The Development of the Addition and Subtraction of Integers: The Case of Jace

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Author Note

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Abstract

Jace participated in a teaching experiment on integer addition and subtraction for 12 weeks. During this teaching experiment, he participated in four individual sessions where he solved various open number sentences. Jace's development, or learning over time, is described for the open number sentence type, $-a + \square = b$ ($a, b > 0$ and $b > a$), using commognitive theory. Although Jace solved this problem type correctly across all four sessions, Jace demonstrated learning. For example, Jace transitioned from drawing number lines to writing horizontal number sentences. Jace transitioned from drawing upon reasoning that made use of movements and distance on a number line to the drawing upon a rule that he invented.

Keywords: integer addition and subtraction, student thinking, development

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The teaching and learning of negative numbers is challenging but foundational to the learning of mathematics. Research on student thinking about integers began in our field by discussing typical struggles students had with integers (e.g., Guerrero & Martinez, 1982), identifying productive contexts, games, or models of integer instruction (e.g., Bell, O'Brien, & Shiu, 1980; Bell, 1982), and identifying problems types for integers (Marthe, 1979) and additive structures in general (Vergnaud, 1982). Not only is the teaching and learning of negative integers difficult, but research on student thinking about negative integers has also been notoriously challenging. Despite the profusion of literature on student thinking about integer addition and subtraction across the decades (e.g., Bell, 1982; Chui, 2001; Gallardo, 2002; Liebeck, 1990; Linchevski & Williams 1999; Marthe, 1979; Mukhopadhyay, Resnick, & Schauble, 1990; Stephan & Akyuz, 2012), our field still has foundational needs in understanding what the learning of integer addition and subtraction looks like over time. As a field, we have made significant progress about the types of reasoning and strategies that children utilize (e.g., Bofferding, 2014; Bishop et al., 2014a, 2014b); however, we still need insight into what the development and learning of integer addition and subtraction looks like.

Student Thinking about Integer Addition & Subtraction

Young children can invent and use productive reasoning about integers (e.g., Bofferding, 2014; Featherstone, 2000; Hativa & Cohen, 1995; Murray, 1985). Children are even capable of inventing their own notation for negative integers (e.g., Bishop, Lamb, Philipp, Schappelle, & Whitacre, 2011). Students have utilized a variety of strategies to solve integer addition and subtraction problems. These strategies include:

- Attaching or detaching the sign to the answer at the end of the problem (e.g., Human & Murray, 1987; Vlassis, 2008)
- Interpreting the negative sign as subtraction (e.g., Gallardo, 1994; Human & Murray, 1987)
- Using analogies to whole numbers (e.g., Human & Murray, 1987; Murray, 1985)
- Drawing upon number line and movements (e.g., Guerroro & Martinez, 1982; Hativa & Cohen, 1995; Human & Murray, 1987; Murray, 1985; Poirier & Bednarz, 1991)

In recent studies on children's thinking about integer addition and subtraction, Bishop et al. (2014a, 2014b) investigated students' reasoning when solving open number sentences with negative integers. Bishop et al. found that some children, before formal instruction on negative integers, denied the existence of negative integers, similar to mathematicians of the past, classifying the problems as impossible. Other children categorized negative integers as a zero. For example, for problems like 3–4 and 2–7 the children would answer 0 in both cases. Some children stated that solutions to open number sentences like 4 + □ = 3 were "not real." Despite these difficulties, Bishop et al. (2014a) found that the children had productive Ways of Reasoning (WoR) that they often used utilized solving these open number sentences. Bishop et al. (2014a) highlighted ordering relations, logical necessity and formalisms, magnitude, computation, and limited as the WoR that children used when solving open number sentences with integer addition and subtraction. Bishop et al. found that the WoR described the ways that children reasoned when solving integer addition and subtraction open number sentences. Building on this, Wessman-Enzinger & Mooney (2014) investigated the ways of thinking about and using integers that children employed when they created contexts for various open number sentences.

