Student-Generated Processes for Developing Mathematical Understanding

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Abstract. While understanding mathematical ideas has been well-articulated from researchers' perspectives, supporting students' mathematical learning benefits from the incorporation of the voices of students. The constructivist grounded theory study reported in this presentation describes how grade 12 students generated and expressed their processes for understanding mathematical ideas within a didactic context. *The Framework for Student-Generated Processes for Understanding Mathematics* illuminates the interrelationships of four processes: *breaking down*, *putting together*, *connecting*, and *writing down*. Students' use of these processes contributed to an increased understanding of mathematical procedures and empowered them in reforming their identity as capable mathematical learners.

I think I'm more open to understanding why something happens and – I understand the reason stuff happens now. It's not just – not just do it. I understand why it's happening. (Laurel)

So once I can connect those two [mathematical principles] then it's like all these doors open up and I start to understand everything. (Elise)

The students participating in the reported study had reformed their intentions for mathematical learning, from memorizing to gain marks to understanding procedures and concepts presented in mathematics class. Most often, they aimed at developing understanding that focused on why procedures worked and at constructing meaning of mathematical ideas through connections as can be seen in the opening statements. Making mathematical ideas for themselves – coming to understand – is what constituted learning for the students.

Scholarship in mathematics education describes *understanding* in a variety of ways (e.g., Hiebert & Carpenter, 1992; Pirie & Kieren, 1989, 1994; Sierpinska, 1994). Students in the study aimed at what Skemp (1976/2006) delineated as relational understanding, developed through personal inquiry into mathematics (Borasi, 1992) as a way to "experience understanding when they can create and manipulate mathematical objects" (Cobb, Wood, Yackel, & McNeal, 1992, p. 598). Standards documents emphasize the importance of students' developing understanding as a means toward proficiency (Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics, 2000, 2014). *Understanding* implies that students are actively developing mathematical ideas for themselves in a personally meaningful way.

Purpose

This report is situated within a broader inquiry. The larger study inquired into students' learning to learn mathematics, as they were developing learning strategies (e.g., categorizing types of questions, identifying key mathematical ideas, creating unit summary sheets) aimed at mathematical understanding. Imbedded within each of these learning strategies, students were generating similar processes for developing understanding. The question framing this facet of the study was: **What processes do students generate and express for developing understanding within didactic high school mathematics classes?**

Mode of Inquiry

I framed the study methodologically using constructivist grounded theory [CGT] (Bryant & Charmaz, 2007; Charmaz, 2009, 2014), which returns to the symbolic interactionist root of grounded theory while looking through a constructivist lens as an interpretive process for inquiring into dynamic phenomena. Within this postmodern orientation, theory is constructed by a researcher on a provisional basis and contingent to the context. There is an "emphasis on processes, making the study of action central" (Charmaz, 2006, p. 9), recognizing that shifts in people's actions and experiences signify growth and changes within the people and their interactions. The researcher, seen as a subjective knower, is immersed in the research setting while co-constructing qualitative data with participants. As data is analyzed abductively, the researcher moves from rich empirical data through levels of abstraction toward developing a mid-range interpretive theory. Processes like coding, memoing, categorizing, theoretical sampling, saturation, and sorting are offered as "systematic, yet flexible guidelines for collecting and analyzing qualitative data … rather than formulaic rules" (Charmaz, 2006, p. 2). CGT supports a focus on students' development of mathematical understanding as a processual phenomenon by interpreting empirical data of students' experiences, rather than applying extant theories. The use of CGT also responds to the growing importance in theorizing to make progress within the field of mathematics education (Hiebert, 1998; Proulx, 2010).

Data Sources

The study occurred in an academically-focused suburban school in a Western Canadian city. Thirteen grade 12 students who were taking a pure mathematics course volunteered to participate in the study. The course was offered in a didactic format where the teacher lectured, students copied out worked solutions to examples, and then students completed similar questions as homework. The students were enrolled concurrently in a course, *Mathematics Learning Skills*, which provided support for their mathematical learning. In the class, students worked on homework and requested help from the teacher. Within the course, I assisted the teacher in providing mathematical support and coaching students to improve their approaches to learning mathematics, simultaneously collecting data.

