Teachers' Development of Learning Trajectories: Engaging in Mathematical Practices

Shelby P. Morge

University of North Carolina Wilmington

Kathleen Lynch-Davis

Appalachian State University

Drew Polly

University of North Carolina Charlotte

David Pugalee

University of North Carolina Charlotte

Abstract

One of the six courses in the North Carolina Elementary Mathematics Add-On Licensure Program has a main focus on the topics of rational numbers and measurement and the high leverage teaching practice of learning trajectories. A major course assignment has the teachers enrolled in the course develop a learning trajectory by describing a goal, laying out a developmental progression, and selecting appropriate tasks to support students’ advancement through that progression. In this paper, we will discuss the course, assignment, what we have noticed about teachers’ difficulties with setting a goal, sequencing the trajectory, selecting appropriate mathematical tasks that are correlated with instances on the trajectory and how we can support them in demonstrating the mathematics teaching practices.

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In North Carolina, faculty from seven campuses collaborated to create an add-on licensure program in elementary school mathematics. This licensure follows the recommendations for elementary mathematics specialists as outlined in the (2010) AMTE document, *Standards for Elementary Mathematics Specialists: A Reference for Teacher Credentialing and Degree Programs.* The EMAoL in mathematics for North Carolina requires the following 18 semester hours of course work:

(3 s.h.) – Number Systems & Operations: K-5 Mathematical Tasks

(3 s.h.) – Geometry & Spatial Visualization: K-5 Assessment

(3 s.h.) – Algebraic Reasoning: K-5 Discourse & Questioning

(3 s.h.) – Rational Numbers & Operations: K-5 Learning Trajectories

(3 s.h.) – Data Analysis and Measurement: K-5 Classroom Interactions

(3 s.h.) – Mathematical Modeling: K-5 Leadership

This set of courses reflects the need for elementary teachers to develop depth of content knowledge in mathematics, but also for them to develop expertise in a set of critical mathematical pedagogical practices. This information will help elementary teachers make sense of the use of learning trajectories as one of the crucial research foundations for the CCSS-M, help them understand elementary mathematics content in a profound way (Ma, 2009), and help them understand and be able to apply mathematical practices as specified in the CCSS-M.

One of the courses has a main focus on the topics of rational numbers and measurement and the high leverage teaching practice as learning trajectories. A major course assignment (see Appendix 1) has the teachers develop a learning trajectory by describing a goal, laying out a developmental progression, and selecting appropriate tasks to support students' advancement through that progression (Sarama & Clements, 2009). This assignment encourages teachers to "establish mathematics goals to focus learning" and "implement tasks that promote reasoning and problem solving which should be a part of every mathematics lesson" (National Council of Teachers of Mathematics (NCTM), 2014, p. 3). Faculty who have taught this course have noticed anecdotally common errors in the teachers' understandings of learning trajectories. In this paper, we will discuss the course, assignment, what we have noticed about teachers' difficulties with setting a goal, sequencing the trajectory, selecting appropriate mathematical tasks that are correlated with instances on the trajectory and how we can support them in demonstrating the mathematics teaching practices.

Literature Review

Learning Trajectories

Simon (1995), a pioneer of the construct of hypothetical learning trajectory, defines his hypothetical learning trajectory model Simon wrote: "The consideration of the learning goal, the learning activities, and the thinking and learning n which the students might engage make up the hypothetical learning trajectory." (Simon, 1995, p. 133) Simon posits that there are three components: the learning goal, the learning activities, and the hypothetical learning process. Not dissimilar from Simon, Clements' and Sarama's (2004) refinement of learning trajectory tends to more explicitly state descriptions of children's' thinking. It is important to note that in both instances, these trajectories do not result in linear interpretations. Children's understanding is different and is not always known and their path for learning and understanding is fluid and not static.

