FUNCTION CONCEPTIONS OF AP CALCULUS STUDENTS

Dr. Adam R. Vrabel, Ed.D.

Lourdes University

Abstract

Functions are one of the most important topics in secondary school mathematics, especially for students who wish to take higher-level mathematics courses beginning with calculus. The prerequisites for Advanced Placement Calculus state that a thorough understanding of functions is needed for those who wish to succeed in the course and pass the AP Calculus Exam. However, research has shown that students' struggles with calculus concepts could be traced to inadequate prerequisite knowledge of functions. Since thousands of students take and pass the AP Calculus Exam every year, it is important to investigate which aspects of functions AP Calculus students generally do and do not understand, and to determine how well their understandings of functions relate to their performance on the exam.

In order to explore this, students from AP Calculus classes in three different schools were tested on their understandings of functions at the end of the course, after they had already taken the AP Calculus Exam. The AP Calculus Exam scores for all participants were then collected and compared to their function understandings. It was found that 1) most participants' understandings of functions were less than sufficient for an AP Calculus course, and 2) there was a high positive correlation between function understandings and scores on the AP Calculus Exam. These results suggest that more work must be done in developing students' understandings of functions in the secondary mathematics curriculum, and greater measures should be taken to ensure that students entering AP Calculus have a sufficient understanding of the prerequisite knowledge.

Statement of the Problem

Functions are one of the most important topics in secondary school mathematics. They make up one of the primary high school standards listed in the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010), and they are featured prominently in the National Council of Teachers of Mathematics' (NCTM) *Principles and Standards for School Mathematics* (2000). There has also been much research that has highlighted their importance to the secondary mathematics curriculum (Baker, Hemenway, & Trigueros, 2001; Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Confrey & Smith, 1991; Dubinsky & Harel, 1992; Eisenberg, 1992; Elia, Panaoura, Eracleous, & Gagatsis, 2005; Gerson, 2008; Knuth, 2000; Leinhardt, Zaslavsky, & Stein, 1990).

While understanding functions is important for all students who wish to graduate high school, it is especially so for those who wish to further their education in mathematics by taking higher level courses, beginning with calculus (Carlson, 1998; Thompson, 1994). The prerequisites for Advanced Placement (AP) Calculus (College Board, 2010) highlight the fact that knowledge and skill with functions is needed for those who wish to succeed in the course:

> "Before studying calculus, all students should complete four years of secondary mathematics designed for college-bound students: courses in which they study algebra, geometry, trigonometry, analytic geometry and elementary functions. These functions include linear, polynomial, rational, exponential, logarithmic, trigonometric, inverse trigonometric and piecewise-defined functions. In particular, before studying calculus, students must be familiar with the properties of functions, the algebra of functions and the graphs of functions. Students must also understand the language of functions (domain and range, odd and even, periodic, symmetry, zeros, intercepts and so on) and know the values of the trigonometric functions at the numbers, 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$, and their multiples." (College Board, 2010, p. 6).

Additionally, some researchers have also discussed the fact that functions are important for students entering calculus. Oehrtman, Carlson, & Thompson (2008) indicated that they spend the beginning of college calculus 1 courses testing and strengthening their students' conceptions of functions, and have said that the time spent doing this "…is crucial for their understanding the major ideas of calculus" (p. 167). Carlson and Oehrtman were also part of the team that developed and validated the *Precalculus Concept Assessment Instrument* (Carlson, Oehrtman, & Engelke, 2010), a 25-item multiple choice exam used to measure students' knowledge of concepts that are central to precalculus and foundational to calculus 1. The majority of the exam's taxonomy is centered on the understanding of function concepts and functional reasoning.

Despite this importance, research has also shown that students at all levels generally have struggled with functions, that the development of a strong sense of functions can take a very long time, and students can maintain several common misconceptions about functions (Carlson, 1998; Eisenberg, 1992; Leinhardt, et al., 1990; Vinner & Dreyfus, 1989). There have also been several studies that have found that students' struggles with calculus concepts could be traced to inadequate prerequisite knowledge of functions (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Clark, et al., 1997; Ferrini-Mundy & Gaudard, 1992; Judson & Nishimori, 2005; Ubuz, 2007). Even students who take and do well in calculus courses have been shown to have a relatively weak or narrow conceptualization of functions (Carlson, 1998).

