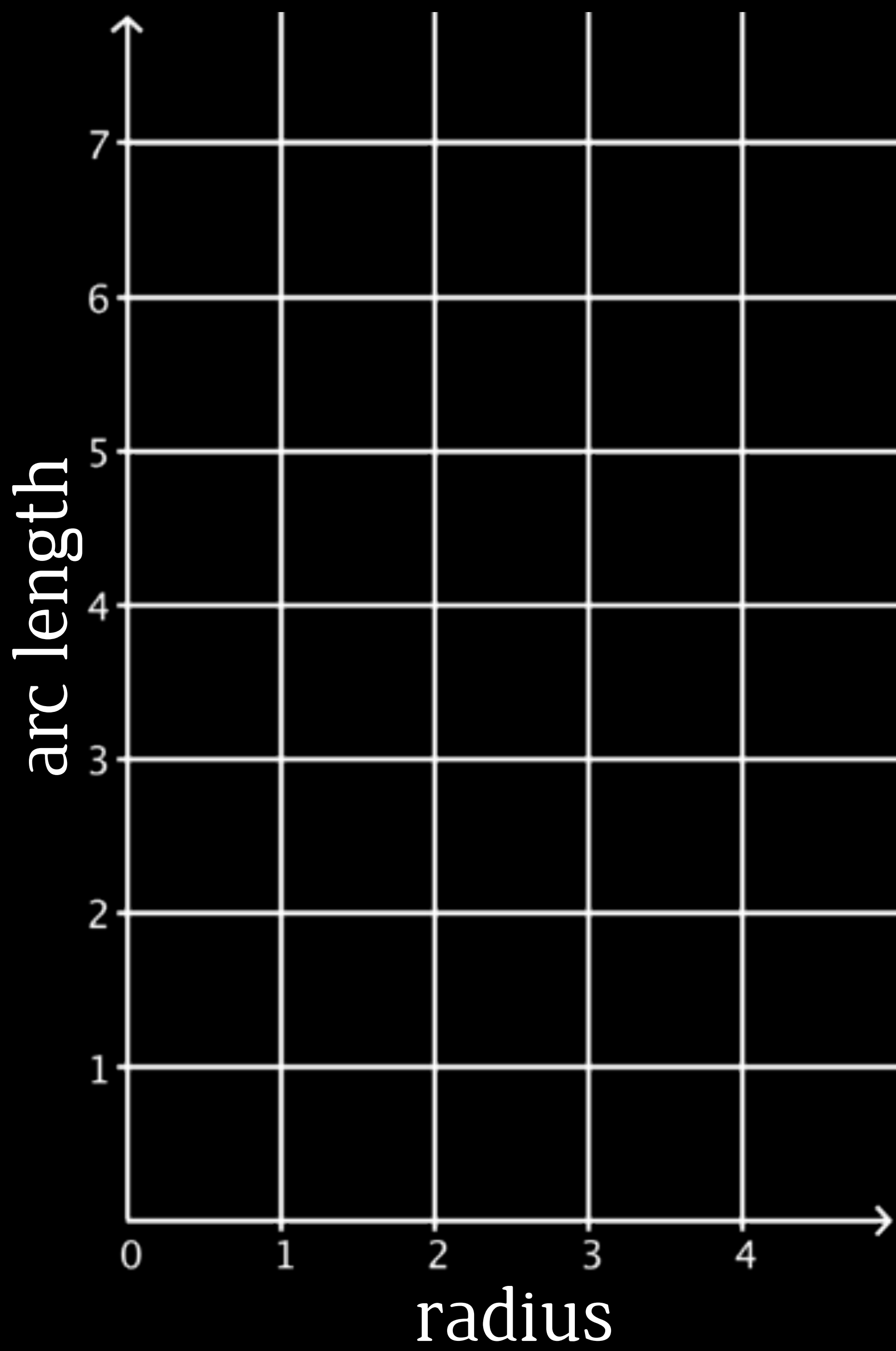
s	r	s/r
7.2	4	1.8
5.4	3	1.8
3.6	2	1.8



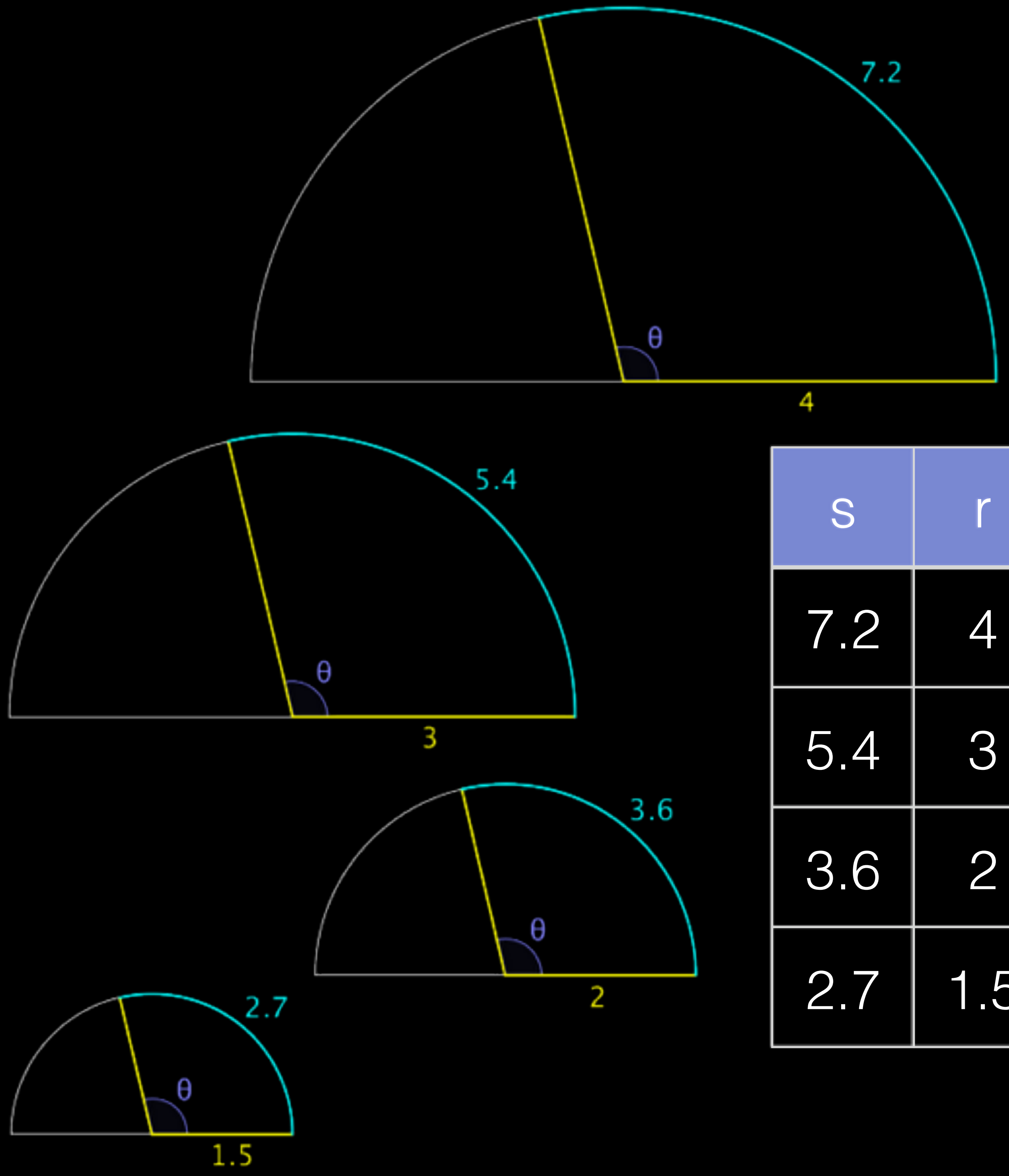
Use the diagrams to complete the table & graph.