The importance of the development of learning trajectories for mathematics conceptual domains is beginning to take a stronghold in the mathematics education community. The Consortium for Policy Research in Education (CPRE) recently published a report on *Learning Trajectories in Mathematics* (2011) espousing that learning trajectories is work that should be undertaken by mathematics educators and valued by funding agencies. Specifically, they call for work on consolidating learning trajectories. Learning trajectories have the potential to develop paths of knowledge that could inform teachers of what they should be teaching and where their students should be along a projected learning path (CPRE, 2011). The National Research Council (NRC) propose that learning trajectories offer “successively more sophisticated ways of thinking about a topic that can follow and build on one another as children learn about and investigate a topic over a broad span of time” (p. 211). Sztajn, Confrey, Wilson, and Edgington (2012) posit that “learning trajectories represent the initial steps toward a theory of teaching that is centered around research on learning” (p.152). Because “there are a lot of similarities among these trajectories, there are also some differences, and researchers tend to defend and advance their own ideas. The field needs to come together to review this work and consolidate it” (CPRE, p. 56).

Consequently, discussions about learning trajectories have also gained traction due to the increased focus on how the Common Core State Standards (CCSSO, 2010) were developed and are structured. Learning trajectories or progressions have become common language in mathematics education. Research on learning trajectories has described them as ranging from a linear path to a landscape of learning (Clements & Sarama, 2004; Confrey, Maloney, Nguyen, Mojica & Myers, 2009; Fosnot & Dolk, 2002; Sarama & Clements, 2009). Regardless of how trajectories or progressions are considered, they are always complex and not always reliable (Empson, 2011).

Learning Trajectories on Rational Numbers

Multiple trajectories appear in the research on the progression of students' thinking about rational numbers (Confrey et al, 2009, Empson & Levi, 2011, Simon & Tzur, 2004, Steffe & Olive, 2010). Confrey, Maloney, etc. map out a trajectory for rational number understanding from kindergarten to fifth grade that begins with fair shares and concludes with ratios, percents, and decimals. Empson and Levi (2011) also focus on developing initial fraction understanding through connections to sharing, explaining that that knowledge learned with understanding is rich in connections and generative. Simon and Tzur (2004) share a rough outline of a trajectory focused on students understanding the relationship between a fraction and an equivalent fraction whose denominator is a multiple of the original fraction. Through lesson implementation, they add tasks and elaborate on the trajectory to better support student understanding. Steffe and Olive (2010) hypothesize that rational number knowledge can emerge as a result of whole number knowledge, which is contradictory to research that students overgeneralize their whole number knowledge and that impedes rational number understanding (McNamara & Schaughnessy, 2010).

Researchers have not identified one established path or best route to student understanding of rational number concepts. In fact, rational number learning trajectories have been referred to as provisional (Simon, 1995) and hypothetical (Simon & Tzur, 2004). The uncertain nature of rational number trajectories may be attributed to the topic of rational numbers and the relationship between the teacher, the students, and their mathematical understandings.

Teachers' Work with Learning Trajectories

The research base on teachers' work with learning trajectories is sparse. After conducting a teaching experiment and a study on a learning trajectory about first grade measurement and addition, Gravemeijer, Bowers, & Stephan (2003) concluded that teachers "will need to construct their own hypothetical learning trajectories on a day-to-day basis..teachers will need to take into consideration their knowledge of their own students, the instructional history of the class, and the end points they envision (p. 64).

Lesh and Yoon (2004) argue that learning trajectories should be viewed more broadly and that not every student has the same learning trajectory based on their backgrounds and experiences. Further, they posit that teachers need to know the mathematical concepts as well a range of mathematical tasks and activities that can support students' achievement of the mathematical goals.

Wilson, Mojica, & Confrey (2013) found that pre-service teachers benefited from working with a learning trajectory on rational number reasoning. Specifically, the pre-service teachers were able to effectively create models of students' thinking and develop a better sense of students' reasoning.

Methods

Research Questions

The study was framed by the following research questions:

1. To what extent were teachers in a graduate program able to construct a learning trajectory for rational number concepts?
2. How did the learning trajectories reflect an understanding of rational number concepts?
3. How did the learning trajectories reflect an understanding of learning trajectories?

Participants

The participants in this study were 41 middle grades and elementary teachers as well as math coaches who were enrolled in the rational number course (described above). Most of the participants are completing course work to obtain their elementary mathematics add-on license. In addition, two of the participants are not currently practicing teachers.

The course in which the assignment is housed is one of six in a sequence of courses leading to an elementary mathematics add-on license in North Carolina. The course was developed by faculty members from two universities, one in the eastern and one in the western part of the state. It was piloted, and since then has been taught at least five times.