However, most of these studies of functions and calculus learning have been primarily conducted at the college level. There has been very little research that focuses on AP Calculus students' understandings of functions. One such study that does is a doctoral dissertation by Kimani (2008), who found that most AP Calculus students have a low or superficial understanding of function transformations, which could suggest that they may have difficulty with functions in general.

Despite these claims, thousands of students take and pass the AP Calculus exam every year. For instance, in 2012 over 159,000 students earned a 3 or higher on the AP Calculus AB exam (College Board, 2012). Research conducted by Eisenberg (1992) and Carlson (1998) suggested that it is possible that a significant number of AP Calculus students are passing the course and exam with a minimal understanding of functions. These claims were supported empirically by Judson and Nishimori (1995). In a study that compared 18 AP Calculus BC students with 26 Japanese students in an equivalent class, they found that "all students lacked a sophisticated understanding of functions," (p. 39) despite demonstrating a strong understanding of calculus concepts. This result conflicted with the notion that a high understanding of functions is a prerequisite to take AP Calculus, as the College Board stated (2010). It is important to note, however, that Judson and Nishimori did not focus on the entire scope of function understandings when they made the above assertion. Instead, they only focused on a few select aspects of functions, such as students' understandings of constant functions and function compositions. Therefore, there is still a need for an investigation of AP Calculus students' understandings of all aspects of functions, as well as a comparison of those understandings to their understandings of calculus concepts by way of their performance on the AP Calculus Exam.

The purpose of this study was to ascertain AP Calculus students' understandings of functions and to compare their understandings with their performance on the AP Calculus Exam. Specifically, the study sought to answer the following research questions:

1) *What do students completing the AP Calculus course know and/or understand about functions?*

2) *To what extent is there alignment between AP Calculus students' understanding of functions and their performance on the AP Calculus Exam?*

Theoretical Frameworks

To help answer these questions, this study used two theoretical frameworks that arose from research about how students learn about and understand functions. The first is NCTM's Five Big Ideas of Functions (Cooney, Beckmann, Lloyd, Wilson, & Zbiek, 2010). In this framework, function understandings were organized around five "Big Ideas" that teachers must focus on in their instruction: 1) The Function Concept; 2) Covariation and Rate of Change; 3) Families of Functions; 4) Combining and Transforming Functions; and 5) Multiple Representations of Functions. These Big Ideas serve as a framework that teachers can use as a way to shape and strengthen their students' learning and understanding of functions. They are also well connected to some of the fundamental concepts of calculus. For example, several aspects of the derivative (Sofronos, et al., 2011) are related or connected to one or more of the Big Ideas:

- The derivative as a rate of change (Big Idea 2)
- The various rules of derivatives (Big Ideas 3 and 4)
- The derivative and antiderivative are inverses of each other (Big Idea 4)
- Multiple representations of the derivative (Big Idea 5)
- The derivative as a limit vs. the derivative as slope. (Big Ideas 1, 2, and 5)

It could be inferred from these properties of derivative just how well connected it is to those of functions, and thus how important it is to have a strong conception of functions in order to better understand the derivative. This is also the case for other important aspects of calculus, such as limits and integrals. Essentially, students with a strong conceptual understanding of functions should succeed in calculus.

In this study, the Big Ideas were used as way to organize function concepts for the purposes of assessing AP Calculus students' understanding of them.

The second theoretical framework is APOS (Action-Process-Object-Schema) Theory (Asiala, et al., 1996), which arose from research that focused on the idea that a student's learning of functions is highly influenced by how he or she *views* functions (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Dubinsky & Harel, 1992). In short, students with an *action view* of functions see them as merely a means for performing a particular action, such as computation. Meanwhile, those with a *process view* see a function as the defining relationship between its input and output. The *object view* of functions and the function *schema* require even more sophisticated thinking. Several researchers (Breidenbach, et al., 1992; Carlson, Oehrtman, & Engelke, 2010; Dubinsky & Harel, 1992; Oehrtman, Carlson, & Thompson, 2008) have claimed that students need at least a process view of functions in order to develop a strong understanding of them, and have used APOS Theory to help explain students' impoverished function sense. Carlson, et al., (2010) asserted that a process view is necessary for students who wish to develop their covariational reasoning (Big Idea #2), which they cited as a critical component to a student's calculus readiness. In this study, AP Calculus students' views of functions were determined in order to help gain further insight into their overall understanding of functions and to see how they related to performance on the AP Calculus Exam

