Convincing Arguments?

Preservice Teachers' Initial and Post-course Views

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Abstract

Recommendations for increased emphasis on mathematical argumentation at all grade levels present particular challenges for the preparation of preservice elementary teachers (PTs). To inform the design and implementation of courses that address these challenges, I interviewed five PTs, eliciting their views of nine mathematical arguments at two points in time, near the beginning of a course emphasizing mathematical argumentation and shortly after its completion. Analyzing their responses using an interpretative, phenomenological approach, I found that they initially preferred arguments in which the arguer thought or acted in accordance with their image of a superior or advanced mathematics student. They therefore preferred arguments in which they perceived the arguer as (a) knowing what to do, (b) finding the correct answer, (c) showing quick solutions with standard mathematical symbols. In contrast, after completing the course, they focused more on (d) understanding the problem, (e) finding solutions that made sense, and (f) using diagrams to explain why solutions worked. They generally showed a substantial shift away from arguments based on empirical evidence or standard mathematical procedures and an increased preference for arguments that explained underlying concepts in terms of actions on quantities.

Background and Purpose

The NCTM (2000) and the Common Core State Standards Initiative (2010) recommend that students engage in mathematical argumentation—constructing mathematical arguments and critiquing the arguments of others—as a regular feature of mathematics instruction at all grade levels. However, research on instruction in U.S. classrooms has shown little evidence of this practice (e.g., Stigler & Hiebert, 1999). Furthermore, studies of preservice elementary teachers (PTs) have revealed weaknesses in their understandings of mathematical justification, even after completing courses that addressed this issue, raising concerns about PTs' readiness to implement this recommendation (e.g., Martin & Harel, 1989; Morris, 2007).

Responding to this problem, many teacher education programs offer courses that provide experiences in mathematical argumentation analogous to those the NCTM and CCSSI recommend for K-12 students. To inform the design and implementation of such courses, I conducted a research study focused on two questions: (a) As PTs compare and evaluate mathematical arguments, what do they initially consider important, and how do their views change by the end of a one-semester course emphasizing mathematical argumentation? (b) What roles do the form and substance of the argument play in PTs' evaluations, and how do these roles change by the end of the semester?

Theoretical Framework

For this study, I classified mathematical arguments according to: (a) the type of task that the argument addressed, (b) the form of the argument, and (c) the substance of the argument.

Argument Construction Tasks

To categorize tasks that elicit mathematical arguments, I adapted Stylianides and Ball's (2008) framework for classifying proving tasks, grouping tasks into four types, depending on the statement under consideration. This focal statement could be either *true* or *false* and either *specific* (involving a single case) or *general* (involving multiple cases). The task in Figure 1, for example, focuses on a true general statement.

A fourth grade class has noticed that 45×32 and 32×45 both have the same answer, 1440. The teacher asks them if every multiplication problem can be done in either order. Will the results always be the same both ways? If so, why?

Caitlyn: Yes, they will always be the same. I saw it in a math book. It's called the commutative property of multiplication. It says $a \times b = b \times a$. That means you can multiply in either order, and the answers will be equal.

Evan: Yes, multiplying both ways will always give the same result. Look at 3×5, for example. That's like counting three rows of dots with five in each row (see pictures at right). If you make it 5×3, that's five rows with three dots in each row. Both are going to have the same number of dots altogether. It's just flipping the picture sideways.

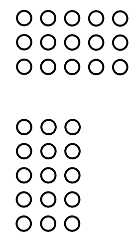


Figure 1. An argument construction task with two sample arguments.

The Form of an Argument

Forms of argument differ according to task-type. For tasks directed at true general statements, I included arguments representing three of Simon and Blume's (1996) categories: (a) appeal to authority, (b) empirical argument, and (c) generic example. Figure 1 includes examples of (a) and (c). For tasks directed at false general statements, I included refutation by counterexample, and for tasks directed at specific statements, I drew on Toulmin (1958), distinguishing arguments with false warrants or warrants without backing from single-case explanatory arguments similar to generic examples.

The Substance of an Argument

Toulmin (1958) cautioned against assessing validity based solely on an argument's form, emphasizing that its substance must also be considered. I therefore followed Skemp (1987) in distinguishing arguments that focus on relationships among symbols from those that explore underlying relationships among the concepts the symbols represent. In Figure 1, for example, Caitlyn's argument focuses on symbol-level relationships, whereas Evan's addresses underlying concepts.