Data Collection

Data collection occurred over four months. After observing each class, I wrote detailed field notes of students' (inter)actions in the class and descriptions of daily informal conversations with the teacher. Students took part in bi-weekly interactive journal writings (Mason & McFeetors, 2002). They responded to prompts about how they were learning mathematics and I replied in order to interact with their ideas, modeling thinking about learning and fostering a relationship with each student. Students were placed into one of three small groups with a focus on developing a learning strategy as a group (transitioning from notes to homework, developing big ideas from completed homework, and studying for unit tests by creating summary sheets). Each small group met for three to five sessions of approximately 30 minutes each. The students also participated individually in two informal interviews as a retrospective look at their progress in shaping their learning strategies. Each interview was approximately 30 minutes and occurred halfway through the study and at the end. While the interactions were intended as multiple sources of data, they also afforded students the opportunities to develop processes of learning to support their mathematical understanding and to notice the qualities of their understanding. Providing these opportunities was framed by Dewey's (1938/1997) notion of experience which is characterized by continuity and interaction and where activity is transformed into experience through the reflective act.

Data Analysis

Using line-by-line coding and the constant comparative method (Glaser & Strauss, 1967), data analysis involved the development of *in vivo* codes. The codes were refined through several passes through the data. Codes such as "applying" and "relating" highlighted students' ways of understanding mathematical ideas. Codes, such as "connecting", were elevated to category labels and represented an abstract understanding of four primary processes. These four categories and their relationships constitute the *Framework for Student-Generated Processes for Understanding Mathematics*, I constructed through the interpretive act of theorizing by exploring the relationships of the categories. The categories and framework are described in the results section below.

Perspective

Within a symbolic interactionist approach, I adopted Blumer's (1954) notion of sensitizing concepts to "merely suggest direction along which to look … providing clues and suggestions" (pp. 7-8). The researcher's sensitivities – what the researcher attends to because of her/his experiences of conducting research, scholarship in the field, and interests – are explored and employed as a starting place in data analysis. My sensitizing concepts developed out of two previous research projects which attended to students' processes of learning mathematics (Mason & McFeetors, 2007; McFeetors, 2003; 2006). The four sensitizing concepts are: intentions, voice, identity, and relationships with sources of knowledge. After each explanation, I provide examples of literature for those who desire a more thorough discussion.

Intentions are internal constructs which give meaning to actions. These thoughts and desires arise from attention to previous experiences and to the consequences of actions, often through reflection. When students are intentional, they are acting with the intentions they have formed and hold, to move toward a particular aim. This aim, as an end-in-view, is fluid and the method of moving toward it contains ambiguity. Intentions point to what students want to do or achieve and a notion of how they might go about doing. So, intentions both mark an aim and a process. (See Bereiter & Scardamalia, 1989; Searle, 1983.) *Voice* points toward having space and confidence to say things and to do so, a reflective stance to make sense of experience through conversation, and being deeply implicated in actively shaping oneself. Voice is dynamic concept, one in which a student's voice is continually being refined through experience and through the voicing of the experience and growth of self. (See Baxter Magolda, 1992; Confrey, 1998.)

Identity is an understanding or sense of self. It is a dynamic processes, where the (re)forming of identity is continually undertaken through experiences and relating with others. While occasionally marked by large shifts, (re)forming identity is more often seen as shaping a way of being in the world and understanding that way of being. Shaping an identity is the ongoing negotiation of a student's relationship with mathematics, learning, schooling, others – identity is malleable and complex. (See Britzman, 1994; Sfard & Prusak, 2005.) *Relationships with sources of knowledge* – such as teachers, peers, and textbooks – point to students' beliefs about knowing and coming to know (epistemological stances) which are inextricably connected to the experiences of learning mathematics. The relationships could include dependence, independence, and interdependence and are often illustrated through examples of where authority in mathematical knowledge lay and through the choice of approaches to learning. (See Belenky, Clinchy, Goldberger, & Tarule, 1997; Chickering & Reisser, 1993.)

Results

The students in the study had a broad repertoire of processes for understanding mathematical content which include: *breaking down*, *putting together*, *connecting*, and *writing down*. These four processes constitute the categories within the *Framework for Student-Generated Processes for Understanding Mathematics* (see Figure 1). The students identified the processes as integral to their sense-making of mathematical ideas and explicitly named (Freire, 2000) these processes.