Methodology

In order to answer the research questions, 85 students from AP Calculus classes in three different schools were tested on their understandings of functions at the end of the course, after they had already taken the AP Calculus Exam. Since taking the AP exam was optional at one of the schools, only 67 of the participants actually took the exam. All participants were administered the *Precalculus Concept Assessment,* or PCA (Carlson, et al., 2010), a 25-item multiple-choice exam designed to assess students' knowledge of concepts that are central to precalculus and foundational to calculus. The majority of the PCA's taxonomy is centered on functional reasoning and the understanding of function concepts, including whether a student has an *action* or *process* view of functions. Additionally, the PCA taxonomy was systematically mapped to the five Big Ideas of Functions in order to show how each item assessed understanding of one or more of the Big Ideas.

The data collected from the PCA included the participants' chosen answers for each item as well as all written work that was done during the exam. Once completed, the PCAs were scored in two different ways. First, each item was coded as *correct* or *incorrect,* based on the answer key for the PCA, which can be found on the PCA website (Arizona Board of Regents, 2007). Second, each item was scored on a 0-4 point scale, with the number of points awarded based on the answer selected and, in some cases, the work produced by the student on that item. Four points are awarded for a correct answer, while incorrect answers received anywhere from 0 to 3 points. Each item on the PCA has its own scoring rubric, which is primarily based on the official analysis done on each item answer by the PCA developers, which can also be found on the PCA website (Arizona Board of Regents, 2007).

Once the PCA items were scored, each participant's understanding of functions was classified as *strong*, *moderate*, or *weak* based on the results. Participants with at least 21 correct answers or 90 points were classified as having a *strong* understanding of functions. Likewise, those with at least 13 correct answers or 70 points were classified as *moderate*, while all other participants were classified as *weak*. Each participant's function view was then determined as either *action* or *process* by scoring the items specifically designed to assess function view. Participants were identified as having a *process* view if they answered at least half of these items (8 of 16) correctly. The point scale was not used in determining function view, as only correct answers could be used as indicators of one having a *process* view (Carlson, et al., 2010). Next, understandings of all five Big Ideas were determined by scoring subsets of items respective to each Big Idea. Each of these understandings was also classified as *strong*, *moderate*, or *weak*. This was done in the same way that the whole PCA was coded. That is, the same scale was used to determine the cut-off scores for classifying the strength of understanding each participant had for each Big Idea. The collection of items aligned with each Big Idea was scored and the total number and the percent of correct answers in that set were counted. Participants were classified as being *strong* in that Big Idea if they scored at least 90% of the total points earned by those items, or answered approximately 84% of the items correctly. Similarly, they were coded as *moderate* for that Big Idea if they scored 70% of the points or answered at least half of the items correctly. Otherwise, they were considered to have a *weak* understanding of that Big Idea. Finally, the AP Calculus Exam scores, measured from 1 to 5, were collected once they were released and compared to the PCA results.

Results

Of the 85 participants in the study, 43 of them (51%) had a *weak* understanding of functions, while only seven (8%) had a *strong* understanding. The average number of correct answers on the PCA was 12.66 (s.d. = 4.584), and the average score on the partial credit scale was 65.73 (s.d. = 15.052), both of which fall just short of a *moderate* score. After running a 1-Way Between-Subjects ANOVA, it was also found that a student's AP Calculus Exam score was highly correlated with his or her function understanding ($F = 13.286$, $p < .001$). Most students with a *strong* understanding of functions had scores of 4 or 5 on the AP Calculus Exam, while most students with a *weak* understanding generally had scores of 1 or 2. Details of the comparison between function understanding and AP Exam scores can be found in Table 1.

Table 1. Breakdown of AP Exam scores for each level of understanding on the PCA.

AP Score	Strong	Moderate	Weak
5	5	5	0
4	1	6	2
3	1	5	7
2	0	2	8
1	0	11	14
AP Average	4.57	2.72	1.9
AP St. Dev.	.787	1.579	.978

There was also a high correlation between students with a *process* view of functions and at least a *moderate* understanding of them. There were 45 participants that displayed a *process* view on the PCA, and all but seven of them (84%) had at least a *moderate* understanding of functions, including all participants who had a *strong* understanding. Meanwhile, participants with an *action* view almost exclusively had *weak* understandings. Of the 40 participants with an *action* view, 36 of them (90%) had a *weak* understanding of functions. Also, almost all students (75%) who scored at least a 3 on the AP exam had a *process* view of functions, while the majority of students who had an *action* view (70%) scored only a 1 or 2 on the exam. See Table 2 for the comparison between function view and AP Exam scores.