Setting, Data Sources, and Methods

Setting

The PTs in the study were enrolled in *Learning and Teaching Number and Operation in the Elementary School,* a course intended to (a) promote a deeper understanding of number and operation, focusing particular attention on developing quantitative understandings of fractions and place value, and (b) connecting this deeper understanding to the learning and teaching of mathematics in the elementary grades. The class followed an inquiry mathematics approach in which the instructor posed challenging problems and established classroom norms that required the PTs to (a) explain their solutions in small-group and whole-class discussions, (b) listen thoughtfully to the explanations (i.e., arguments) of others, and (c) pose questions to classmates if their explanations required clarification. The problems often led to conflicting solutions, and the instructor neither ratified correct solutions nor dismissed incorrect ones. Instead, he encouraged class members to resolve their differences by carefully considering the arguments presented and determining which ones made sense. The nearly constant engagement in presenting, questioning, and evaluating mathematical arguments made this course an ideal setting for the study.

Data Sources

Data sources included: (a) PTs' written responses to a preliminary survey and (b) transcripts of 50-minute initial and post-course interviews with five selected PTs. Each of these focused on *Thinking about Students' Explanations*, a set of four problems designed to elicit PTs' views of mathematical arguments from various categories in the theoretical framework. Each problem showed an argument construction task and two or three sample arguments. (See Figure 1.) PTs were asked to decide which they found convincing and to give reasons for their choices. By using these problems in the initial survey, I obtained an overview of the arguments PTs preferred, allowing me to identify issues that merited further investigation and select interviewees who could shed light on them. By returning to the same problems in the initial interview, I could investigate their views of these arguments more deeply, and by revisiting them in the post-course interview, I could explore how their views had changed. The complete problem set from *Thinking about Student's Explanations* appears in Appendix A.

Methods of Analysis

Reading transcripts from each interviewee separately, I added notes to passages that addressed interviewees' reasons for preferring one argument to another. Based on my notes, I developed codes for emergent themes, organized them under superordinate themes and theory-driven themes, and wrote a case report on each interviewee, which I shared with its subject to obtain feedback on my interpretations. After completing this process for all five interviewees, I used the case reports as a reduced data set for cross-case analysis, eventually obtaining nine superordinate themes, six theory-driven themes and 60 subthemes. Each superordinate theme was supported by data from at least three interviewees. (See Table 1 for some examples.)

Findings

What PTs Considered Important

Interviewees initially valued mathematical arguments in which the arguer thought or acted in accordance with their image of a superior or advanced mathematics student. They therefore preferred arguments in which they perceived the arguer as (a) knowing what to do, (b) finding the correct answer, (c) showing quick solutions with standard mathematical symbols. In contrast, after completing the course, they emphasized: (d) understanding the problem, (e) finding solutions that made sense, and (f) using diagrams to explain why solutions worked. In the sections that follow, I briefly describe and illustrate themes from the initial survey and first interview along with contrasting themes from the post-course interview.

Table 1

Subthemes, Descriptions, and Examples Illustrating the Theme, "Getting the Correct Answer."

Subtheme	Description	Example
• Correct therefore convincing	Interviewees cited the correct answer as a reason for endorsing a particular argument.	"I chose Flavia's response, because she explained each of her steps, reduced fractions, and got the correct answer" (Corey, Preliminary Survey).
• Not best but still correct	Interviewees stated that some arguments they did not prefer were still correct.	"I didn't like [Evan's argument] as much, but it is correct. I just preferred Caitlyn's and Dawn's" (Ermida, Interview 1).
• Correct in context	Interviewees indicated that conflicting answers should be considered correct, due to the context in which they appear.	"At this point, Heather is right. From what she's given and what she has learned, she is correct" (Ermida, Interview 1).

Knowing what to do and understanding the problem. As the interviewees talked about the problems in *Thinking about Students' Explanations* in their initial interviews, they demonstrated a shared initial belief that, when presented with such problems, students should ideally know what to do and automatically pursue a course of action that would lead to the correct answer. From this viewpoint, problems are essentially unproblematic. Errors result, not from failure to understand and interpret problems correctly, but from students' failures to remember and follow the correct procedure. In the post-course interviews, however, the PTs devoted much more time and attention to analyzing the problem and relating it to the mathematical ideas they had explored over the course of the semester. Corey's comments in Table 2 illustrate these two contrasting positions.