These ways of thinking about and using the integers are called the Conceptual Models for Integer Addition and Subtraction (CMIAS) or ways of thinking and learning about negative integers—Bookkeeping, Counterbalance, Translation, Relativity, and Rule (Wessman-Enzinger & Mooney, 2014). The *Bookkeeping* conceptual model is described as a utilization of integers as gains and losses, where zero represents neither a gain nor a loss. The *Counterbalance* conceptual model is described as a neutralization use of integers, where zero represents neutralization. The *Translation* conceptual model is described by vector or directed movements of the integers, where zero either represents a referent point or no movement. The *Relativity* conceptual model is described by the use of integers in relative positions or orderings with an unknown referent, where zero represents the unknown referent. The *Rule* conceptual model is described as the use algorithms or invented rules.

Although there is increased interest into student thinking about integer addition and subtraction as a field and ways to describe student strategies or reasoning (e.g., Bishop et al., 2014; Bofferding, 2014; Wessman-Enzinger & Mooney, 2014; Whitacre et al., 2012, 2014), little is known about how students think and learning about integer addition and subtraction over extended time. Specifically, we need to know more about how students' thinking and learning about integer addition and subtraction evolves over time. This research brief paper addresses this gap in the literature by providing an account into the development of student thinking and learning about integer addition and subtraction.

Theoretical Perspective on Learning

Learning can be perceived as a change in mathematical discourse (Sfard, 2008). With commognitive theory, learning is defined as a "process of changing one's discursive ways in a certain well-defined manner" (Sfard & Avigail, 2006, p. 4). In reference to negative integers, Sfard and Avigail (2006) stated, "a person who learns about negative numbers alters and extends her discursive skills as to become able to use this form of communication in solving mathematical problems" (p. 4). Although the negative integers may be a different kind of abstraction and require instructional experiences for students (Fischbein, 1987), students do not create a new mathematical discourse or participate in an entirely new learning experience. Rather, they are likely to modify their mathematical discourse about whole numbers to accommodate the negative integers. These changes in their mathematical discourses are evidence of their learning. Sfard considers discourse as a communication with oneself, thinking as communicating (Sfard, 2008) influenced by learning experiences. Thus, thinking mathematically is mathematical discourse. And, learning can be described by the changes in this discourse. Sfard points to important components of mathematical discourse: word use, visual mediators, narratives, and routines.

Sfard (2008) classifies a discourse as mathematical is if the discourse includes language that is mathematical. *Word use* refers to how mathematical words are used in discourse. Discourses are often focused about a medium, a concrete object, or artifact. As a part of mathematical discourse, *visual mediators* are produced. Visual mediators with integers may be the mathematical symbols employed by students or the drawings they use to discuss their thinking or solve a problem. Sfard and Avigail (2006) state that these visual mediators are “part and parcel in the act of communication, and thus of the cognitive processes themselves” (p. 7). Sfard (2008) defines a *narrative* as, “any sequence of utterances framed as a description of objects, that is subject to endorsement or rejection with the help of discourse-specific substantiation procedures” (p. 134). Narratives include mathematical definitions, theories, theorems, and properties formed as students interact with the integer addition and subtraction. Narratives can be endorsed or rejected. That is, a student may develop a narrative that is rejected later. *Routines* refer to the set of repetitive patterns in mathematical and nonmathematical activities. This includes the mathematical activity of the participants as they substantiate their mathematical narratives. Sfard (2008) points to the repetitive characteristics of discourse as routines. The idea is that some routines may be inherent and not explicitly communicated as an expectation. Another aspect of routines is identification of when and how the routines occur.

Word use, visual mediators, routines, and narratives comprise four components of discourse from a commognitive theoretical standpoint. Each of the components, although listed separately, relates to the other components. For example, word use may work in conjunction with visual mediators, through a routine, to communicate a narrative. These tenets of commognition are synergistic and work together to describe students’ discourses.