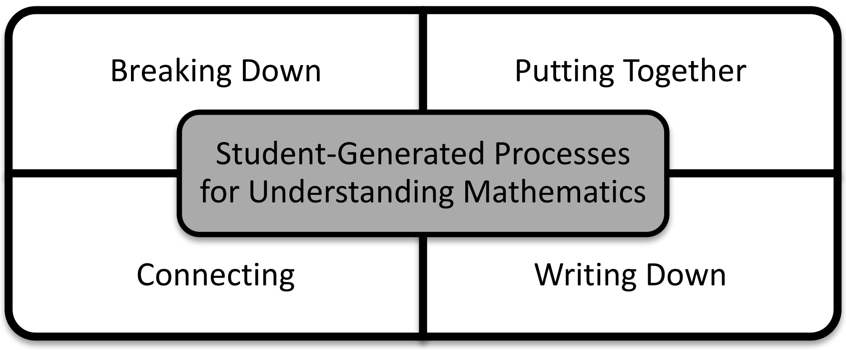

Figure 1. Framework for Student-Generated Processes for Understanding Mathematics

Breaking Down

Students developed the process of *breaking down* as they were making meaning of a symbolic system recorded as worked solutions in their homework or notes. Elise offered *breaking down* as a way by which she made sense of all the identified component parts of a single lesson, working from a completed homework assignment. She explained that when she used the learning strategy of identifying key mathematical ideas, it "broke down what it actually meant. So, I understand what to do when I have a question." Grace echoed the idea and added that "it's easier to understand." *Breaking down* as a way of learning mathematics held potency for the students, so much so that Grace incorporated and used the phrase numerous times to explain her mathematical learning.

The students saw the mathematics teacher's presentation of a lesson as a whole in which the component parts were largely inaccessible. Vanessa pointed to the difficulty to access component parts when she commented on not understanding the notes after class or being able to use them to begin a homework assignment. In response, Vanessa worked with Teresa to create transition pages that broke down the content from notes given to them in mathematics class. As an example, they worked on *breaking down* the component parts of sigma notation. As both students clarified the mathematical terminology, they recorded three different ways of referring to the lower and upper limits of a series written in sigma notation. They were making meaning of the mathematical symbols and terms through their multiple ways of recording and explaining the component parts of sigma notation. The process of *breaking down* provided the opportunity for the students to make sense of each of the mathematical elements within a lesson by generating the meaning of each element for themselves.

Some of the students in the study develop summary sheets, a set of papers which summarized the content over a whole unit. Starting one lesson at a time, Chelsea pointed out that she could "break up ideas … I don't try to write too much on a sticky note … I kind of separate the ideas." She would identify a singular idea to write about on a sticky note. In the use of sticky notes, the separating of ideas was both visual and tactile, allowing Chelsea to see one specific idea at a time. It was in deciding what mathematical idea she would write on each sticky note that Chelsea was *breaking down* component parts of each lesson. The students needed the opportunity to see and make sense of all the component parts of a lesson, ascertaining these for themselves, before they built the parts back up into big ideas for a single lesson.

Putting Together

Not only was *breaking down* the mathematical ideas in a lesson important for the students, but the action that often followed was to reassemble the mathematical ideas into a coherent whole. The students named this process for making sense of mathematical ideas as *putting together*. Nadia explained, "If I have the main idea and then everything that could possibly be underneath that and I continue doing that for all the different information, then you can see that I really have everything put together." After *breaking down* ideas from a lesson, Nadia explained what it meant to collect the fragments of mathematical ideas, reconstructing the cohesive whole that had been presented in class by the teacher. It is important to note that the scale of *putting together* existed at the level of a singular lesson within a unit. Within the students' mathematics class, a lesson usually focused on a single topic (e.g., Fundamental Counting Principle, reflections of functions, laws of logarithms).

Chelsea emphasized that in order to understand grade 12 mathematics topics, "you have to put it all together." In creating a summary sheet for a whole unit, Chelsea first worked through each lesson individually. As she did this, she was it as an opportunity where the mathematical ideas "come together … 'cause they look like separate ideas if they're – and then if they do combine, then, at the end, just do that." Even the prompt she posed to herself, "Oh, where can I put this?" indicates that she was considering how to *put together* ideas in a lesson as she created a summary sheet. In this way, *breaking down* content to understand elements of a lesson was not an isolated process for understanding, but was done in a way that allowed the students to combine the individual elements that were now sensible to them.