Table 2. Breakdown of AP Exam scores for each function view.

AP Score	Process	Action
5	10	0
4	8	1
3	6	7
2	5	5
1	11	14
AP Average	3.03	1.81
AP St. Dev.	1.577	.962

It was also found that no Big Idea was significantly understood more than any other, but the understanding of Big Idea 5 (Multiple Representations) was generally lower than any of the other Big Ideas. Finally, the results of 1-Way Between-Subjects ANOVA tests run on all five Big Ideas across AP Exam scores indicated that there were mostly significant differences between the scores of students with different levels of understanding for each Big Idea. The only exceptions to this were for comparisons between *moderate* and *weak* students for Big Idea 2 (p = .103), Big Idea 3 (p = .582), and Big Idea 4 (p = .06). Chi-square tests for independence (df = 4) were also run for all Big Ideas, and all five tests were found to be significant. That is, understandings of each Big Idea were related to AP Exam scores. The full comparisons between each Big Idea and AP Exam scores are found in Table 3.

Table 3. Total participants to get each exam score for each level of understanding of each Big Idea.

AP Score	Big Idea 1			Big Idea 2			Big Idea 3			Big Idea 4			Big Idea 5		
	S	M	W	S	M	W	S	M	W	S	M	W	S	M	W
5	6	4	0	7	2	1	7	2	1	3	6	1	5	4	1
4	1	8	0	1	5	3	1	4	4	1	7	1	1	4	4
3	2	6	5	1	8	4	3	5	5	1	7	5	0	7	6
2	0	6	4	0	3	7	1	0	9	0	7	3	0	1	9
1	0	14	11	0	9	16	1	10	14	0	12	13	0	5	20
X^2 Test	$X^2 = 31.346$, $p < .001$			$X^2 = 39.012$, $p < .001$			$X^2 = 29.042$, $p < .001$			$X^2 = 16.437$, $p = .037$			$X^2 = 36.621$, $p < .001$		

Note: All X^2 tests were significant.

Function Understandings Needed for Success in AP Calculus

There was a very strong correlation between the participants' understandings of functions and their performance on the AP Calculus Exam. This correlation reinforces the idea that knowledge and understanding of functions is what is needed in order to succeed in an AP Calculus course (College Board, 2010). However, some researchers have given more specific parameters for just what aspects of functions are needed for calculus success. In particular, Oehrtman, et al. (2008) argued that a process view of functions (from Big Idea 1) and strong covariational reasoning skills (from Big Idea 2) are essential to understanding the primary concepts of calculus. First, of the 40 students with a *process* view who took the AP Exam, 24 of them (60%) scored a 3 or higher. However, of the 27 students with an *action* view who took the exam, just one student scored higher than a 3, and only 7 others (26%) scored a 3 (refer to Table 2). Therefore, it could be said that a process view is necessary for success in calculus, but it is not sufficient.

As for covariation, 24 of the 32 participants (75%) who scored at least a 3 on the exam had at least a *moderate* understanding of Big Idea 2 (refer to Table 3). All but three of these students also had a *process* view of functions. However, there were also 9 students who scored less than 3 on the exam despite having both a *moderate* understanding of Big Idea 2 and a *process* view of functions. So perhaps a *moderate* understanding of covariation is not a strong enough predictor for success in calculus. Instead, it could be the case that having both a *process* view and a *strong* understanding of covariation is both necessary and sufficient for success in calculus. The results of this study support this theory, as 7 of the 9 students (78%) who had both a *process* view and a *strong* understanding of Big Idea 2 scored a 5 on the exam, with the other two scoring a 4 and a 3, respectively. Since every student with a *strong* understanding of Big

Idea 2 also had a *process* view of functions, it is likely that the latter is necessary in order to develop the former. This is supported by the idea that a strong indicator of a *process* view is the ability to identify and explain how a function's input and output are related, and that part of that relationship is how the input and output covary. In general, the results of the study support the claims of Oehrtman, et al. (2008), in that both a process view of functions and high skill with covariational reasoning is essential to success in calculus.