Research Question

The goal of the larger study was to explore the development student thinking about integers and operations with integers when promoting various conceptual models, or ways of thinking, about integers. This paper will focus on describing the learning one of the participants, Jace, for a particular open number sentence problem type $-a + \square = b$, where $a < b$. Specifically, the research question for this research brief that will be addressed is:

What learning emerged as Jace solved the open number sentence, $-a + \square = b$, where $a < b$?

Methodology

Participants and Setting

Three fifth grade students from a rural Midwest school participated in a teaching experiment (Steffe & Thompson, 2000) for 12 weeks. Fifth graders were selected as participants because the *Common Core State Standards* (National Governors Association [NGA] & Council of Chief State School Officers [CCSSO], 2010) indicate that instruction for negative integers begin in Grades 6 and finish in Grade 7. Grade 5 participants were selected to allow for the instructional experiences within this study to be these students' first experiences with negative integers, while also being as close to the NGA and CCSSO recommended instructional age. Jace was selected as the participant to be reported on in this paper because he participated in nearly all of the group sessions.

Data Collection and Sources

The students completed two written pre-tests and two written post-tests. Each student participated in eight individual sessions and nine group sessions. The outline of the data collection is shown below (see Figure 1). All individual sessions and group sessions were videotaped and transcribed. Other sources of data include all of the students' written work, a teacher-researcher journal, and witness field notes. The individual sessions were broken into two parts. The first part presented contextual problems supporting integer addition and subtraction to the students. The second part presented open number sentences about the addition and subtraction of integers to the students. Group sessions were immersed in contexts that promoted some CMIAs: Bookkeeping, Counterbalance, Translation, and Relativity.

<table>
<tr><td>Jace Individual Session 1a
(Pre-Assessment): Context</td><td>Alice Individual Session 1a
(Pre-Assessment): Context</td><td>Kim Individual Session 1a
(Pre-Assessment): Context</td></tr>
<tr><td>Jace Individual Session 1b
(Pre-Assessment):
Number Sentences</td><td>Alice Individual Session 1b
(Pre-Assessment):
Number Sentences</td><td>Kim Individual Session 1b
(Pre-Assessment):
Number Sentences</td></tr>
<tr><td colspan="3">Group Session 1</td></tr>
<tr><td colspan="3">Group Session 2</td></tr>
<tr><td colspan="3">Group Session 3</td></tr>
<tr><td>Jace Individual Session 2a:
Context</td><td>Alice Individual Session 2a:
Context</td><td>Kim Individual Session 2a:
Context</td></tr>
<tr><td>Jace Individual Session 2b:
Number Sentences</td><td>Alice Individual Session 2b:
Number Sentences</td><td>Kim Individual Session 2b:
Number Sentences</td></tr>
<tr><td colspan="3">Group Session 4</td></tr>
<tr><td colspan="3">Group Session 5</td></tr>
<tr><td colspan="3">Group Session 6</td></tr>
<tr><td>Jace Individual Session 3a:
Context</td><td>Alice Individual Session 3a:
Context</td><td>Kim Individual Session 3a:
Context</td></tr>
<tr><td>Jace Individual Session 3b:
Number Sentences</td><td>Alice Individual Session 3b:
Number Sentences</td><td>Kim Individual Session 3b:
Number Sentences</td></tr>
<tr><td colspan="3">Group Session 7</td></tr>
<tr><td colspan="3">Group Session 8</td></tr>
<tr><td colspan="3">Group Session 9</td></tr>
<tr><td>Jace Individual Session 4a
(Post-Assessment): Context</td><td>Alice Individual Session 4a
(Post-Assessment): Context</td><td>Kim Individual Session 4a
(Post-Assessment): Context</td></tr>
<tr><td>Jace Individual Session 4b
(Post-Assessment):
Number Sentences</td><td>Alice Individual Session 4b
(Post-Assessment):
Number Sentences</td><td>Kim Individual Session 4b
(Post-Assessment):
Number Sentences</td></tr>
</table>

Figure 1. Outline of data collection.