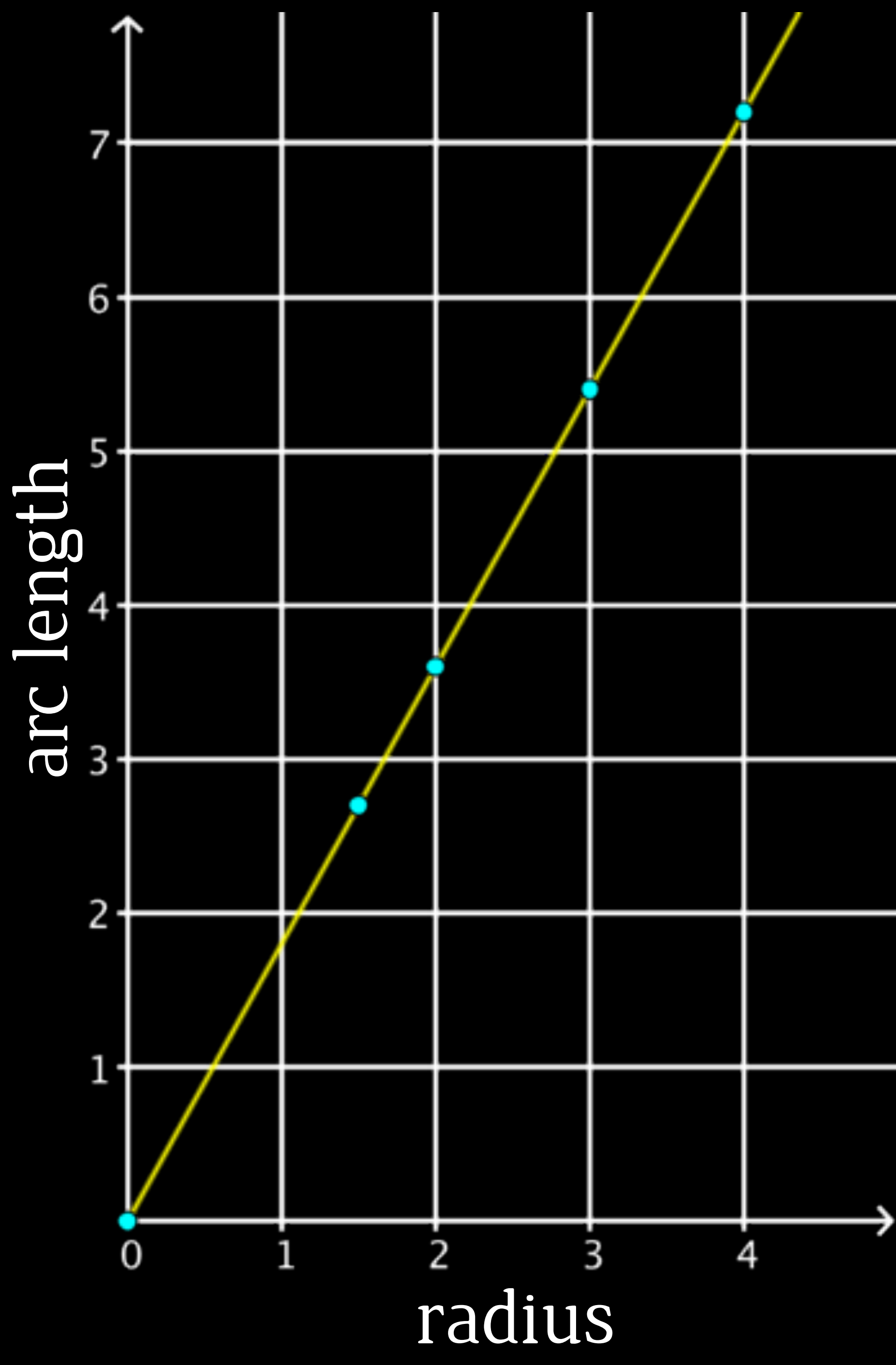
s	r	s/r
7.2	4	1.8
5.4	3	1.8
3.6	2	1.8
2.7	1.5	1.8



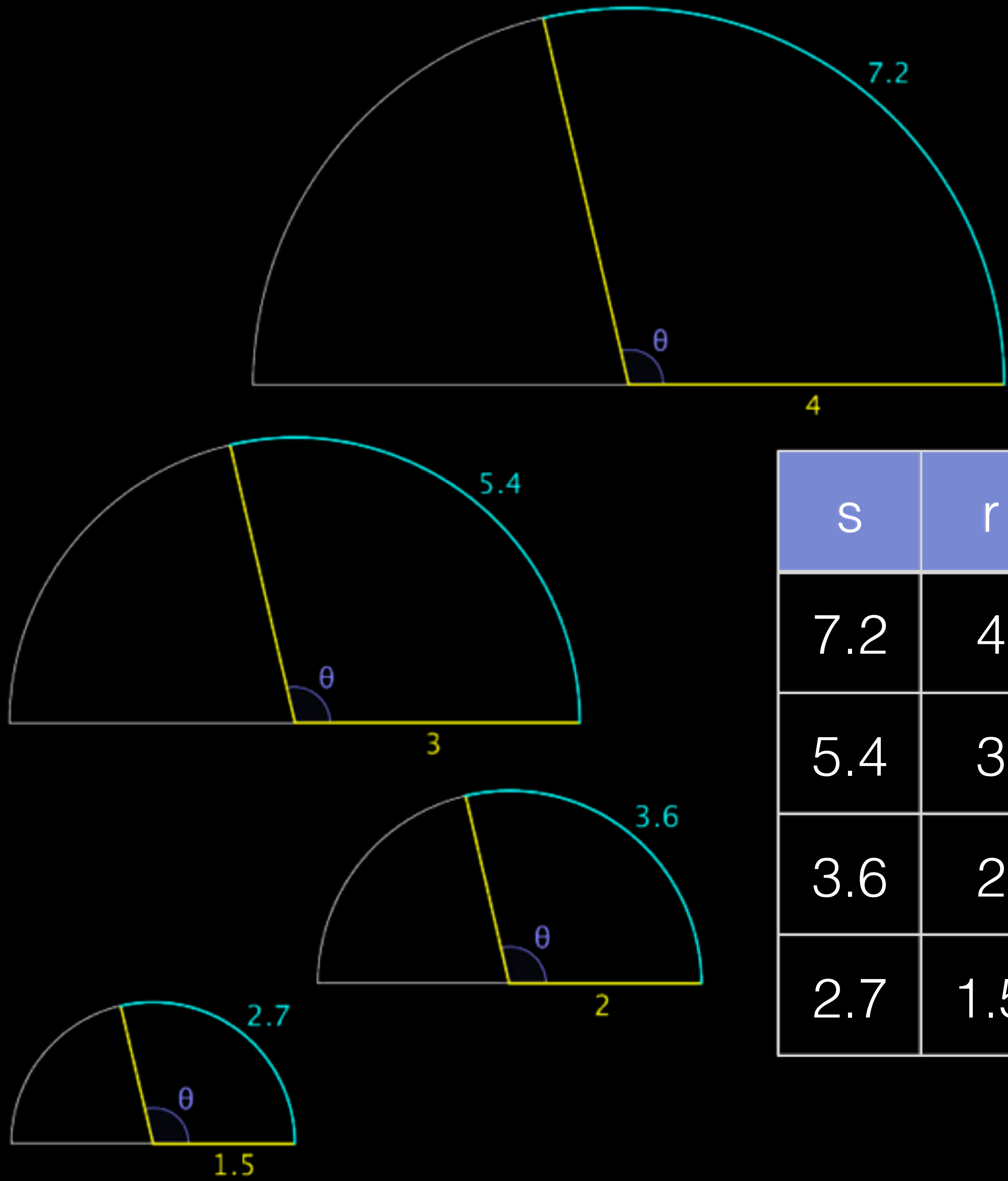
Use the diagrams to complete the table & graph.