Data Sources and Data Analysis

The "Introduction to Learning Trajectories" assignment was collected from two of the faculty teaching the course. . After the courses were taught and grades were submitted, faculty reviewed the submitted assignments. Faculty members teaching the course wanted to know to what extent teachers were able to develop learning trajectories for rational number concepts. Using an inductive qualitative approach, individual faculty members reviewed the assignments submitted by teachers in their course for three categories based on assignment guidelines, goals, developmental path, and identifying appropriate tasks. Faculty rated each student's paper as emerging, developing, or proficient in each of the three categories. The categories are defined in the table below. For a more detailed explanation of the assignment see Appendix 1.

	Emerging	Developing	Proficient
Goals - Describes a goal and explains why it is an important concept for students to understand.	All parts of this section have been thoroughly addressed which means that responses are thoughtful, clear and to the point.	All parts of this section have been adequately addressed which means that responses are not all clear and to the point and/or can benefit from additional further elaboration	One or more parts of this section have not been addressed or are not adequately addressed which means that responses are cursory and/or difficult to ascertain
Developmental Path -1) Indicates a reasonable timeline for reaching the goal 2) Identifies content knowledge that students will be expected to develop at that stage 3) Each level is more sophisticated than the last and contributes to understanding of original goal			
Identifying Appropriate Tasks – 1) Two specific mathematical tasks are described that are matched to the levels of thinking 2) Tasks help students learn the ideas and practice the skills needed to master the level of thinking and push students to reflect on the intended concepts and relationships			

Table 1. Categories Defined

After rating each paper, faculty looked at overall understandings and misunderstandings and identified common themes. Common themes and patterns for each category were vetted and examples were identified to support the themes.

Findings

Most teachers in this study struggled with this assignment. The table below shows the number of students who were rated at each category.

	Emerging	Developing	Proficient
Goals	12	15	14
Developmental Path	12	16	12
Identifying Appropriate Tasks	19	9	12

In addition to ratings, common themes and patterns emerged in the data in each of the categories described above.

Goals

In the assignment, the teachers are asked to select an overall goal and describe why it is important for their students to learn. This gives us some insight into the teachers' understanding of the mathematical topic and how it is connected to other areas of mathematics. In addition, it demonstrates teachers' abilities to write student learning goals.

We have found that the teachers are challenged by describing an overall goal and often select a vague goal or a very specific Common Core Standard. An example of a proficient goal: "Students will be able to understand that two fractions are equivalent (equal) if they are the same size, or on the same point on a number line. This is cited in Common Core Mathematics as objective 3.NF.3a. In third grade, students have been exposed to fractions as a geometry goal in kindergarten through second grade. In second grade students have learned to partition into equal parts of halves, thirds, and fourths. They have also learned that three thirds, four fourths, etc., is a whole. They have also learned that shapes of the same size can be partitioned differently, and still show fourths, thirds, and halves. This is what leads into third grade equivalency understanding. Students will need to see that equivalency must be among objects or sets of objects that are the same size, or located on the same point on a number line. This is a topic of great importance for students to understand, so that in the future when they are working with adding, subtracting, multiplying and dividing of fractional parts they can see why fractions can be renamed (simplified, common denominators, percentage, etc.) while working through mathematical computations. Knowledge of equivalency in fractions will also help them in comparing fractions and ordering fractional parts as they enter the fourth grade."

Developmental Path

Next, the teachers are required to create a progression that will support their students in reaching the overall goal. An example of a proficient path on equipartitioning was:

- Understand the term ***equal***
- Equipartition collections by attributes
- Equipartition a whole into correct number of parts, equal-sized parts, exhaust the whole
- Justify reasoning of equipartition
- Partition and iterate whole, ½, and ¼ and name part in reference to whole
- Justify the changes in partition size in reference to the number of shares required
- Partition an unequal share among persons sharing, i.e. 5 cookies with 4 children

However, teachers still faced difficulty with creating a progression or path. They often list sub goals in the progression that are not necessary, such as the need to simplify fractions or convert between improper fractions and mixed numbers when their overall goal is for students to be flexible with rational number operations. In addition, the steps in the teachers' progressions are often not built upon each other and highlight their misunderstandings of fraction concepts. For example, one teacher generated the following progression: a) Students will determine equivalent fractions using a number line. b) Students will produce visual models using sets and part/whole models to represent equivalent fractions. c) Students will express whole numbers as fractions and understand that 3/3=3/1, 4/4/=4/1, etc. d) Students will determine the greater or less fraction by utilizing visual models. This progression was meant to support students in "explaining fractions and comparing fractions when reasoning by their size while producing visual models and determining equivalent fractions using those visual models."