It should also be mentioned that function view makes up a large part of understanding Big Idea 1. In fact, every student classified as having a *process* view of functions had at least a *moderate* understanding of Big Idea 1, and all students with a *strong* understanding of Big Idea 1 also had a *process* view. More information about the ties between function understanding and success in AP Calculus is revealed when the focus is switched from function view to the level of understanding of Big Idea 1, as now the difference between *strong* and *moderate* understandings of Big Idea 1 can be accounted for. For example, the average exam scores for participants with a *strong* and *moderate* understanding of Big Idea 1 (not including the small subset of *moderate* students with an *action* view) are 4.44 and 2.61 respectively, a difference of almost 2. When also accounting for covariation, there were 12 students who took the exam and were classified as *strong* in at least one of Big Ideas 1 and 2. For these students, the average AP Exam score was 4.5. For the 19 students with *moderate* understandings of both, the average score was only 2.47, a difference of over 2. This further supports the claims of Oehrtman et al. (2008), while shedding more light on the degree to which students should understand the function concept and covariation. It seems as if a good understanding of both is necessary, while a strong understanding of at least one is essential to success in calculus.

One of the other Big Ideas that is likely necessary for calculus success is Big Idea 5, *Multiple Representations.* All but one of the students classified as *strong* in Big Idea 5 scored a 5 on the AP Exam (with the remaining student earning a 4), and 29 of the 34 students who were *weak* in Big Idea 5 scored a 1 or 2 on the exam (refer to Table 3). Therefore, it seems as if students proficient in working in multiple representations of functions are more likely to have success in calculus. This is most likely due to the notion that fluency with different representations of functions is indicative of a strong understanding of functions in general (Cooney, et al., 2010; Kaput, 1998; Keller & Hirsch, 1998; Moschkovich, et al., 1993), and includes the ability to easily move between representations and to choose an appropriate representation to work in.

Finally, while it is important to note which ideas of functions seem to be most related to success in AP Calculus, the overall results suggest that at least good understandings of all aspects of functions are correlated with strong performances in calculus. There were 21 participants who were not classified as *weak* in any of the five Big Ideas of functions. Of these participants, 17 of them (81%) scored a 3 or higher on the exam, 13 of whom (76%) scored at least a 4, and the average score for these students was 3.67. In contrast, of the 46 participants who were *weak* in at least one Big Idea, only 15 of them (33%) scored at least a 3 on the exam, only 6 of whom scored a 4 or a 5 (13%), and the average score was merely 2.02. This is an important result, as it reinforces the notion that functions are a key prerequisite to understanding and succeeding in calculus. This breakdown of AP Scores between students who had at least one *weak* Big Idea and those who did not can be seen in Table 4.

It should be noted that the specific relationships between understandings of different function concepts and performance on the AP Calculus Exam are all based on the students' understandings after having taken a full year of calculus, and it is not known whether these correlations would have been found if the participants' understandings of functions were measured at the beginning of the course instead. It is difficult to speculate on how much taking AP Calculus influenced the function understandings of the participants at the end of the course.

Table 4. AP Scores of students who did or did not have at least one *weak* Big Idea.

	N	**5**	**4**	**3**	**2**	**1**	**μ**	**σ**
No Weak BIs	21	8	5	4	1	3	3.67	1.39
≥ 1 Weak BIs	46	2	4	9	9	22	2.02	1.19

Conclusions, Implications, and Further Research

There were two major ideas that can be taken from the findings in this study. First, students with a good understanding of functions at the end of an AP Calculus course tend to score higher on the AP Calculus Exam than those without one. The second idea is that many students completing AP Calculus still need to significantly improve their understandings of functions, and they tend to have difficulty with many of the concepts described above.

Current research has explained the various ways in which students understand each of the different aspects of functions, and discussed common misconceptions and difficulties that students often have with functions (e.g., Leinhardt, et al., 1990; Oehrtman, et al., 2008; Thompson, 1994). The Big Ideas of functions were developed from much of this research as a way for teachers to organize their teaching of functions (Cooney, et al., 2010). There have also been studies that determined college calculus students' understandings of functions (e.g., Carlson, 1998; Carlson, et al., 2010). The current study is significant in that it is the first that produces similar insight at the high school level, and in AP Calculus in particular. It provides information about the importance of the understanding of functions for AP Calculus students, as well as which aspects of functions appear to be the most related to success in AP Calculus. Given the vast number of students taking AP Calculus every year, learning how to improve their understandings and readiness for the course is critical, especially since some colleges and universities have stopped accepting AP scores for credit, claiming that many former AP students enter college with a lack of the foundational knowledge and skills needed to succeed in their programs (Ben-Achour, 2013).