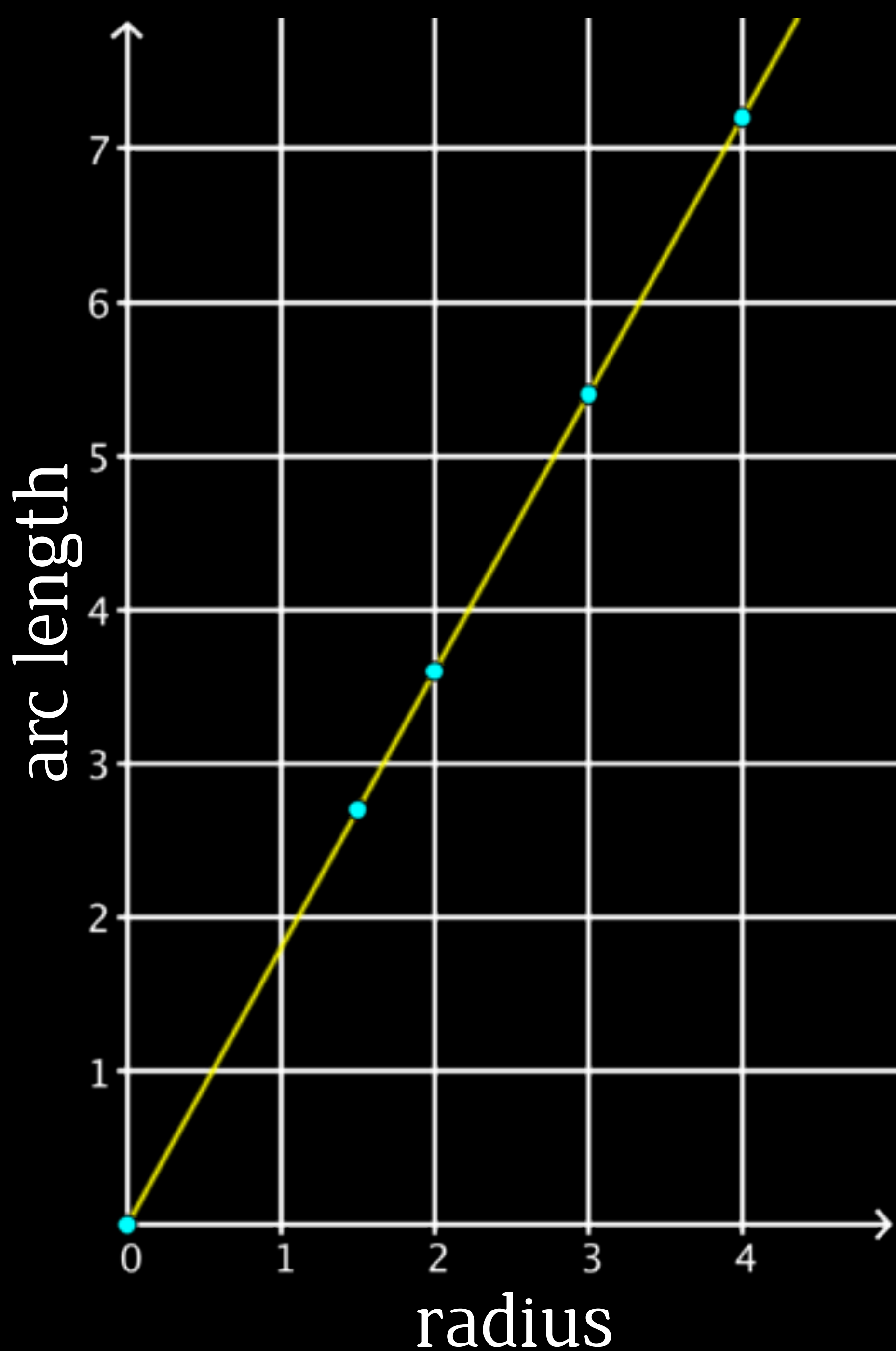
s	r	s/r
7.2	4	1.8
5.4	3	1.8
3.6	2	1.8
2.7	1.5	1.8



We see that the arc length varies directly as the radius and θ is the constant of proportionality!