Table 2

Contrasting Themes: "Knowing What to Do" and "Understanding the Problem"

Knowing What to Do (Early in the Course)	Understanding the Problem (Post-Course)
In elementary school, ... you get it drilled in your brain, so when you see it, you know it. Rather than having to figure it out for [ourselves], we knew these things. We had to know automatically, so [we] could build off that base, on to further math. (Corey) [Georgia] forgot to realize that, when adding fractions, the common denominator remains the same. (Corey)	[The problem] states that this [indicating the two jugs in Georgia's picture] is the whole, that both of these create the whole. … They are combining two one-gallon jugs together, which [means], if you have one-eighth of one gallon and three-eighths of the other gallon, when combined, you would have four-sixteenths. (Corey)

Finding the correct answer and finding solutions that make sense. When comparing arguments near the beginning of the semester, the interviewees emphasized correct answers in several different ways. When presented with two or more arguments that reached the same conclusion, one that they viewed as correct, they often avoided choosing one argument as superior, saying instead that they were all equally convincing. When I pressed them to make a choice, they tended to express it as a personal preference, rather than an objective evaluation, noting that the arguments they did not choose were still correct. In other situations where the arguments reached conflicting conclusions, they emphasized that their preferred argument had the correct answer—or at least they believed so. In the post-course interviews, however, they placed more emphasis on whether the answer made sense in the context of the problem, rather than whether it resulted from following a procedure correctly. Excerpts from Grace's interviews illustrate these tendencies (see Table 3). I should also point out that, in her initial interview, she stated that she could not remember whether you were supposed to add the denominators when adding fractions. However, during her post-course interview, she seems to know that this is the standard procedure but rejects it because it fails to make sense in this context.

Table 3

Contrasting Themes: "Finding the Correct Answer" and "Finding Solutions that Make Sense"

Finding the Correct Answer (Early in the Course)	Finding Solutions that Make Sense (Post-Course)
All three of them explained where they got it [their answer] from and why they did it that way. So that's why I thought they were all equally convincing. (Grace) I do think maybe [Beth's] answer is more convincing, but they both are correct. (Grace) I liked Georgia's answer. ... I didn't think [Flavia's argument] was convincing, because I didn't think it was correct. (Grace)	G: To me, it makes sense that it's four-sixteenths. But maybe it's just confusing me, because if there was no word problem, and you were just [asking me to] add one-eighth and three-eighths, then … I would come up with four-eighths. ... M: Okay. So maybe that's the question. Is this a problem where you should add fractions in the usual way? G: No. Because it really doesn't make sense to me, if you're adding two jugs of water and just all a sudden have the same amount.

Showing quick solutions with symbols and explaining why with diagrams. The participants initially preferred brief arguments that used "numbers"—standard mathematical symbols—to show how the answer was obtained. In the post-course interview, they often rejected these "shortcuts" if they failed to understand why they worked or if they thought the arguer lacked this understanding. Instead, they preferred arguments that used diagrams to explain why solutions worked. Table 4 illustrates these ideas with excerpts from Diana's and Corey's interviews. Diana's notion of what it means to understand a solution method had apparently changed so much that she could not imagine having claimed to understand Andy's method. Regarding Corey's shift from arguments using "numbers" to those with diagrams, I should note that it was the combination of a diagram and a satisfactory explanation that earned his endorsement. He rejected some arguments with diagrams when they failed to justify why the solution worked. Making a similar point, Grace suggested that the diagram in Evan's argument was potentially unnecessary.

> I think Evan's is the best, because he drew [a diagram]. But even if he didn't, he still understands. He gave an example and showed why it worked, because you're just switching the order or doing three rows of five and not five rows of three, but it's all going to be the same amount. (Grace, Post-course Interview)

Table 4

"Showing Quick Solutions with Symbols" and "Explaining Why with Diagrams"