Table 1 shows the different problem types of integer addition and subtraction open number sentences that involve negative integers. Many of these open number sentence types were given to the students during Individual Sessions 1b, 2b, 3b, and 4b. Because the students became more efficient with working with integer addition and subtraction over time, the amount of open number sentences and different problem types that they could complete during the interview session increased over time (see Figure 2). Because of this, there are some problem types that are consistent across all four sessions and other problem types that are not consistent over all four sessions.

Table 1

Problem Types of Integer Addition and Subtraction Open Number Sentences with Negative Integers

Types of Problems	Open Number Sentence
Addition (one negative integer given)	$-a + b = \square$
Case 1: a, b > 0 and a > b	$a + -b = \square$
Case 2: a, b > 0 and b > a	$-a + \square = b$
	$a + \square = -b$
	$\square + -a = b$
Addition (two negative integers given)	$-a + -b = \square$
Case 1: a, b > 0 and a > b	$-a + \square = -b$
Case 2: a, b > 0 and b > a	$\square + -a = -b$
Subtraction (all positive integers given)	$b - a = \square$
Only Case: a, b > 0, a > b	$b - \square = a$
Subtraction (one negative integer given)	$-a - b = \square$
Case 1: a, b > 0 and a > b	$a - -b = \square$
Case 2: a, b > 0 and b > a	$-a - \square = b$
	$a - \square = -b$
	$\square - -a = b$
	$\square - a = -b$
Subtraction (two negative integers given)	$-a - -b = \square$
Case 1: a, b > 0 and a > b	$-a - \square = -b$
Case 2: a, b > 0 and b > a	$\square - -a = -b$

Individual Session 1b	Individual Session 2b	Individual Session 3b	Individual Session 4b
-20 + 15 = □	-16 + 4 = □	-18 + 12 = □	-20 + 15 = □
12 + -16 = □	20 + -33 = □	15 + -24 = □	12 + -16 = □
-4 + □ = 10	-6 + □ = 15	-3 + □ = 14	-4 + □ = 10
-7 + □ = -2	-6 + □ = -1	-9 + □ = -3	-7 + □ = -2
□ + -3 = 7	□ + -2 = 17	□ + -4 = 13	□ + -3 = 7
□ + 13 = -5	□ + 19 = -4	□ + 25 = -2	□ + 13 = -5
-8 + -7 = □	-12 + -5 = □	-17 + -6 = □	-8 + -7 = □
-2 + □ = -10	-4 + □ = -19	-5 + □ = -21	-2 + □ = -10
□ + -9 = -16	□ + -9 = -21	□ + -9 = -17	□ + -9 = -16
10 – 12 = □	5 – 9 = □	12 – 18 = □	10 – 12 = □
1 – □ = 3	4 – □ = 6	3 – □ = 4	1 – □ = 3
-5 – 4 = □	-9 – 8 = □	-5 – 3 = □	-5 – 4 = □
2 – -3 = □	3 – -4 = □	1 – -3 = □	2 – -3 = □
-1 – □ = 8	-2 – □ = 9	-2 – □ = 10	-1 – □ = 8
2 – □ = -10	6 – □ = -10	4 – □ = -12	2 – □ = -10
□ – -1 = 6	□ – -1 = 4	□ – -2 = 5	□ – -1 = 6
□ – 8 = -5	□ – 9 = -3	□ – 6 = -2	□ – 8 = -5
-15 – -4 = □	-11 – -2 = □	-12 – -4 = □	-15 – -4 = □
-12 – □ = -13	-15 – □ = -16	-10 – □ = -11	-12 – □ = -13
	□ – -3 = 2	□ – -3 = 1	□ – -2 = 1
	□ – -4 = 0	□ – -5 = 0	□ – -3 = 0
	12 + □ = 8	15 + □ = 9	17 + □ = 8
		8 + □ = -5	6 + □ = -2
		□ + 2 = 0	□ + 4 = 0
		-4 – 10 = □	-2 – 8 = □

Figure 2. Open number sentences provided in the Individual Session 1b, 2b, 3b, and 4b.