The findings of this study have several implications for teachers and developers of the secondary mathematics curriculum. First, they reinforce the idea that developing students' conceptualization of functions is very important in preparing them to take calculus. It is recommended that high school mathematics departments place an emphasis on the teaching of functions, with an aim toward especially developing a process view of functions, covariational reasoning skills, and skill with interpreting, producing, and moving between different representations of functions. Precalulus teachers may wish to consider using NCTM's Big Ideas of Functions as a means to organizing their teaching of functions, as this is what they were initially designed for (Cooney, et al., 2010). Additionally, teachers of AP Calculus may choose to implement a placement exam like the PCA for students who wish to enter the course. Recall that the PCA has shown to be a strong predictor of success in calculus, even though it was not specifically designed to be a placement exam (Carlson, et al., 2010). Implementing such an exam would help limit the number of students who take the course without a well-developed understanding of functions.

This study also leaves room for future research in order to address some of the limitations of the study as well as any new questions generated from the findings. For example, the participants in this study all took the PCA *after* they had taken the AP Calculus Exam. It is possible that their understandings of functions could have changed in some capacity from the time they entered the course. In learning calculus, some students may have strengthened their function understanding. It is also possible that some may have had their understandings and skills with certain aspects of functions regress. Therefore, a similar but more comprehensive study could be conducted in which AP Calculus students' function understandings are measured at the beginning of the course as well as at the end. Multiple interviews could also be conducted with a subset of students at given intervals during the year to gain further insight into their understandings. Again, their conceptualizations of functions would be compared to their scores on the AP Exam, and also with their performance in the class itself. The findings of such a study would help provide secondary mathematics teachers with further information about students' levels of proficiency with functions as they are entering AP Calculus, and it would give them an idea of how well the course itself influences their understandings of functions. That is, it would help answer the question, "How much learning of functions occurs *during* a calculus class?" Another possible future study would be a similar investigation of AP Calculus BC students. It would be expected that those who take the BC course generally have a stronger understanding of functions than those in the AB course. The results would help highlight some of the similarities and differences between AP Calculus AB and BC students.

Finally, a larger study that focuses on the development of function understanding from the end of algebra 2 to the end of precalculus could also be very informative. How does a student's understanding of functions actually change over time? Does function understanding grow any faster with the use of one curriculum or textbook as opposed to another? The findings of such a study could help teachers and curriculum developers identify aspects of functions teachers need to focus on how they might be organized and presented.

References

Arizona Board of Regents (2007). *Precalculus Assessment (PCA).* Retrieved January 23, 2013 from http://mathed.asu.edu/instruments/pca/index.shtml.

Asiala, M. Brown, A. DeVries, D. J., Dubinsky, E., Matthews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In: E. Dubinsky, A. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education* (Vol. 2, pp. 1-32). Providence, RI: American Mathematical Society.

Asiala, M., Cottrill, J., Dubinsky, E., & Schwingendorf, K. E. (1997). The development of students' graphical understanding of the derivative. *The Journal of Mathematical Behavior, 16*(4), 399-431.

Baker, B., Hemenway, C., & Trigueros, M. (2001). On transformations of basic functions. In: H. Chick, K. Stacey, & J. Vincent (Eds.), *Proceedings of the 12th ICMI Study Conference on the Future of the Teaching and Learning of Algebra* (Vol. 1, pp. 41-47). University of Melbourne.

Ben-Achour, S. (2013, February 20). More colleges stop giving credit for AP exams. *Marketplace.* Retrieved January 19, 2014 from http://www.marketplace.org/topics/life/education/more-colleges-stop-giving-credit-ap-exams.

Breidenbach, D., Dubinsky, E., Hawks, J. & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics, 23*(3), 247-285.

Carlson, M. (1998). A cross-sectional investigation of the development of the function concept. *Research in Collegiate Mathematics Education III, Conference Board of the Mathematical Sciences, Issues in Mathematics Education, 7*(2)*,* 114-162.

Carlson, M., Oehrtman, M. & Engelke, N. (2010). The precalculus concept assessment: A tool for assessing students' reasoning abilities and understandings. *Cognition and Instruction, 28*(2), 113-145.

Clark, J., Cordero, F., Cottrill, J. Czarnocha, B., DeVries, D., St. John, D., Tolias, G. & Vidakovic, D. (1997). Constructing a schema: The case of the chain rule. *Journal of Mathematical Behavior, 16*(4), 345-364.