Identifying appropriate tasks

The teachers are then asked to identify tasks that match each sub goal in the progression. The discussion of tasks and the cognitive demand of tasks is the focus of a previous course in the add-on licensure program. Teachers tended to do better with this part of the assignment. An example of a proficient rated paper included the following "hexagon sandwiches" task to help students identify unit fractions.

> Use different combinations of pattern blocks that will build toppings for each yellow hexagon sandwich.
> -How many different sandwiches can you make?
> -To be different, a sandwich must use a different set of blocks from all other sandwiches.
> -Using your "Hexagon Recording Sheet", record each solution by tracing each of the different pattern blocks you are using to completely cover a hexagon sandwich.
> -On each solution, record fractional parts, using unit fractions.
> Example: 1/2 + 1/6 + 1/6 + 1/6 = 1 sandwich
> -How will you know when you have found all of the different ways to create toppings for the sandwiches? (no repeats and none missing)
> -Share solution strategies in groups and with the whole class.
> -How do you know when you have found all of the possible ways to add toppings to the hexagon sandwiches?

However, course participants, even with this prior knowledge, struggle with identifying appropriate tasks in their learning trajectory or progression. For example, one teacher provided the following task as an example to support students in using their understanding of multiplication to comprehend that a/b X c/d = ac/bd in simplest form. "Bonnie and Jared are having a birthday cake. Their mother cuts Bonnie's cake into ninths and Jared's cake into eighths. If you eat two slices, how much cake have you eaten? Whose cake would you need to eat two slices from to eat the most cake? The least? Explain your thinking and show your work."

Conclusions

In conclusion, teachers' development of rational number learning trajectories for the "Introduction to Learning Trajectories" assignment revealed flaws in their ability to establish a student learning goal, generate sub goals that build upon each other and support student understanding of the learning goal to form a coherent developmental path, and select tasks that correlate to the sub goals and encourage conceptual and procedural knowledge of the student learning goal. We question if the variation that appears in research on progressions of students' thinking about rational numbers (Confrey et al, 2009, Empson & Levi, 2011, Simon & Tzur, 2004, Steffe & Olive, 2010) is reflected in the complications that elementary teachers face when asked to look at their grade level standards and create a trajectory for student learning. In fact, this may be hindered because many districts provide explicit pacing or curriculum guides to teachers, so the work of defining a learning trajectory is no longer necessary.

The results of this assignment may also highlight deficiencies in the teachers' understandings of the mathematical concepts and lack of knowledge about tasks and activities that can support students' achievement of the mathematical goals (Lesh & Yoon, 2004). It was apparent in our courses and in the trajectories that K-2 teachers struggle to understand what the important foundational content knowledge, while upper elementary teachers may still be handicapped by their procedural understanding of the rational number content. This further highlights that a one-semester course on rational number and learning trajectories may not be sufficient to provide teachers with enough opportunities to be proficient in both the creation of a learning trajectory and the conceptual understanding of rational number at the elementary level.

Implications

For learning trajectories to be useful for teachers (the audience in which they are intended for), initial trajectories will need to be accessible. Teachers need to be able to unpack trajectories, modify those trajectories, and based on their mathematical understandings and knowledge of their students (Gravemeijer, Bowers, & Stephan, 2003) provide learning opportunities for their students. As mathematics teacher educators, we need to develop experiences for teachers to assist them in unpacking this knowledge and provide opportunities that will allow teachers to successfully use trajectories in their work with students. We are seeking strategies to scaffold this work with the teachers and possibly revise the "Initial Thinking about Learning Trajectories" assignment to encourage their capabilities in establishing mathematics goals to focus learning and implementing tasks that promote reasoning and problem solving (NCTM, 2014).

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Appendix 1 Assignment Guidelines and Rubric

Assignment #2 – Initial Thinking about Learning Trajectories

As teachers, learning trajectories help us answer several questions.