s	r	s/r
7.2	4	1.8
5.4	3	1.8
3.6	2	1.8
2.7	1.5	1.8

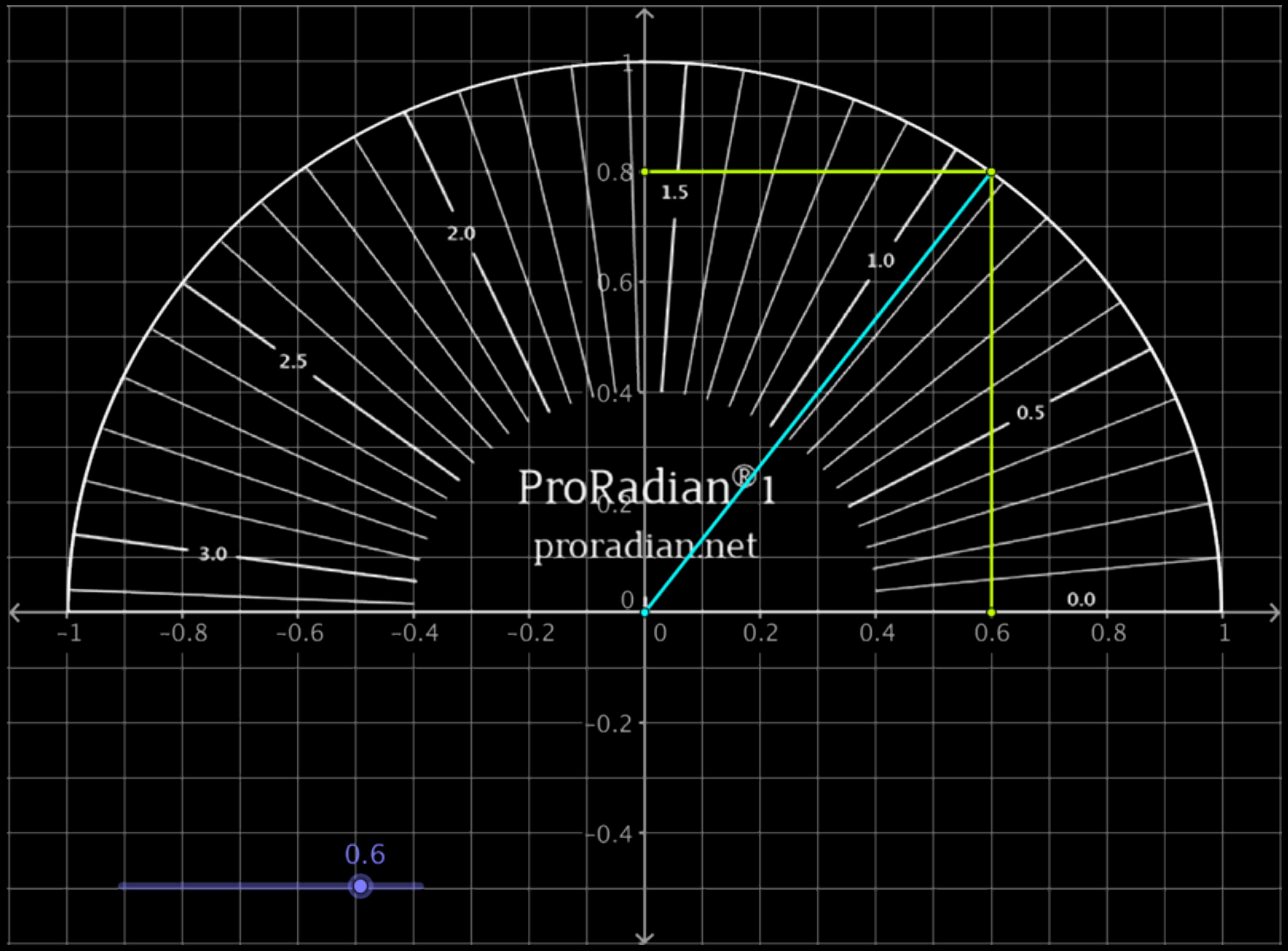


Handout Activity 6

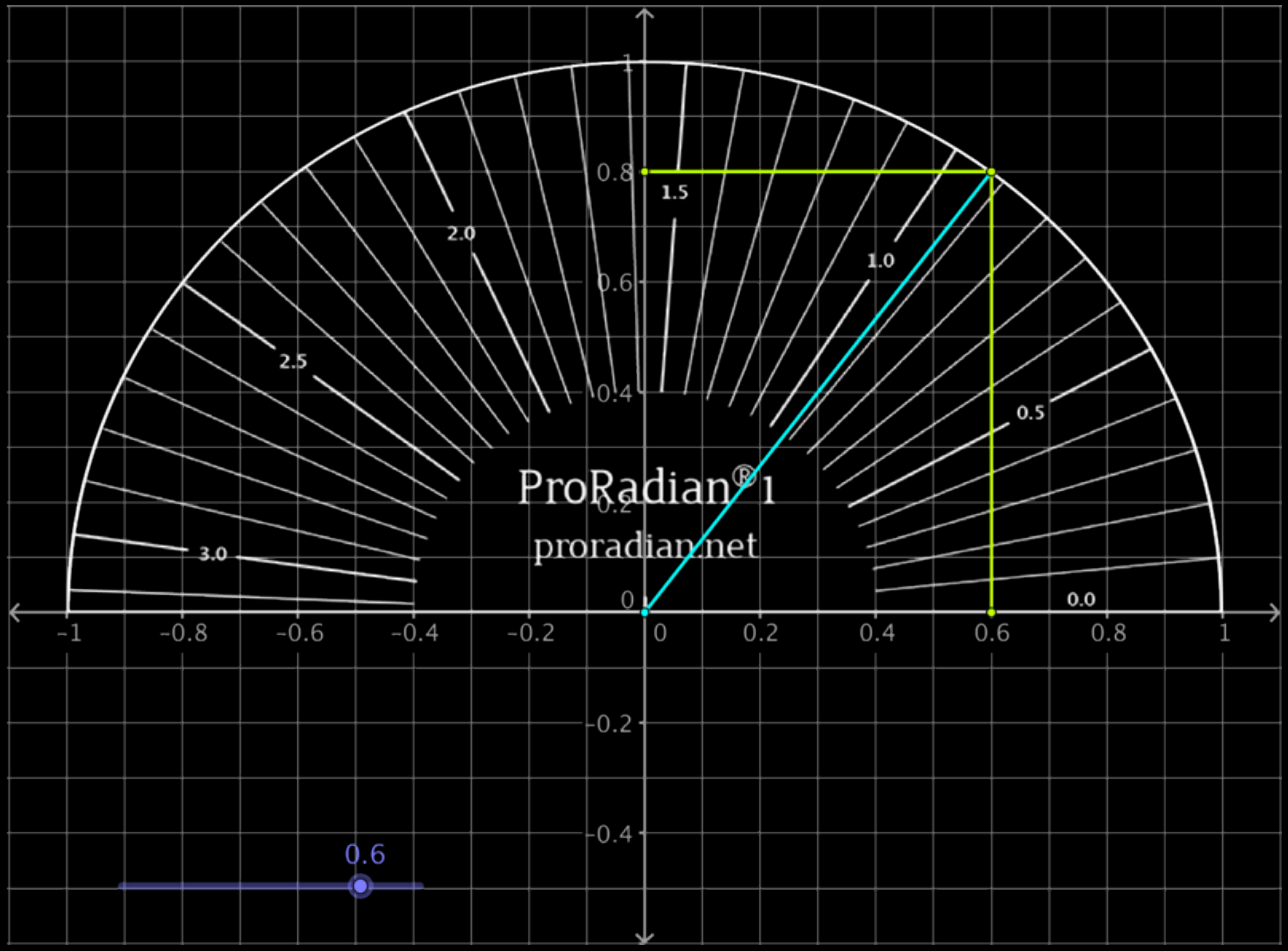
Inverse Trig Functions

Inverse Trig Functions

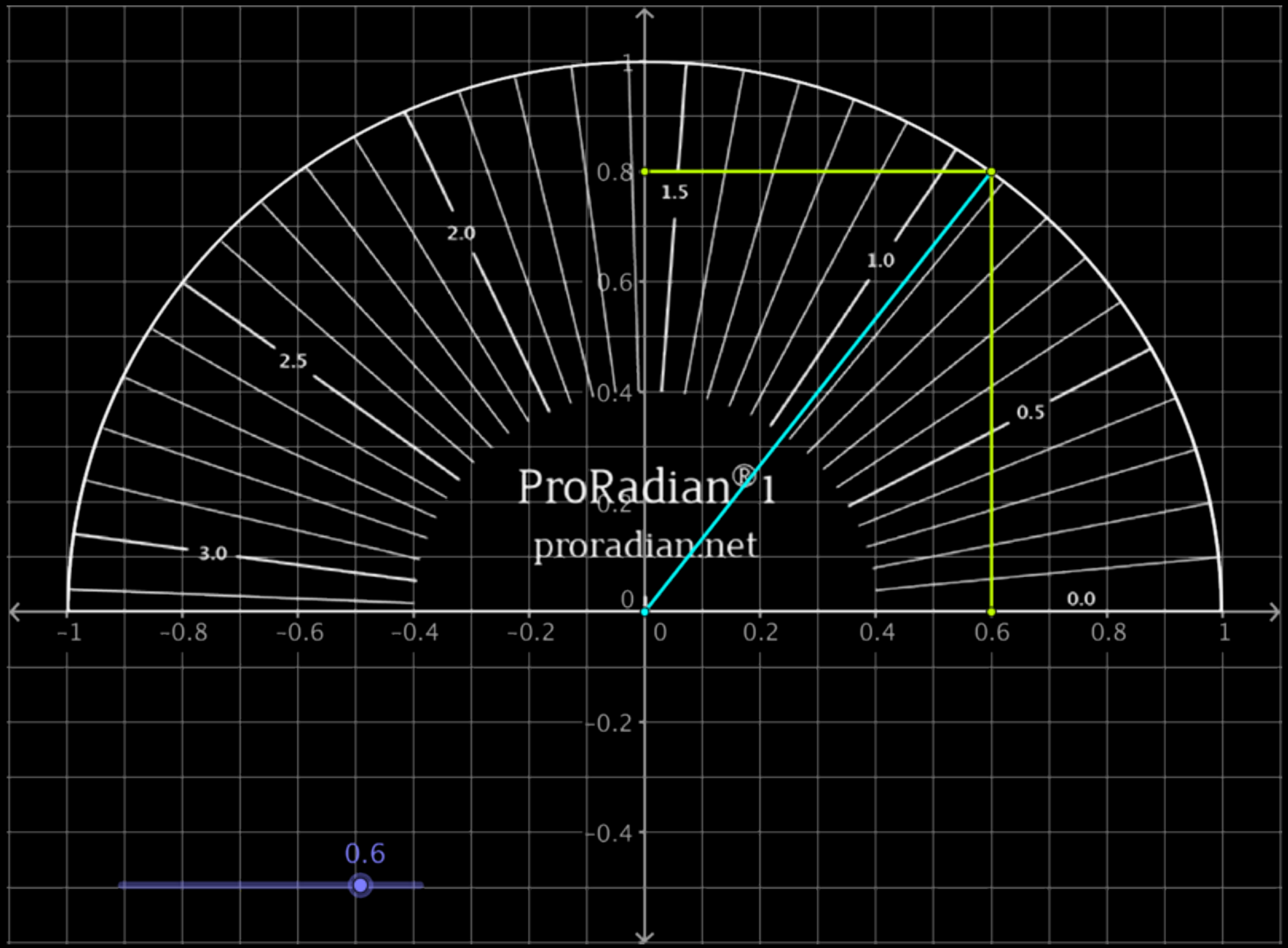
Place ProRadian1 on a Cartesian grid, where one unit is equal to the radius. Align the center with the origin, and you can "read" the inverse sine and cosine values.



ProRadian® 1
proradian.net
0.0
0.5
1.0
1.5
2.0
2.5
3.0
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
-0.2
-0.4
0.6



