#306 Griffith Even the Weird Kids Have Parents

When we begin to study Integral Calculus, we like to start in a place that is familiar to the students. We then approach these first few days in three steps:

1. Look at the integral as area

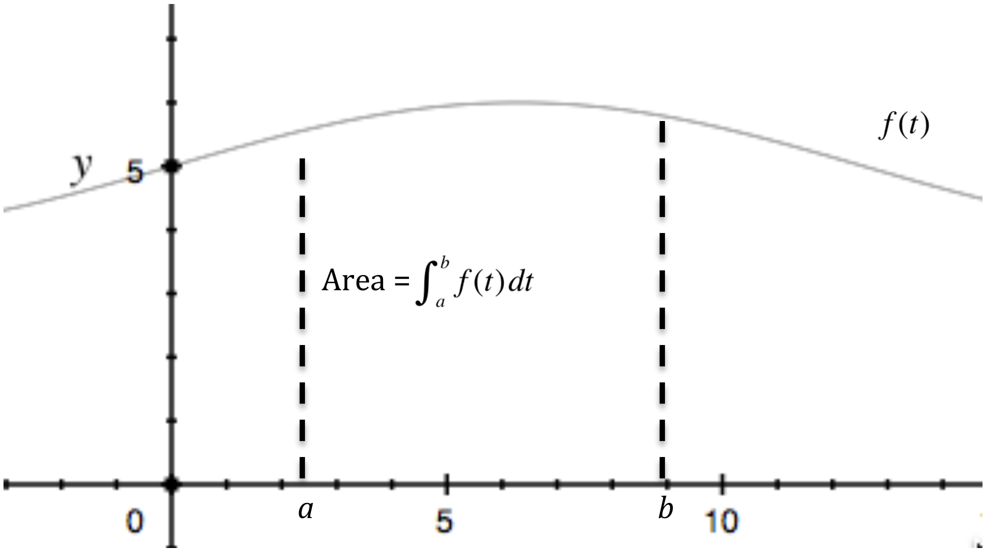
2. Look at two basic functions

3. Make general conclusion

We want students to have an intuitive understanding of the FTC and then see the proof in action.

We start with a continuous function, $y = f(t)$ on an interval $[a, b]$. We can define the area between f and the t-axis on this interval as the definite integral $\int_a^b f(t)\,dt$

Thus, we have $A(x) = \int_a^b f(t)\,dt$



We first evaluate this by using limits of sums. We then construct the definite integral from those sums.

Is there more to the integral than just area?

We will take a look at a two examples to help answer that question.

We are familiar with distance, rate, and time. What does $d = rt$ have to do with the FTC?

Funny you should ask…

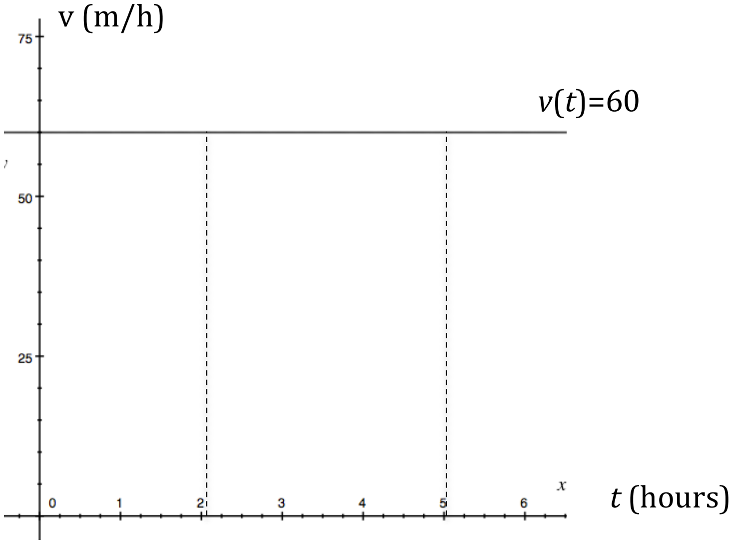
Let's start with a constant velocity, $v(t) = 60\ mph$ and the time period from $t = 2$ to $t = 5$.
We know that the distance traveled is 180 miles.

We know that when t=5, we travel 300 miles and when t=2, we travel 120 miles. The change in position is 180 miles.

Using our $rt = d$ formula,

$$60t\big|_{t=5} - 60t\big|_{t=2} = 300 - 120 = 180$$

Graphically, we see



When we find the area of the rectangle, we have (3h)(60m/h)=180miles.

(1) The area under the curve is 180

(2) The change in position is 180

(3) velocity is the derivative of position

If we let the upper limit be x, we have the area under $v = 60$ from $t = 2$ to $t = x$.