Interestingly, the written records of many of the students' learning strategies represented the action of *putting together* in their final form. While this is a visible record for an observer to notice and to think about, the students' use of *putting together* pointed to their activity of how mathematical ideas within a lesson was being enacted while in process, rather than pointing to the static document at the end. The process of *putting together* highlights the notion that the students were generating mathematical ideas for themselves as a way of making sense of the ideas presented in their mathematics class. The whole of the mathematical ideas in the lesson were now of the student's making and having, rather than the teacher's providing.

Connecting

When students were *breaking down* and *putting together*, they were developing relationships among mathematical ideas within a lesson. The scale of the lesson was a single mathematical topic. The relationship-building, however, was not limited to connections within lessons as students named *connecting* as the process for understanding that occurred when they made connections across a unit or multiple lessons. The scale of *connecting* was larger in comparison to *putting together*. Rather than making sense of the intricacies of a specific procedure within a lesson as with *putting together*, students were able to identify relationships between previously learned ideas and new mathematical content in *connecting*.

Nadia recognized *connecting* ideas across units was important for her sense-making because "when we start on the next chapter and it's a continuation on from the previous one and it kind of, it all builds off the previous chapters." She also recognized *connecting* ideas across grades when she pointed out the fundamental counting principle is connected to "tree diagrams, that's even in junior high .. it kind of just builds off of that." Nadia was expressing an emergent understanding of fundamental counting principle in seeing why the structure of it worked in light of her previous knowledge of tree diagrams. Vanessa extended Nadia's connection when Teresa introduced the factorial notation to her in a small group session and Vanessa exclaimed, "Oh, I get it! … It's like this, we don't want to draw, like a tree diagram so then you actually, oh okay. Yeah, yeah. It's like faster." Vanessa was in the moment of figuring out factorial notation and was developing a justification for the computation and notation. *Connecting* mathematical ideas across lessons enabled the students to generate relationships that rendered the mathematical ideas sensible. In this way, the process of *connecting* afforded learning for understanding.

Danielle explained the process of *connecting* when making unit summary sheets: "I don't really connect them to, till the end. … then I get it more. … It's mostly in my head. I just make those connections by myself. And then --, usually I remember them, though." Danielle would record individual ideas on sticky notes, and then move the sticky notes around to demonstrate connections for "a constant reminder of why are we doing something." Danielle repeatedly emphasized the justification, or the "why," of a procedure when she was generating her own connections. Ashley demonstrated her *connecting* directly on her summary sheets as she would "highlight or draw after in arrows how these ideas connect to the different parts of the chapter." For these students, *connecting* meant they were generating relationships for themselves, not by taking on the teacher's structuring of content, but by generating an interconnected structure of the content that made sense to them.

Students were building several different types of connections. They were *connecting* mathematical ideas within units; for example, understanding the inverse quality of exponential and logarithmic functions. They were *connecting* mathematical ideas across multiple units; for example, understanding tree diagrams as the underlying structure for the fundamental counting principle. They were *connecting* mathematical ideas with experiences outside of mathematics class; for example, Pascal's Triangle and creating mazes. Although an outside observer might see *putting together* and *connecting* as the same process, for the students these processes differed in their uses and in their results because of the difference in scale of the context of the relationships created. For each of these *connecting* activities, the students demonstrated that they were generating relationships among ideas for themselves. The students' authority in making sense of mathematical ideas emerged through this process.

Writing Down

Another process for making meaning of mathematical ideas that the students named is that of *writing down*. While occasionally used in a casual sense, *writing down* carried with it the intensity and intentionality of understanding mathematical ideas. The students would commit to paper and commit to words only those mathematical ideas that they had rendered sensible. The choice of language is distinct from scribing notes in class explained by Shane as being "like a drone – copied down the notes" or by Grace as "you gotta take all the notes. Gotta take good notes." In fact, Grace continued on to explain that in addition to taking the teacher's notes that she would also "write down little notes for myself." She named these little notes as "side notes." Grace explained the side notes as, "It's writing the numbers and then beside it why you did it." The explanation of the steps for a worked solution was the focus of what Grace was *writing down* beside the symbolic steps provided by the mathematics teacher.

