#306 Griffith Even the Weird Kids Have Parents

1. Look at the integral as area
2. Look at two basic functions
3. Make general conclusion

We use limits of sums (RRAM, LRAM, MRAM) to introduce the integral as area. We define $\int_a^b f(x)\,dx$ as the area between *f* (*x*) and the *x*-axis on the interval [*a, b*]. Is there more to the integral than just area? You bet.

We will take a look at a two examples to help answer that question.

Let's start with a constant velocity, $v(t) = 60\ mph$ from *t* = 2 to *t* = 5. We know that the distance traveled is 180 miles.
When we find the area of the rectangle, we have (3h)(60m/h)=180miles.
(1) The area under the curve is 180 (2) The change in position is 180 (3) velocity is the derivative of position

If we let the upper limit be *x*, we have the area under *v* = 60 from *t* = 2 to *t* = *x*.
Using the area formula from Geometry, the area would then be $A(x) = 60(x-2) = 60x - 120$
Notice that the derivative of 60*x* – 120 is 60.
Is there a connection between area under the curve and the derivative ? Once again, you bet!

Let's take a look at a well-known function, $f(t) = 2t$. Most students are comfortable with this fellow.

Let $F(x)$ be the function that gives that area under the curve between *t* =1 and *t* = *x*. We say that $F(x) = \int_1^x 2t\,dt$

Using Geometry:

$$F(x) = \frac{1}{2}(x-1)\big(2(1) + 2(x)\big)$$

$$F(x) = \frac{1}{2}(x-1)(2+2x) = \frac{1}{2}\left(2x + 2x^2 - 2 - 2x\right)$$

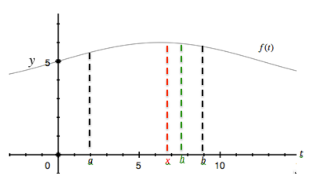
$$F(x) = x^2 - 1$$

Thus, we can write: $F(x) = \int_1^x 2t\,dt = x^2 - 1$.

Do you recognize anything?
Let's take the derivative of both sides: $F'(x) = 2x$
The derivative of the area function is *f*, or the area function is an antiderivative of *f*. This is not a coincidence….

Let *f* be a continuous function on [*a, b*]

$\int_a^b f(t)\,dt$ is the area between *f* and the *x*-axis on the interval [*a, b*]



Let *F*(*x*) be the area between *a* and *x*, where $a \le x \le b$.

The area under curve between two points is

$$F(x) = \int_a^x f(t)\,dt$$

Let us think about the derivatives here:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h} = \lim_{h \to 0} \frac{\int_x^{x+h} f(t)\,dt}{h}$$

We can use a proof that involves avg. val.

This means that if $F(x) = \int_a^x f(t)\,dt$ then $F'(x) = f(x)$

We can also write this as $\frac{d}{dx}\big(F(x)\big) = \frac{d}{dx}\left(\int_a^x f(t)\,dt\right) = f(x)$ What is the big deal with this?

Every continuous function *f* is the derivative of another function *F*

Also, we can think about *G*(*x*), where *G*(*x*) = *F*(*x*) + C

This leads to the FTC Part2: $\int_a^b f(x)\,dx = F(b) - F(a)$, where *F* is an antiderivative of *f*.

Again, the connection between derivative and integrals…

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

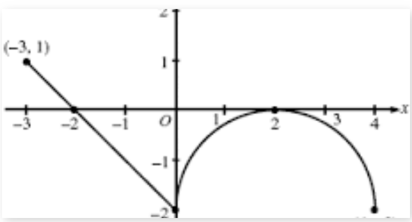
$$F(b) = F(a) + \int_a^b f(x)\,dx$$

$$F(a) = F(b) - \int_a^b f(x)\,dx$$

Let's look at a few AP style problems that reinforce a current point of emphasis:

1. An object moves horizontally with initial postion $x(1) = 3$. The velocity is given by $v(t) = \sin\left(\frac{\pi}{3}t\right)$ $t \geq 0$. What is the position at t=5?

$$\int_1^5 v(t)\,dt = x(5) - x(1) \qquad x(1) + \int_1^5 v(t)\,dt = x(5)$$

2. 

The graph of a function f consists of ...
Let g be the function defined by

$$g(x) = \int_0^x f(t)\,dt$$

We want our students to recognize this type of problem and immediately write: $g' = f$

3. (a) Given $L(20) = 43$, $L'(t) = 0.5\sqrt{t}\sin(.07t)$ Find $L(25)$

$$\int_{20}^{25} L'(t)\,dt = L(25) - L(20)$$

(b) Given $\frac{dx}{dt} = \frac{\sqrt{t+7}}{e^{2t}}$ and $x(2)=1$. Find $x(4)$

$$\int_2^4 x'(t)\,dt = x(2) - x(4)$$

(c) Given $g(x) = \int_1^x f(t)\,dt$ Find $g(2)$, $g(-2)$, $g'(-3)$, $g''(-3)$

$$g' = f$$

4. Water enters a take at a constant rate of 7 liters/hr. Water leaks out at the rate of $\sqrt{2t-5}$ liters/hr. $0 \leq t \leq 24$ hours. At t = 0, tank holds 28 liters of water.

(a) How many liters of water leak out in the first three hours?

(b) Write an expression for $L(t)$, the total numbers of liters of water in the tank at time t.

We notice two clues in the stem of the problem "rate" and "liters/hr. This tells us we are dealing with a derivative. Now, we know that $\int rate = \Delta volume$. The FTC says that the first three hours will have a volume change of $\int_0^3 \sqrt{2t-5}\,dt$. The students can then evaluate the integral. This is another use of the FTC.

In part (b), we are asked for the total amount of water in the tank at time t. Students might be tempted to find an antiderivative and then set up an equation. We do not want them to do that. We want them to think about the situation:

L(t) = initial amount + water in – water out

$L(t) = L(0) + \int_0^t L'(x)\,dx = 30 + \int_0^t \left(7 - \sqrt{2x-5}\right)dx$ this is what we want!