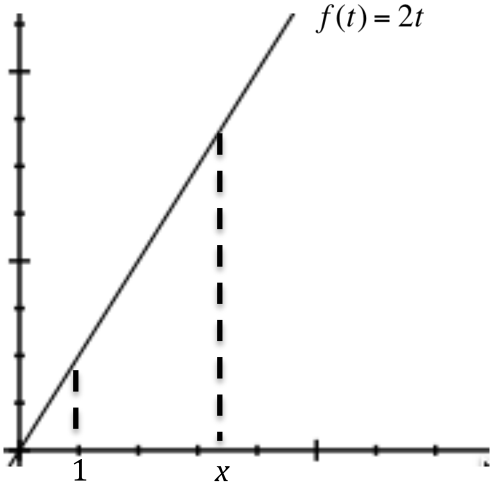
Using the area formula from Geometry, the area would then be $A(x) = 60(x-2) = 60x - 120$

This means that if I want to find the distance traveled, I take distance traveled in x hours, $60x$, and subtract the distance traveled in two hours, 120. This gives me the change in distance between $t = 2$ and $t = x$. Does this make sense?

Notice that the derivative of $60x – 120$ is 60.

Is there a connection between area under the curve and the derivative ? You bet!

Let's take a look at a well-known function, $f(t) = 2t$. Most students are comfortable with this fellow and they know that the derivative is 2. We can work with y = 2*t*, and then students can handle the general proof that is found in almost all textbooks:



Let $F(x)$ be the function that gives that area under the curve between *t* =1 and *t* = *x.*

We say that $F(x) = \int_1^x 2t\,dt$

Let's use Geometry to help us find this area:

$$F(x) = \frac{1}{2}(x-1)(2(1)+2(x))$$

$$F(x) = \frac{1}{2}(x-1)(2+2x) = \frac{1}{2}(2x+2x^2-2-2x)$$

$$F(x) = x^2 - 1$$

Thus, we can write: $F(x) = \int_1^x 2t\,dt = x^2 - 1$.

Do you recognize anything?

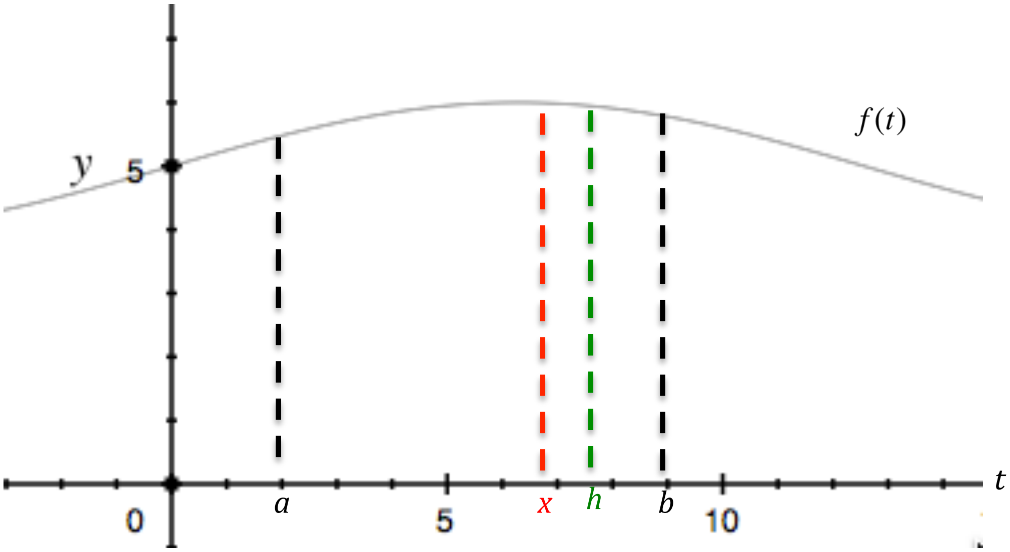
Let's take the derivative of both sides: $F'(x) = 2x$

The derivative of the area function is *f*, or the area function is an antiderivative of *f.*

This is not a coincidence….

Let f be a continuous function on [*a, b*]

$\int_a^b f(t)\,dt$ is the area between f and the *x*-axis on the interval [*a, b*]



Let *F*(*x*) be the area between *a* and *x*, where $a \le x \le b$.

The area under curve between two points is $F(x) = \int_a^x f(t)\,dt$

Let us think about the derivatives here:

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h}$$

This last expression is the average value formula for integrals. It says that $f(c)$ exists for some *c* between *x* and *x* + *h*.

We write $f(c) = \frac{1}{x+h-x}\int_x^{x+h} f(t)\,dt = \frac{1}{h}\int_x^{x+h} f(t)\,dt$

Since c is between x and *x*+h, as *h* approaches zero, *c* approaches *x*. Thus $f(c)$ will approach $f(x)$.

Now, we can say $F'(x) = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h} = \lim_{h\to 0} f(c) = f(x)$

This means that if $F(x) = \int_a^x f(t)\,dt$ then $F'(x) = f(x)$

We can also write this as

$$\frac{d}{dx}\left(F(x)\right)=\frac{d}{dx}\left(\int_a^x f(t)\,dt\right)=f(x)$$

What is the big deal with this?

Every continuous function f is the derivative of another function F

Definite integral is an antiderivative.

