Why Can You Switch the Order When You Multiply?

Children's Applications and Justifications of the Commutative Property of Multiplication

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Abstract

This interview study explores how 19 students from grades 4 through 12 attempt to justify the commutative property of multiplication. Harel and Sowder's (1998) taxonomy of proof schemes is used as a general framework for interpreting students' justifications. Students showed evidence of symbolic, authoritative, empirical, quasi-transformational, and transformational proof schemes. An important relationship was noted between students' ability to articulate why it makes sense to multiply to enumerate the number of objects in an array, and the production of a transformational justification of the commutative property. Two types of conceptions of the commutative property emerged: syntactic and structural. The results of this study informed the writing of several "conceptual learning goals" (Lobato, 2013) related to how students ought to understand the commutative property.

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The commutative property of multiplication (CPM) is a fundamental property of operations. The Common Core State Standards for Mathematics indicate that students should understand and apply the CPM beginning in third grade (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010, p. 23). It is not clear *how* the standards writers intended for students to understand the property, though they note that "one hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity *why* a particular mathematical statement is true or where a mathematical rule comes from" (p. 4).

This study, then, explores how students actually use and understand the CPM. Identifying the ways in which students think about the property can contribute to our knowledge of how they understand multiplication. This research can also help us articulate "conceptual learning goals" (Lobato, 2013) which specify how we believe students *should* understand the CPM. This study sits at the intersection of research focused on student development in the domain of number and operations, and research into the development of algebraic thinking. As Kaput (2008) writes, "Our activity can be termed *algebraic* when we are stating these properties explicitly and examining their generality—not when we are using them tacitly" (p. 13). When students use the CPM, they are exhibiting skill in the domain of number and operations. When they investigate the property, and justify why it works, they are engaging in algebraic thinking.

Background and Theoretical Framework

This study draws from research into several aspects of children's understanding of multiplication, including the relative difficulty of different problem types (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Vergnaud, 1983), the strategies children use to solve problems (Ambrose, Baek, & Carpenter, 2003; Carpenter et al., 1999), and the schemes of action and operation that underlie children's strategies (Confrey, 1994; Steffe & Cobb, 1998; Steffe, 1988, 1994). Several studies previously investigated children's use and understanding of the CPM. Two studies found that populations who do not study multiplication in school (e.g., indigenous groups in Africa and Brazilian street sellers) are unlikely to apply the CPM, while those who do study multiplication in school routinely apply it (Petitto & Ginsburg, 1982; Schliemann, Araujo, Cassundé, Macedo, & Nicéas, 1998). Various studies found that young children generally do not apply the CPM when solving contextual problems in which the multiplier and multiplicand are clearly defined, though they may apply it to number-only problems (Ambrose et al., 2003; Baek, 2007; Carpenter et al., 1999; Nunes & Bryant, 1995; Vergnaud, 1983). Some authors suggest that children may find it more intuitive to apply the CPM to problems involving an array context (Ambrose et al., 2003; Carpenter et al., 1999; Nunes & Bryant, 1995). Battista and colleagues, however, caution that children do not initially see arrays as multiplicative structures; this is something they must construct over time (Battista, Clements, Arnoff, Battista, & Van Auken Borrow, 1998). Finally, two studies describe some of the justifications of the CPM produced by children in grades 3 and 6. Children justified the property by citing a rule, generating examples, and using array-based justifications (Bastable & Schifter, 2008; Valentine, Carpenter, & Pligge, 2005).

Methods

In this study, nineteen students spanning grades 4 through 12 were interviewed for the purpose of learning more about how students understand the CPM. The study considered the following research questions: (1) Do students use the commutative property? If so, in what circumstances? (2) How do students justify the commutative property? (3) Is there a relationship between the strategies that children use to solve multiplication problems and their justifications of the commutative property?