Data Analysis

The data comes from the four Individual Sessions 1b, 2b, 3b, and 4b, in which Jace solved open number sentences for the addition and subtraction of integers. Jace's responses (i.e., word use-transcripts, visual mediators-drawings) to each open number sentence determined the units of data. In this study, word use and visual mediators were examined by looking at the verbal interactions and drawings produced by the student. The word use and visual mediators were used to describe the students' narrative, or way of reasoning to solve the open number sentence. Routines in this study were described by comparing specific word use, visual mediators, and narratives to the overall word use, visual mediators, and narratives. Changes in this word use, visual mediators, narratives, and routines were examined across sessions. Because these are the tenets of discourse and learning is described as a change in discourse, describing these changes highlights learning. A number sentence that Jace solved correctly across all four interviews was selected to report on for this paper.

Results

Across the Individual Sessions 1b, 2b, 3b, and 4b, Jace became better at solving open number sentences (see Figure 3). He increased from getting 50% of the open number sentences correct in Session 1b to 98% of the open number sentences correct in Session 4b. In Figure 3 the open number sentences are matched up by problem type across the four sessions. Jace's correct answers are in green and his incorrect answers are in red. All of the answers that Jace provided during the session are listed. For example, in Individual Session 4b for -15 – -4 = □, Jace first stated -19, which was incorrect. He then changed his answer to -11, which was correct. Both of these solutions, -19 and -11, are listed in the cell, but because Jace's final answer was correct, it considered that he answered that open number sentence correctly.

Individual Session 1b		**Individual Session 2b**		**Individual Session 3b**		**Individual Session 4b**	
-20 + 15 = □	-5	-16 + 4 = □	-12	-18 + 12 = □	-6	-20 + 15 = □	-5
12 + -16 = □	-4	20 + -33 = □	-13	15 + -24 = □	-9	12 + -16 = □	-4
-4 + □ = 10	14	-6 + □ = 15	21	-3 + □ = 14	17	-4 + □ = 10	14
-7 + □ = -2	5	-6 + □ = -1	5	-9 + □ = -3	-6, 6	-7 + □ = -2	5
□ + -3 = 7	10	□ + -2 = 17	19	□ + -4 = 13	17	□ + -3 = 7	10
□ + 13 = -5	-18	□ + 19 = -4	-23	□ + 25 = -2	-27	□ + 13 = -5	-18
-8 + -7 = □	-1	-12 + -5 = □	-17	-17 + -6 = □	-23	-8 + -7 = □	-15
-2 + □ = -10	-8	-4 + □ = -19	-15	-5 + □ = -21	-16	-2 + □ = -10	-8
□ + -9 = -16	-7	□ + -9 = -21	-12	□ + -9 = -17	-8	□ + -9 = -16	-7
10 – 12 = □	-2	5 – 9 = □	4, -4	12 – 18 = □	-6	10 – 12 = □	-2
1 – □ = 3	not possible	4 – □ = 6	not sure	3 – □ = 4	not sure	1 – □ = 3	-2
-5 – 4 = □	-1	-9 – 8 = □	-1	-5 – 3 = □	-2	-5 – 4 = □	-1, 9
2 – -3 = □	-1	3 – -4 = □	-1	1 – -3 = □	-2	2 – -3 = □	5
-1 – □ = 8	7	-2 – □ = 9	11	-2 – □ = 10	12	-1 – □ = 8	9, not sure
2 – □ = -10	8	6 – □ = -10	-16, 16	4 – □ = -12	16	2 – □ = -10	12
□ – -1 = 6	-7, not sure	□ – -1 = 4	5	□ – -2 = 5	7	□ – -1 = 6	5
□ – 8 = -5	3	□ – 9 = -3	6	□ – 6 = -2	4	□ – 8 = -5	3
-15 – -4 = □	-11	-11 – -2 = □	-13	-12 – -4 = □	-8	-15 – -4 = □	-19, -11
-12 – □ = -13	-1	-15 – □ = -16	-1	-10 – □ = -11	not sure	-12 – □ = -13	1
□ – -2 = 1	3	□ – -3 = 2	5	□ – -3 = 1	4	□ – -2 = 1	-1
		□ – -4 = 0	4	□ – -5 = 0	5	□ – -3 = 0	-3
		12 + □ = 8	-4	15 + □ = 9	-6	17 + □ = 8	-9
		5 + □ = -3	-8	8 + □ = -5	13, -13	6 + □ = -2	-8
				□ + 2 = 0	-2	□ + 4 = 0	-4
				-4 – 10 = □	-6	-2 – 8 = □	-10
Percent Correct	50%		~57%		64%		92%