1. What objectives should we establish for our students?
2. Where do we start?
3. How do we know where to go next?
4. How do we get there?
5. How do we know when we are there?

Part 1: Review the readings "Teaching Math in the Primary Grades: The Learning Trajectories Approach" and Young Mathematicians at Work, Chapter 2 (along with other articles you have read) as references.

Part 2: Select a specific topic relate to rational numbers from the grade level you teach.

1. **Goal or topic:** What is your goal or topic? What does it mean or look like at your grade level? Why is it an important topic for students to learn?
2. **Developmental Path/Common Core Standards:** Select a timeframe to look at your topic (9-week period, academic year, K-5 curriculum, etc.) List the standards that address or are related to the topic at each segment of the timeframe. Break it into at least 5 segments.
3. **Instructional tasks:** Describe two activities or tasks that would help address the topic at each segment. The activities or tasks should be directly related to the standards listed in column 2.

Part 3: Then respond to the reflection questions listed below the table.

<table>
<tr><td colspan="3">1. Goal or topic:
Describe your goal or topic and explain why it is an important concept for students to understand.</td></tr>
<tr><th>Time Frame</th><th>2. Developmental Path/Common Core Standards</th><th>3. Instructional Tasks</th></tr>
<tr><td></td><td>Indicates content knowledge that students will be expected to develop grade level. What does the topic look like at each grade level?</td><td>1.Two specific mathematical tasks are described that are matched to the levels of thinking
2.Tasks help students learn the ideas and practice the skills needed to master the level of thinking and push</td></tr>
</table>

		students to reflect on the intended concepts and relationships

Part 3: Reflection Questions:

1. Based on your mathematical knowledge, does the current trajectory for the topic you selected make sense (as indicated in the Common Core Standards)? Explain why or why not using examples.
2. What additional prompts/questions or scaffolding might you include to help your students meet the expected landscape of learning goals? Provide examples to describe how you will attempt to move all students in your classroom along the developmental path.

Assignment 2 Rubric			
Assignment Component	**Exemplary**	**Proficient**	**Unsatisfactory**
Goal - *Describe your goal or topic and explain why it is an important concept for middle school students to understand.* **– 3 points**	All parts of this section have been *thoroughly* addressed which means that responses are thoughtful, clear and to the point. (3 pts)	All parts of this section have been adequately addressed which means that responses are not all clear and to the point and/or can benefit from additional further elaboration. (2 pts)	One or more parts of this section have not been addressed or are not adequately addressed which means that responses are cursory and/or difficult to ascertain (1 pt.)
Developmental Path – 1. Indicates content knowledge that students will be expected to develop at that stage 2. Each level is more sophisticated than the last and contributes to understanding of original goal	All parts of this section have been *thoroughly* addressed which means that responses are thoughtful, clear and to the point. (6pts)	All parts of this section have been adequately addressed which means that responses are not all clear and to the point and/or can benefit from additional further elaboration (4pts)	One or more parts of this section have not been addressed or are not adequately addressed which means that responses are cursory and/or difficult to ascertain (2 pts)

– 6 points			
Tasks – 1. Two specific mathematical tasks are described that are matched to the levels of thinking **2.** Tasks help students learn the ideas and practice the skills needed to master the level of thinking and push students to reflect on the intended concepts and relationships **- 9 points**	All parts of this section have been *thoroughly* addressed which means that responses are thoughtful, clear and to the point. (9 pts)	All parts of this section have been adequately addressed which means that responses are not all clear and to the point and/or can benefit from additional further elaboration (6 pts)	One or more parts of this section have not been addressed or are not adequately addressed which means that responses are cursory and/or difficult to ascertain (3 pts)
Reflection Questions – 1. Responses demonstrate reflective thought 2. Include explanations and justifications for statements **- 6 points**	All parts of this section have been *thoroughly* addressed which means that responses are thoughtful, clear and to the point.	All parts of this section have been adequately addressed which means that responses are not all clear and to the point and/or can benefit from additional further elaboration	One or more parts of this section have not been addressed or are not adequately addressed which means that responses are cursory and/or difficult to ascertain
Spelling/Grammar – 1 point	No spelling or grammatical errors		Includes spelling or grammatical errors
Total = 25 points			