During the interviews, students solved a range of contextual and number-only multiplication problems and described their strategies. Then, students were asked to justify their uses of the CPM. The interviews were semi-structured, (Bernard, 1988), which allowed me to adapt the interview to students across a wide range of ages and mathematical abilities, and to make multiple attempts to elicit students' most sophisticated justifications of the CPM. The interviews were transcribed, and the transcripts were analyzed to identify strategies the student used to solve multiplication problems, instances in which the CPM was applied, and how the student justified the CPM. Students' informal strategies were categorized using categories adapted from Ambrose et al. (2003) and Carpenter et al. (1999). A process of open coding was used to determine categories of justification of the CPM, although this process was also informed by the research literature (e.g., Bastable & Schifter, 2008; Harel & Sowder, 1998; Nunes & Bryant, 1995; Valentine et al., 2005).

Results

All students used the CPM at least once when solving number-only problems. Far fewer students used the property when solving contextual problems. Because all students used the property at least once, all students were asked to justify the CPM. Harel and Sowder's (1998) taxonomy of proof schemes was used as a framework for interpreting students' justifications of the CPM. Students articulated authoritarian, symbolic, empirical, and transformational justifications for the CPM. Justifications classified as quasi-transformational or transformational proof schemes revealed the most about students' understanding of multiplication and the commutative property. Quasi-transformational justifications included (a) a balancing conception (e.g., 5 groups of 7 is the same as 7 groups of 5 because with 5 groups, there are fewer groups and more in a group, while with 7 groups, there are more groups, but fewer in each group); (b) a justification based on equivalent partial products (e.g., 8×13=13×8 because both are 80+24); and (c) a partition model (e.g., 9×6=6×9 because 54 can be partitioned into 6 segments of 9, or into 9 segments of 6). Transformational justifications included array-based justifications and justifications based on a rectangular area model. Few relationships were observed between students' strategies for solving multiplication problems and their justifications of the commutative property. A significant relationship was noted, however, between students' ability to articulate why it makes sense to multiply to enumerate the number of objects in an array, and the production of an array-based justification of the CPM. If students were not able to see the rows and columns of the array as iterable, composite units, they were not able to articulate how an array could represent both $a \times b$ and $b \times a$.

Discussion

Two types of conceptions of the CPM emerged from this study: syntactic and structural. In a syntactic conception, the property is a rule dictating how the symbols in an expression can be manipulated while still maintaining equivalence. There are two parts to the syntax: you can switch the order of the factors, and you get the same answer either way. Structural conceptions of the CPM draw on mental imagery connected to students' conceptions of multiplication. Additionally, structural conceptions incorporate the "logical necessity" (Simon, 2011) of the numerical equivalence of the two products.

The results of this study informed the writing of several "conceptual learning goals" (Lobato, 2013) related to how students *ought* to understand the CPM. The goals specify the meanings, mental images, ideas, connections, ways of comprehending situations, and explanations that comprise a given concept. The first goal names the understanding necessary for a syntactic understanding of the CPM. The remaining goals articulate aspects of understanding necessary for a structural understanding of the CPM:

1. When solving number-only problems, students should understand that the commutative property means that the factors can appear in either order without changing the product.
2. Students should be able to articulate a justification of the commutative property that is based on an array model. The justification should make use of the row-and-column structure of the array and identify composite units in the rows and/or columns. For example, students should be able to explain that $5\times7=7\times5$ because 5 groups of 7 and 7 groups of 5 can both be identified in the array. Students' justifications may rely on rotating the array, so that an array that initially has 5 rows of 7 becomes an array with 7 rows of 5; or students' justifications may note that if you see the rows as groups, there are 5 groups of 7, but if you see the columns as groups, there are 7 groups of 5.
3. Students should be able to articulate a justification of the commutative property based on a rectangular area model. They should understand that by rotating a rectangle, the side that was the base becomes the height, and the side that was the height becomes the base. The amount of area enclosed by the rectangle, however, is unaffected by the rotation.
4. Students should understand that the commutative property is a property of the operation of multiplication, and thus can be applied to any multiplication problem,

including problems in which the multiplier and multiplicand are clearly defined by the problem context.

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