Figure 3. Jace's answers to open number sentences.

Overall, the proportion of problems that Jace solved correctly improved over the 12-week period. Although Jace's performance improved, it is notable to observe how long it took for Jace to make sense of the some subtraction problems types despite the support of conceptually-based group sessions. Some problem types remained difficult for Jace throughout the sessions (see, e.g., -5 – 4 = □). Yet, Jace solved other problem types successfully across the four sessions (see., e.g., -4 + □ = 10). Jace's learning for the problem type -a + □ = b (Case 2: a, b > 0 and b > a) across the four individual open number sentence sessions (i.e., Sessions 1b, 2b, 3b, 4b) are highlighted next. This open number sentence type was selected because although Jace solved it correctly across all four sessions and his discourse about it changed.

Describing the Learning of -a + □ = b (Case 2: a, b > 0 and b > a)

Jace solved the problem type -a + □ = b (Case 2: a, b > 0 and b > a) correct across all four sessions. Although he answered this problem type correctly across the sessions (see Figure 3), how Jace solved this varied across the sessions (see Figures 4, 5, 6, and 7).

Figure 4 illustrates Jace's learning of problem type $-a + \square = b$ (Case 2: $a, b > 0$ and $b > a$) by including Jace's transcripts (word use), drawings (visual mediators), reasoning for solving the open number sentence (narratives), and describing how much Jace used that narrative and type of visual mediator in that particular individual session (routines).

Word use. Jace's word use in Session 1b began with discussing how to draw a number line to model this number sentence. Jace began with, "I'm going to do the number line thing again." He then described the actions of his drawings. Jace's word use in Session 2b began with solving 16 + 5. As he continued his explanation, he transitioned into talking about how to use the number line in the latter part of the explanation. This differs from Session 1b, where his word use was initiated with number line discussion rather than serving as a justification. Then, in Session 3b, none of Jace's word use included moving about a number line or distances on a number line. In Session 4b, Jace was efficient in his word use for explaining how to solve the open number sentence. His word use centered around generalizations for solving this type of problem. Across the four sessions, Jace called positive integers either "whole numbers" or "regular numbers."