$$\cos^{-1}(0.6) \approx 0.93$$



$$\sin^{-1}(0.8) \approx 0.93$$

Radian Measure & Trig

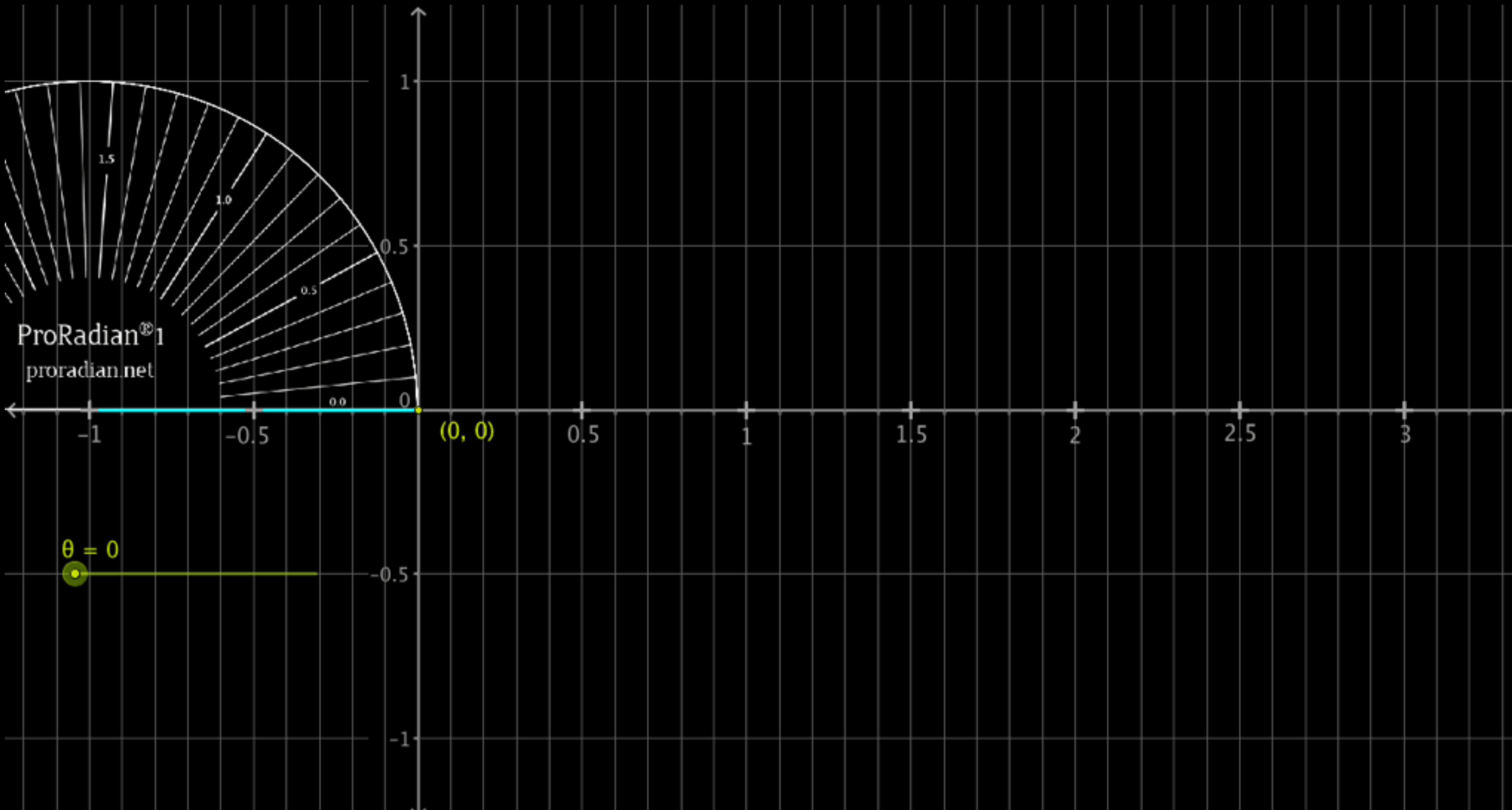
CCSS.MATH.CONTENT.HSF-TF.A.2

Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.