An aspect that was important to the students in *writing down* was that they used words they perceived as their own. When I asked Grace further about her side notes, she was emphatic that, "Oh, it's in my words!" Vanessa concurred that, "when you write something in your own words and own – like, how you'll get it, not how someone else will, you kind of know the material better." A possible insight as to why the students felt that putting mathematical ideas in their own words was important is that by doing so they were active as opposed to simply enacting something they were told to do.

When Danielle explained how she was going about creating her summary sheet, she mentioned, "And so then I just wrote everything down individually. Make sure I understand them." In this way, Danielle was saying that she would only commit to paper that which she understood. Often *writing down* was seen as a culminating activity by the students. Even so, ideas were sometimes written down in a provisionary manner, marking the understanding the student had at that moment as a dynamic activity. Kylee explained that at the end of class, "I just wrote down … the main important points" to represent her emergent understanding. Later in the unit, she would refine her understanding by *writing down* mathematical ideas on cue cards in order to, as she explained, "I'll write down … to make sure I understand and then – it's all definitely quite an active process." Kylee's *writing down* was dynamic as she continually refined the mathematical ideas she was recording as she made sense of them for herself. The students would commit to words only those ideas that they had rendered sensible.

Discussion

The students engaged in a generative activity of making mathematical ideas for themselves. While the mathematical ideas were presented in class by their teacher through representing worked examples, the students viewed this as a whole conceptual object. This whole was an asserted fact, one that belonged to the teacher and to the domain of mathematics to be given with the expectation that it would be taken. The students identified that their activity in response was to memorize concepts and procedures. Within the learning strategies, though, students were *breaking down* the whole into the component mathematical ideas. Once the elements were identified and each rendered sensible, the students would reassemble the mathematical ideas to reconstruct a cohesive whole. It was in this reassembling, through the processes of *putting together* and *connecting* that the students were generating the mathematical ideas for themselves. Woven throughout these three processes, the students were *writing down* mathematical ideas that held meaning for them. Kylee explains that "writing it down is definitely learning it in my own ways." Rather than passively receiving the mathematical content from their teachers, the students were transforming the ideas (Borasi & Siegel, 1990). Taken together, each of the four processes for understanding the students shaped and used afforded opportunities for meaning-making. It cannot be overstated that that processes for understanding reported were also developed by the students, and held potency because they were student-generated.

More profoundly, I had glimpses into how the students were beginning to see themselves as individuals who could understand mathematics. Through generating processes and developing an understanding of mathematics, the students saw themselves as sources of knowing, as they were able to shape mathematics ideas through verbalizing their understanding and ways of coming to understand. Grace reported, "You can tell how I'm improving. ... I actually understand what I'm doing." Many of the students had comments that echoed Grace's, demonstrating that they were seeing themselves as capable of understanding mathematics and capable of learning mathematics. The students expressed a desire to understand mathematical ideas. They acknowledged that a memorization-based approach had left them struggling. In contrast to their context, they engaged in an inquiry approach independently (Houssart, 2001) that enabled them to transform the mathematical ideas through the processes they generated: *breaking down*, *putting together*, *connecting*, and *writing down*. They were authoring mathematical ideas for themselves.

The results of this study advance our possibilities for student action in mathematics class which leads to understanding mathematical ideas and to developing identities as capable mathematical learners. In conclusion, I offer four points of significance to consider:

1) *Intentionality*: The students' success in learning for understanding has significance for how students can be supported in developing mathematical understanding regardless of class context. The study explicates how students reformed their intentions for learning, which can in turn be encouraged with other students.
2) *Processes for Understanding*: The students' naming of the processes they generated and their elucidating of the development is innovative, providing important insights into the nuances of such sense-making processes that might otherwise be overlooked. The results emphasize that students' personal construction of processes for understanding is just as important as their construction of mathematical ideas.
3) *Voicing Ways of Learning*: A substantial contribution is that the students were empowered as learners in becoming aware of the ways they are coming to understand mathematical ideas. This developed as the research design allows for "eliciting and gathering evidence of student understanding" (NCTM, 2014, p. 56). Implications for research design are also raised, where *hermeneutic listening* (Davis, 1996) frames a way of noticing that attends to tentative shifts in students' epistemological positioning.
4) *Future Research*: The framework provides an analytic tool in order to interpret the experiences of students in other classrooms who aim at mathematical understanding, inviting other mathematics educational researchers to offer further student-generated processes.

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