Individual Session 1b -4 + □ = 10	Individual Session 2b -6 + □ = 15	Individual Session 3b -3 + □ = 14	Individual Session 4b -4 + □ = 10
J: I'm going to do the number line thing again. (Draws line with two tic marks at each end). I will just put negative ten here because that's all we really need. And, I will put ten right here. And zero right here. So to get from negative four to ten, you would have ...Well, you could do this first. You could put a four right here. (Draws a four above zero. Then draws a connecting line from four to ten). And, from regular four to ten would be six. And if you added four, from zero to ten. That would be just four there (draws four above the connecting line from 0 to 4). Four plus six equals ten. If you added another four, which is right here, (draws a connecting line from 0 to -4), Then that would be fourteen. So, negative four plus fourteen equals ten.	J: (Writes horizontally 15 + 6 = 21). I did fifteen plus six because the answer ... Since fifteen is a whole number and then that would be just regular fifteen, but you have to add six more because the six goes ... Hold on. Here I will draw you one. (Draws a number line.) So that would be negative six right here (draws the negatives to the right) and fifteen right here (draws the positives to the left with 15 and -6 each an equal distance from 0 in the drawing.) It would be fifteen (draws an arch from 15 to 0 and writes 15 above the arch). Plus (draws a "+" above the zero) another six (draws an arch from 0 to 6 with 6 above the arch). T: Ok. So what's the answer that goes in the box? J: Ah negative ... Wait. Twenty-one (Writes 21 in the box.) Just regular twenty-one.	J: (Draws a vertical problem first. Vertically writes 14 + 3 = 17. Then, draws an arrow to the box.) T: Ok. Can you tell me what the answer in the box is? J: Seventeen. T: Ok. Can you tell me what you were thinking? How you figured that out? J: Because negative three is basically box (points at box) minus three. So, I did fourteen plus three and I got seventeen. And, seventeen minus three (points at -3) equals fourteen (points at fourteen). It's kind of like the commutative property. T: Oh. Ok. Can you explain the commutative property? J: You just flip it around and you still get the same answer. Like fourteen minus three, I mean fourteen plus three is seventeen. And, seventeen minus three is fourteen.	J: (Draws a horizontal problem. And writes 14 in the box.) T: Ok. How'd you get 14? J: Fourteen minus four equals ten. Fourteen plus negative four or negative four plus fourteen will equal ten. Because when you take a negative number and add it to a regular number, you are just subtracting. Instead of negative four plus fourteen, you can do fourteen minus four.

Figure 4. Jace's word use for solving -a + □ = b (Case 2: a, b > 0 and b > a).

Visual mediators. Jace's visual mediators changed across the four sessions. In Session 1b, Jace drew an empty number line with three distances highlighted. In Session 2b, Jace again drew a number line. However, this time Jace only used two the distances on both sides of the zero on the number line, rather than multiple distances. This change may point to Jace becoming familiar with using distances between or to zero to make sense of integer addition and subtraction. In Session 3b, Jace did not draw a number line. Instead, Jace drew only a vertical number sentence. In Session 4b, Jace drew only a horizontal number sentence. This my point to Jace no longer needing to draw upon the number line and becoming more efficient.

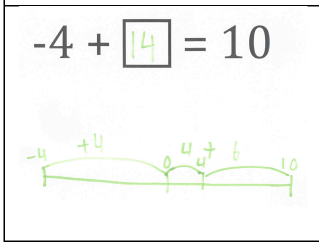
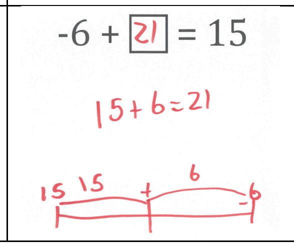
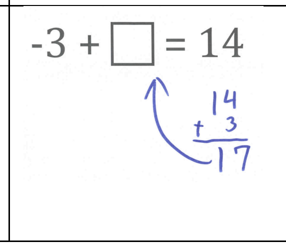
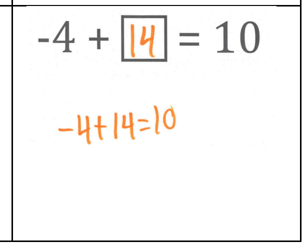
Individual Session 1b -4 + □ = 10	Individual Session 2b -6 + □ = 15	Individual Session 3b -3 + □ = 14	Individual Session 4b -4 + □ = 10
			

Figure 5. Jace's visual mediators for solving -a + □ = b (Case 2: a, b > 0 and b > a).