College Board. (2010). *AP Calculus course description.* Retrieved August 14, 2011 from http://www.collegeboard.com/student/testing/ap/sub_calab.html.

College Board (2012). *Student Score Distributions.* Retrieved October 16, 2012 from http://media.collegeboard.com/digitalServices/pdf/research/student_score_distributions_2012.pdf.

Common Core State Standards Initiative. (2010). Common Core State Standards for Mathematics. Retrieved August 11, 2011 from http://www.corestandards.org/the-standards.

Confrey, J. & Smith, E. (1991). A framework for functions: prototypes, multiple representations, and transformations. In: R. Underhill & C. Brown (Eds.), *Proceedings for the thirteenth annual meeting of Psychology of Mathematics Education-NA* (pp. 57-63). Blacksburg, VA: Virginia Polytechnic Institute Press.

Cooney, T. J., Beckmann, S., Lloyd, G. M., Wilson, P. S., & Zbiek, R. M. (2010). *Developing essential understanding of functions for teaching mathematics in grades 9-12.* Reston, VA: NCTM.

Dubinsky, E. & Harel, G. (1992). The nature of the process conception of function. In: G. Harel & E. Dubinsky (Eds.), *The concept of functions: aspects of epistemology and pedagogy* (pp. 85-106). Washington, DC: Mathematical Association of America.

Eisenberg, T. (1992). On the development of a sense for functions. In: G. Harel & E. Dubinsky (Eds.), *The concept of functions: aspects of epistemology and pedagogy* (pp. 153-174). Washington, DC: Mathematical Association of America.

Elia, Panaoura, Eracleous, & Gagtsis (2006). Relations between secondary pupils' conceptions about functions and problem solving in different representations. *International Journal of Science and Mathematics (5),* 533-556.

Ferrini-Mundy, J., & Gaudard, M. (1992). Secondary school calculus: preparation or pitfall in the study of college calculus? *Journal for Research in Mathematics Education, 23*(1), 56-71.

Gerson, H. (2008). David's understanding of functions and periodicity. *School Science and Mathematics, 108*(1), 28-38.

Judson, T. W., & Nishimori, T. (2005). Concepts and skills in high school calculus: an examination of a special case in Japan and the United States. *Journal for Research in Mathematics Education, 36*(1), 24-43.

Kaput, J. J. (1998). Representations, inscriptions, descriptions, and learning: A kaleidoscope of windows. *Journal of Mathematical Behavior, 17*(2), 265-281.

Keller, B. A. & Hirsch, C. R. (1998). Student preferences for representations of functions. *International Journal of Mathematical Education in Science and Technology, 29*(1), 1-17.

Kimani, (2008). *Calculus students' understandings of the concepts of function transformation, function composition, function inverse and the relationships among the three concepts.* Unpublished doctoral dissertation, Syracuse Unversity, Syracuse, NY. Retrieved August 1, 2011, from Proquest Dissertations and Theses database.

Knuth, E. J. (2000). Student understanding of the Cartesian connection: An exploratory study. *Journal for Research in Mathematics Education, 31*(4), 500-507.

Leinhardt, G., Zaslavsky, O., & Stein, M. (1990). Function, graphs and graphing: Tasks, learning and teaching. *Review of Educational Research, 60*(1), 1-64.

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In: T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 69-100). Hillsdale, NJ: Erlbaum.

NCTM. (2000). *Principles and standards for school mathematics.* Reston, VA: NCTM.

Oehrtman, M. C., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and practice in undergraduate mathematics* (pp. 27-42). Washington, DC: Mathematical Association of America.

Sofronos, K. S., DeFranco, T. C., Vinsonhaler, C., Gorgievski, N., Schroeder, L. & Hamelin, C. (2011). What does it mean for a student to understand the first-year calculus? Perspectives of 24 experts. *The Journal of Mathematical Behavior, 30*(2)*,* 131-148.

Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. In E. Dubsinky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in collegiate mathematics education I. (*Vol. 4*,* pp. 21-44). Providence, RI: American Mathematical Society.

Ubuz, B. (2007). Interpreting a graph and constructing its derivative graph: stability and change in students' conceptions. *International Journal of Mathematical Education in Science and Technology, 38*(5), 609-637.

Vinner, S. & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Educaiton, 20*(4), 356-366.