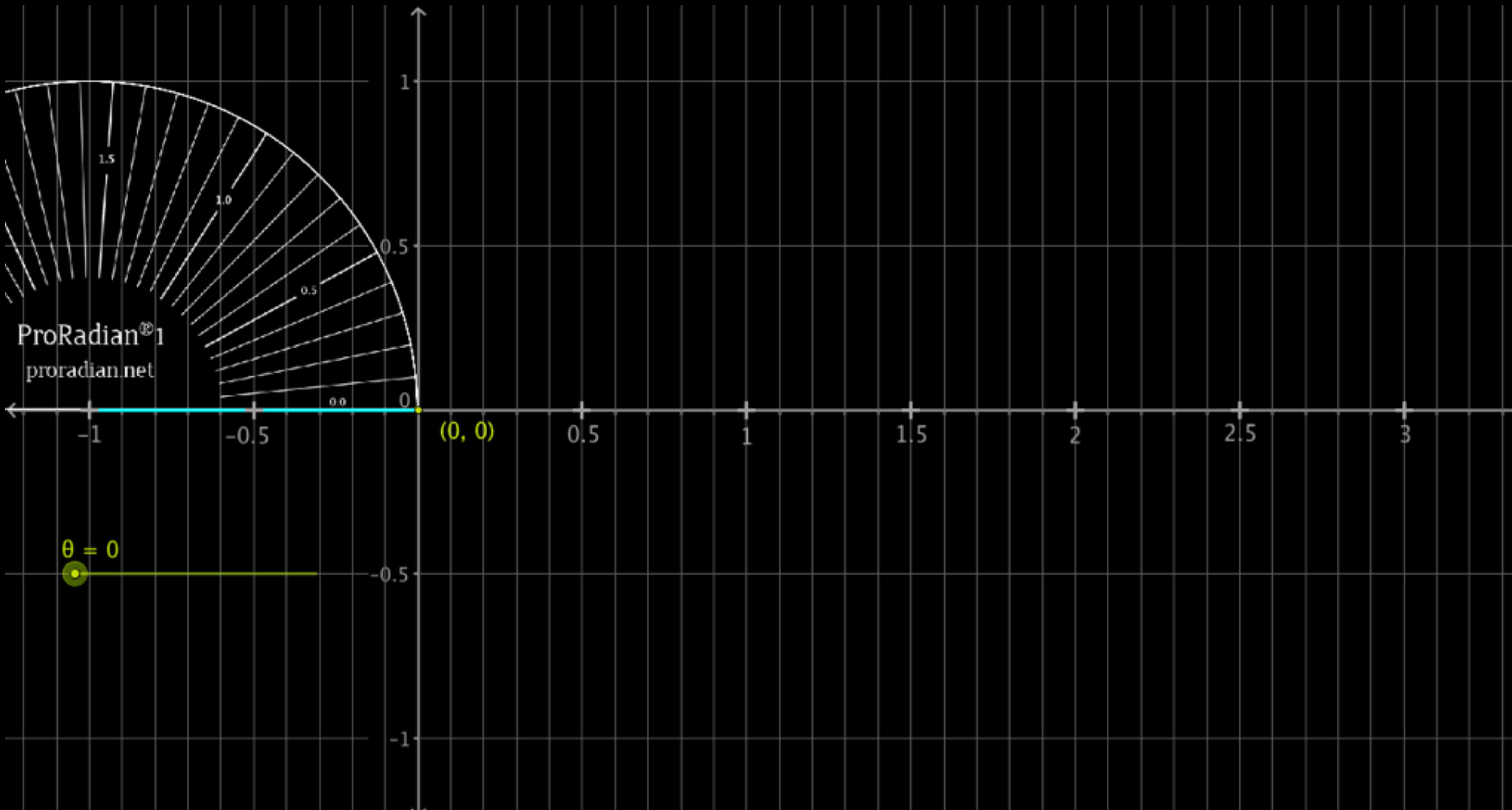
Radian Measure & Trig

extension of trigonometric functions to all real numbers

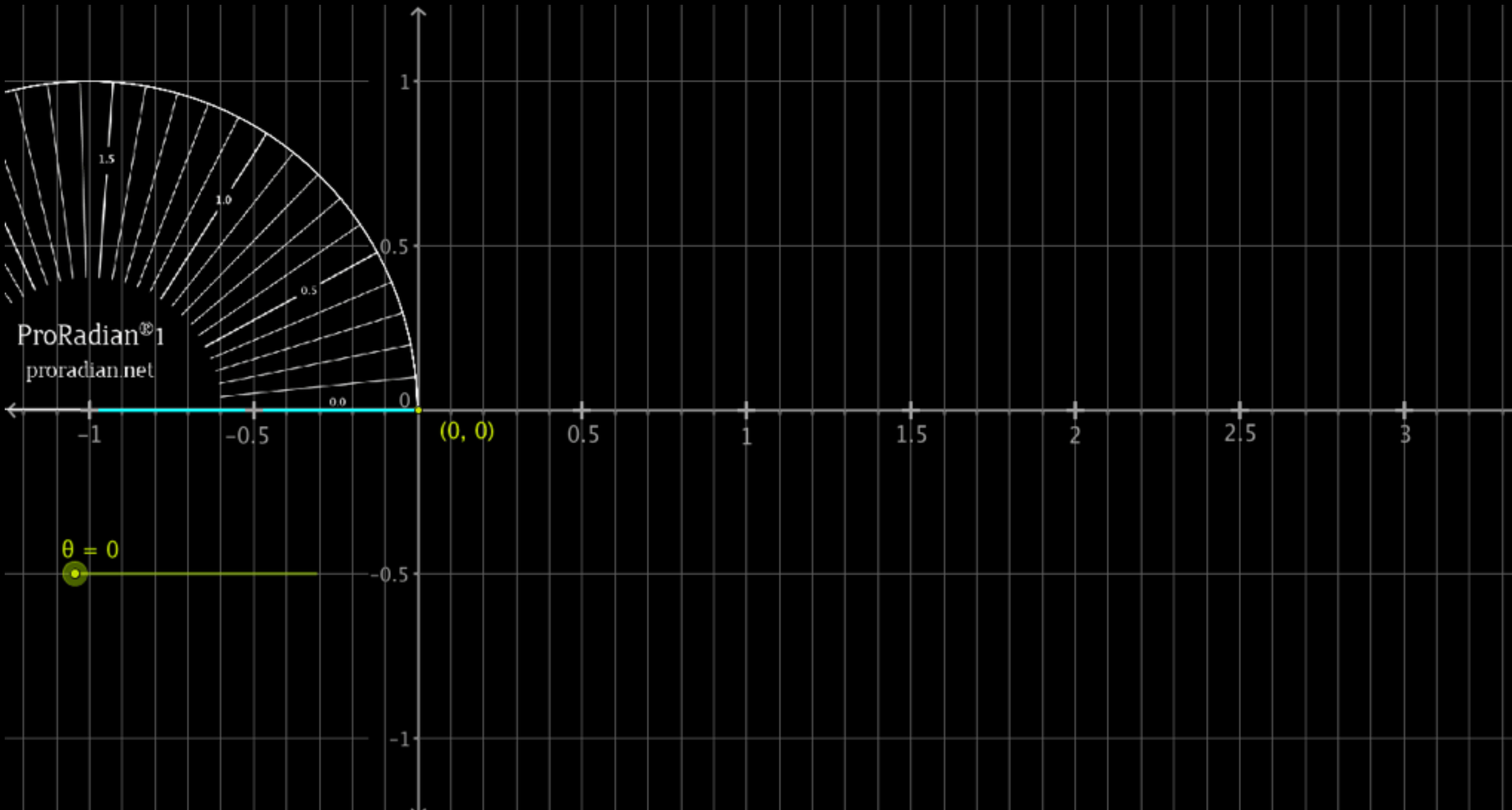
Handout Activity 7



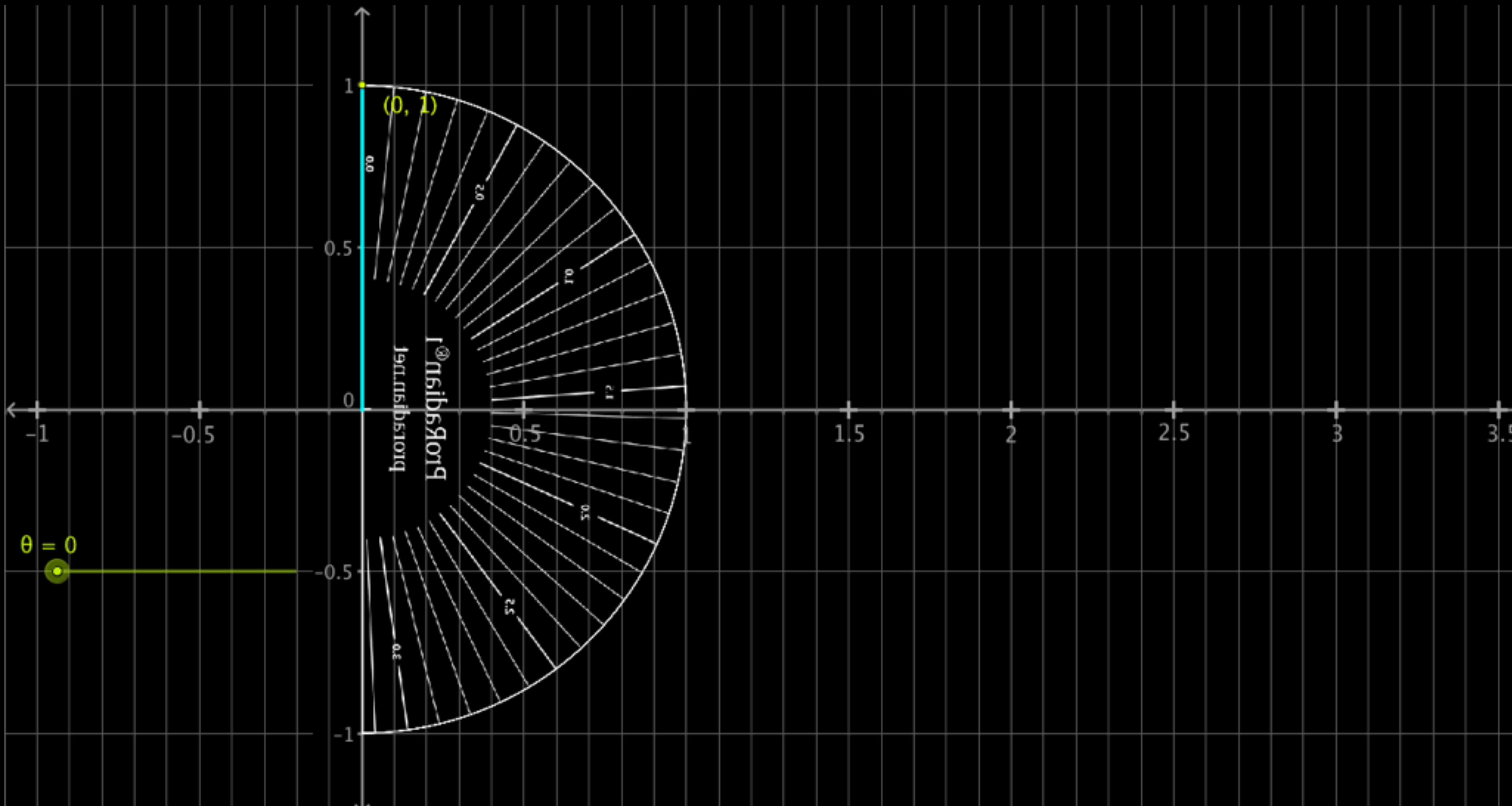
Record the “Verticals”