Narratives. Jace's narratives changed across the session as well. In Session 1b, it was considered that Jace used translation between numbers and the distances on a number line to solve the open number sentence. In Session 2b, it was considered that Jace again used distanced on a number line. However, in Session 2b Jace seemed to be also drawing upon some algebraic reasoning by changing the structure of the number sentence. By Session 3b, Jace no longer used movements between numbers, but only used algebraic reasoning, or a structure change, to solve this open number sentence. In Session 4b, Jace again used algebraic reasoning, but also used a rule that he had developed and constructed an analogy to whole numbers. Given that Jace used movements and distances on a number line in the first session and a rule he developed paired with an analogy and algebraic reasoning in the last session, this may point to Jace becoming more flexible with his reasoning.

Individual Session 1b -4 + □ = 10	Individual Session 2b -6 + □ = 15	Individual Session 3b -3 + □ = 14	Individual Session 4b -4 + □ = 10
Movement & Distances on a Number Line	Algebraic Reasoning Movement & Distances on a Number Line	Algebraic Reasoning	Algebraic Reasoning Analogy Rule

Figure 6. Jace's narratives for solving -a + □ = b (Case 2: a, b > 0 and b > a).

Routines. Jace's routines transitioned across the session. He transitioned from utilizing thinking that was less typically utilized in his sessions to drawing upon thinking that he used frequently. Similarly, Jace did not use number lines that frequently (25% in Session 1b and 13% in Session 2b) and transitioned to writing vertical or horizontal number sentences, which he drew frequently (92% of the time in both Session 3b and 4b). Changing from drawing numbers lines, which he didn't do often, to writing number sentences, which he did do frequently, may point to Jace becoming more familiar with this particular open number sentence type or utilizing different types of reasoning where the number line is not needed as much.

Individual Session 1b -4 + □ = 10	Individual Session 2b -6 + □ = 15	Individual Session 3b -3 + □ = 14	Individual Session 4b -4 + □ = 10
In this session, Jace used a number line in 5 of the 20 open number sentences, or 25% of the time. In this session Jace used movement and distances on a number line 30% of the time.	In this session, Jace used a number line in 3 of the 23 open number sentences, or 13% of the time. In this session, Jace used algebraic reasoning, or a structure change of the number sentence, 26% of the time. Jace used movement and distances on a number line 22% of the time.	In this session, Jace used vertical number sentences in 23 of the 25 open number sentences, or 92% of the time. In this session, Jace used algebraic reasoning, or a structure change of the number sentence 40% of the time.	In this session, Jace used horizontal number sentences in 23 of the 25 open number sentences, or 92% of the time. In this session, Jace used algebraic reasoning, or a structure change of the number sentence, 32% of the time. He made an analogy to a different number sentence 68% of the time. He drew upon a rule he invented 60% of the time.

Figure 7. Jace's routines for solving -a + □ = b (Case 2: a, b > 0 and b > a).

Conclusions

The data in this study supports previous findings that students invent their own strategies for addition and subtraction of integers (Bishop et al., 2014a) and often draw upon their whole number reasoning (Bofferding, 2014). Although developmental research with integers has been completed with cross-sectional studies (e.g., Bishop et al., 2014a), these results extend the previous literature by providing a different developmental perspective by providing descriptive insight into student learning about integers over time for a particular open number sentence type. Jace solved the open number sentence type, $-a + \square = b$ (Case 2: $a, b > 0$ and $b > a$), correctly across the four sessions. Yet, he demonstrated learning as he transitioned drawing upon translations and distances on a number line to an established procedure he developed.

Educational Importance of the Research

The results of this study have instructional implications. Although students may obtain correct solutions to integer addition and subtraction problems, students are still learning about the integers. Learning about integer addition and subtraction transcends operations and answers, including conceptualizations. Additionally, learning integer addition and subtraction takes substantial time. NGA and CCSSO recommendations include introduction and mastery of all operations in the seventh grade. Results reported here suggest that students may need more time to obtain strong conceptually based understandings of integer addition and subtraction, especially if we wish to support student invented strategies.

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