From FDWK:

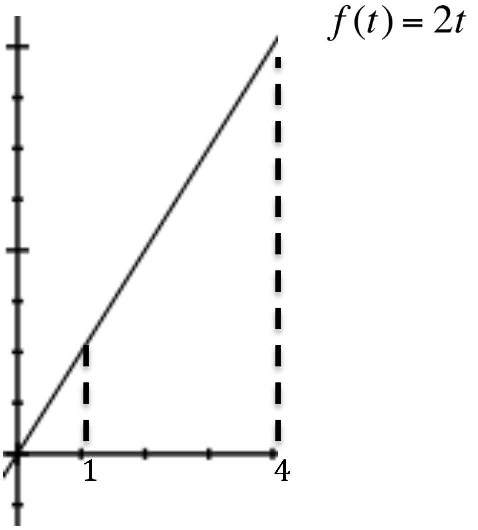
It says that every cont. function is the derivative of some other function, namely $\int_a^x f(t)\,dt$. It says that every cont. function has an antiderivative. And it says that integration and differentiation are inverses of one another. If any equation deserves to be called the FTC, this is surely the one.

This is profound.

And don't call me Surely.

What else can we conclude? Well, let's return to our friend, $y = 2t$ and see how we use the well-known antiderivative, $x^2 + C$

We will give the upper limit a value. Let us think about the area under the curve $y = 2t$ between $t = 1$ and $t = 4$:



Using our knowledge from Geometry, the area is $\frac{1}{2}(3)(2+8)=15$.

Now, think about $F(x)=x^2+C$. We know that $F(4) = 16 + C$ and $F(1) = 1 + C$. And $F(4) - F(1)=15$. Notice the at the C's go away.

Coincidence? Not at all.

This leads to the FTC Part2: $\int_a^b f(x)\,dx = F(b) - F(a)$, where *F* is an antiderivative of *f*.

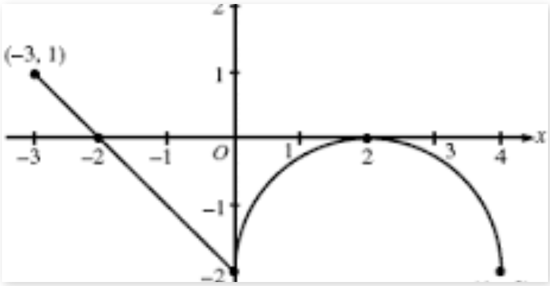
Again, the connection between derivative and integrals…

Let's look at a few AP style problems that reinforce a current point of emphasis:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$
$$F(b) = F(a) + \int_a^b f(x)\,dx$$
$$F(a) = F(b) - \int_a^b f(x)\,dx$$

- An object moves along a horizontal line with initial position *x*(1) = 3. The velocity of the object at time $t \geq 0$ is given by $v(t) = \sin\left(\frac{\pi}{3}t\right)$
- What is the position at time *t* = 5?

$$\int_1^5 v(t)\,dt = x(5) - x(1)$$
$$x(1) + \int_1^5 v(t)\,dt = x(5)$$



The graph of a function f consists of …
Let g be the function defined by

$$g(x)=\int_0^x f(t)\,dt$$

We want our students to recognize this type of problem and immediately write: $g'=f$

On a recent exam, 9 of the 45 points of the free response section were awarded for direct use of the FTC:

- 1(d) Given $L(20)=43$, $L'(t)=0.5\sqrt{t}\sin(.07t)$ Find $L(25)$

$$\int_{20}^{25} L'(t)\,dt = L(25)-L(20)$$

- 2(b) Given $\frac{dx}{dt}=\frac{\sqrt{t+7}}{e^{2t}}$, $x(2)=1$ Find $x(4)$

$$\int_2^4 x'(t)\,dt = x(2)-x(4)$$

- 3(a)(b) Given $g(x)=\int_1^x f(t)\,dt$

  Find $g(2)$, $g(-2)$, $g'(-3)$, $g''(-3)$

$$g'=f$$

Finally, we have a question that allows to include an integral in the expression for the function. A problem like this appeared a few years ago.

- **Water enters a take at a constant rate of 7 liters/hr. Water leaks out at the rate of** $\sqrt{2t-5}$ **liters/hr. $0 \leq t \leq 24$ hours. At $t = 0$, tank holds 28 liters of water.**
- **(a) How many liters of water leak out in the first three hours?**
- **(b) Write an expression for *L(t),* the total numbers of liters of water in the tank at time *t.***

We notice two clues in the stem of the problem "rate" and "liters/hr. This tells us we are dealing with a derivative. Now, we know that $\int rate = \Delta volume$. The FTC says that the first three hours will have a volume change of $\int_0^3 \sqrt{2t-5}dt$. The students can then evaluate the integral. This is another use of the FTC.

In part (b), we are asked for the total amount of water in the tank at time *t*. Students might be tempted to find an antiderivative and then set up an equation. We do not want them to do that. We want them to think about the situation:

L(t) = initial amount + water in – water out

$L(t) = L(0) + \int_0^t L'(x)dx = 30 + \int_0^t \left(7 - \sqrt{2x-5}\right)dx$ this is what we want!