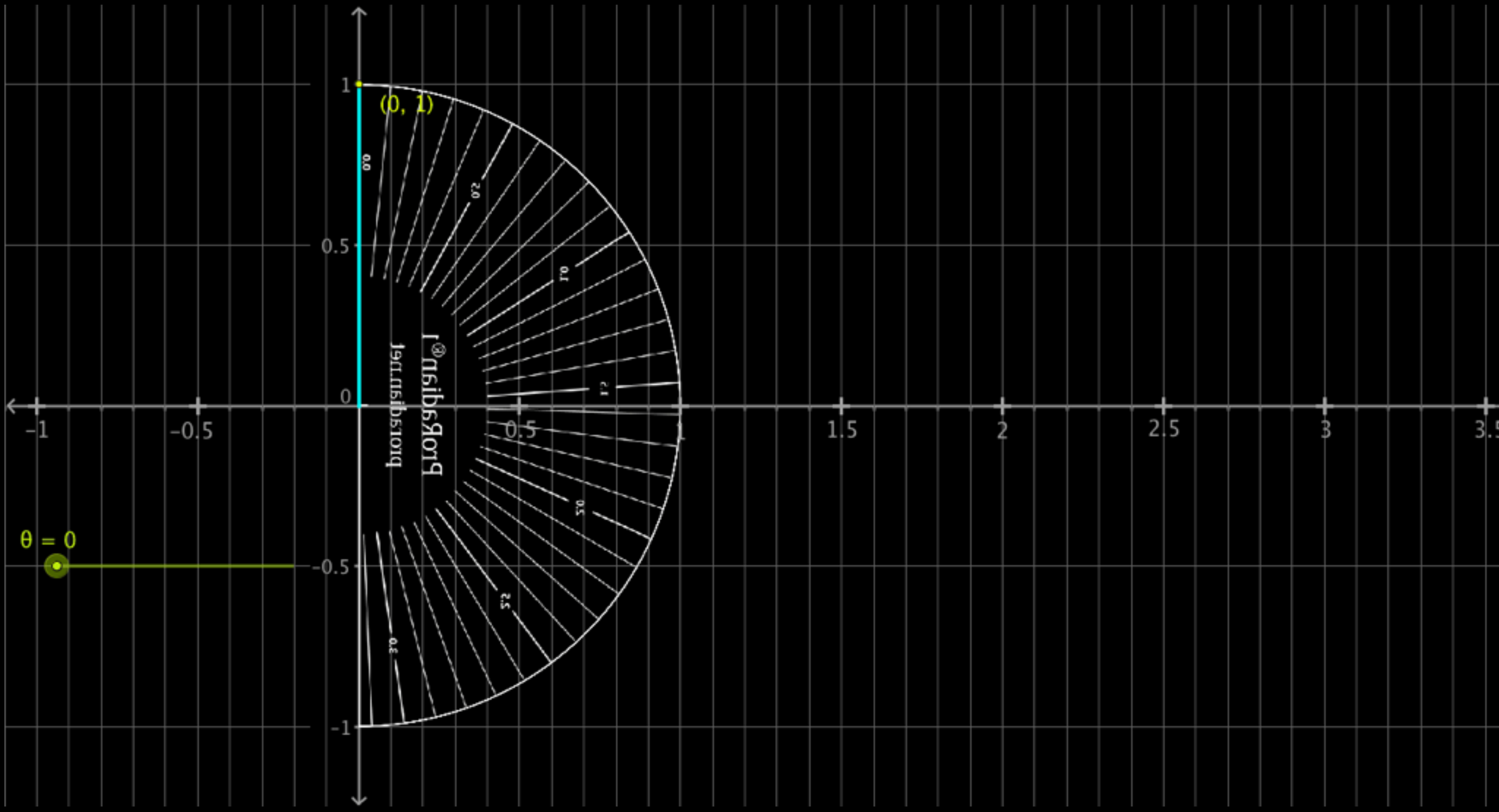
Record the "Verticals"



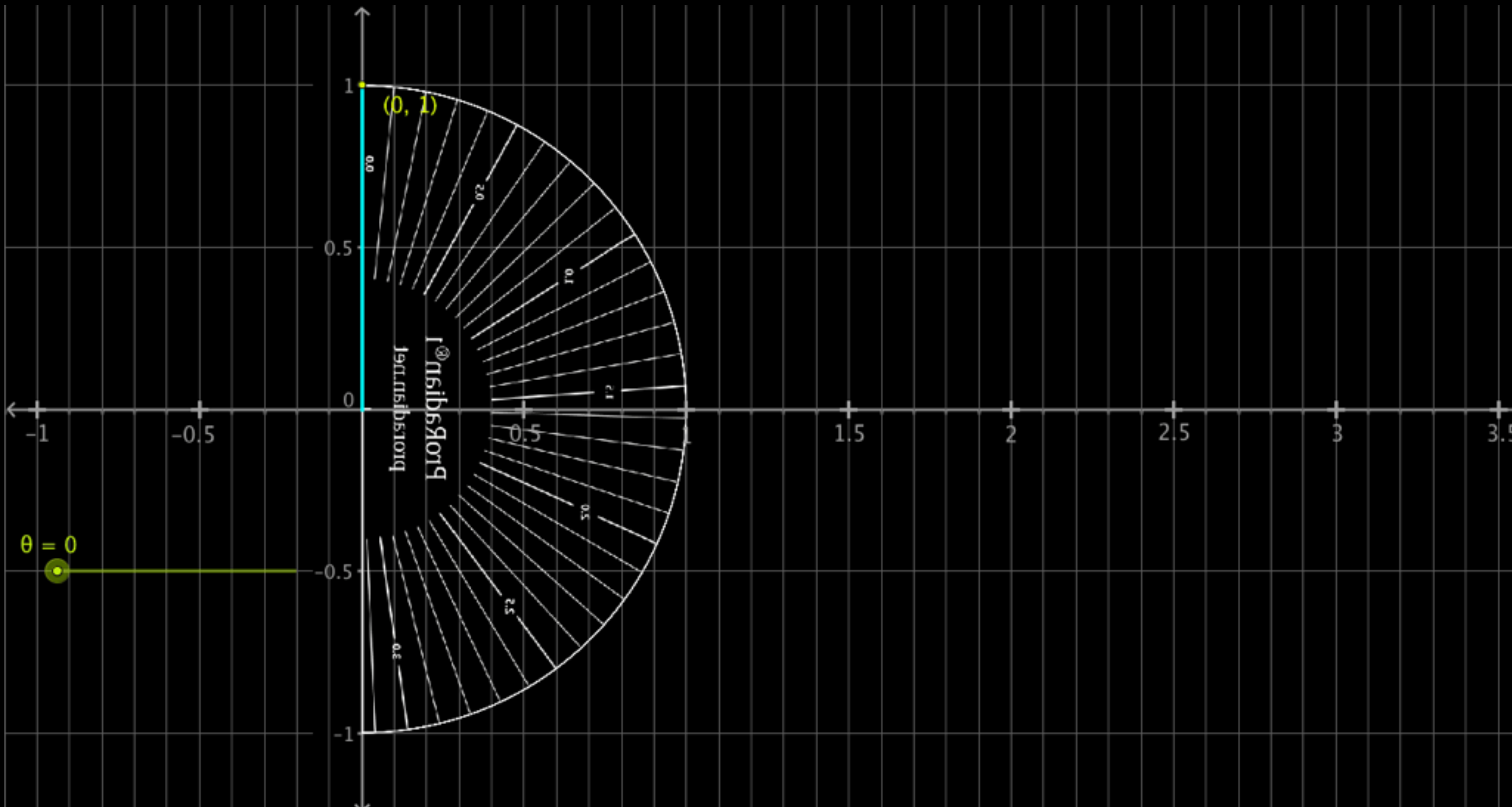
Handout Activity 7



Record the “Horizontals”



Record the “Horizontals”



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