Showing Quick Solutions with Symbols (Early in the Course)	Explaining Why with Diagrams (Post-Course)
D: I would cross-multiply it [like Andy]. I feel like that's the more advanced [approach]. That's the shortcut to it. Once you understand the basics—you understand why you're doing that—you can do the shortcut … if you understand the reasoning behind it. M: Okay. … Do we know what the reasoning is behind that method? D: We were taught it in school … taught that this was the process that you use to get it and to do it. (Diana)	D: I like Beth's more, because I don't understand Andy's that well. ... I didn't know if that was just some shortcut? I didn't know the logic behind it. M: Well, you did use the word "shortcut," if I recall, but you said the shortcut was okay as long as you understood the procedure. D: Right! And I didn't understand the procedure, so I didn't try the shortcut. … I guess now I just like Beth's better. I can't understand why I would have said I understood that, if I had no idea. (Diana)
I like Flavia's and Dawn's because of the way they're using the numbers to distinctly show their answers. … I'm not much of a visual learner. I don't like using representations to show what I want; I like to just see the numbers and work it out in my mind. And when they are doing it here [in their explanations], they're showing that. They're showing the steps to it. They're showing "two times three equals six," and then they're reversing it—"three times two equals six"—with Dawn. (Corey)	I like Evan's best … because it has a representation. Now, after the [Number and Operation] class, I just love representations and drawings, because I feel it's a visual aid to show students that is very easy to connect to their knowledge. And I like the way [Evan] does it. He shows three times five is the same as three rows with five in each row, and then five times three is the same as creating five rows with three in each row. It's just flip-flopping it, and it shows that it will work in any situation. (Corey)

The Argument's Form

Shifts toward more explanatory arguments. In comparison to the data from the beginning of the semester, the post-course interviews showed two substantial shifts: (a) from empirical arguments to generic examples and (b) from arguments based on purely procedural solutions (warrants without backing) to more explanatory single-case arguments. The examples in Table 4 illustrate these two trends. Corey initially preferred Dawn's empirical argument for the commutativity of multiplication but grew to prefer Evan's generic example, and Diana initially favored Andy's procedural approach but switched to Beth's explanatory single-case argument.

Appeal to authority. For other types of arguments, changes in PTs' views were not so clearly positive. For example, Caitlyn's argument, an appeal to authority, initially earned strong approval from only one interviewee, Ermida, but she continued to endorse it at the end of the semester. Table 5 shows her view in contrast to Corey's.

Table 5

Contrasting Views of Appeal to Authority

Ermida (Post-Course)	Corey (Post-Course)
Caitlyn's is convincing, because that's a mathematical property she understands, and that would be a higher [level] than what Evan was thinking. … Because if she didn't understand the basics down here [indicating Evan's argument] … then she wouldn't have understood [what she read in the textbook].	Just because she can say, "A times B equals B times A," doesn't mean she actually understands why it works. It's just saying, "Oh, this is a rule; it works." … It doesn't have any reasoning behind why it works. It's just saying, "I saw this once; it works."

On a related note, four interviewees initially endorsed Flavia's argument, a procedural solution based on a false warrant. By the end of the semester, all of them shifted to endorse Georgia's explanatory argument. However, some struggled to understand and explain why

Flavia's procedure failed to produce the correct answer. At one point in this process, Grace's frustration led her to exclaim, "I wish someone would just tell me the answer!" which certainly suggests a willingness to accept the word of authority, at least in some situations.

Counterexamples. Finally, when presented with a false general statement, none of the interviewees produced a counterexample at the beginning of the semester. Only one did so in the post-course interview. However, when I suggested a potential counterexample, the others recognized it and immediately understood its role in proving the statement false. Some interviewees, however, expressed unconventional ideas about the potential for counterexamples to general statements in mathematics, viewing all such generalizations as working hypotheses—considered true only until counterexamples emerged to show otherwise. Diana, for example, expressed this viewpoint in relation to her notion of being "open-minded" (see Table 6).

Table 6

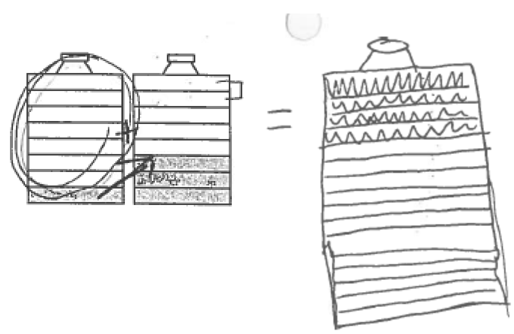
Open-mindedness and the Potential for Counterexamples to Mathematical Generalizations

Diana (Early in the Course)	Diana (Post-Course)
Farther along in their education, they'll realize it doesn't always work. … You add two negatives together, and you're going to have a smaller number. … That's why I liked [Ivy's] answer more. … I like that Ivy is more open-minded about it and willing to accept that it may change later, as you learn new things. D: [Discussing Andy's argument] I feel that it's trial and error, that you use your method …until you learn otherwise. … The key is to have an open mind that later it might change. M: So things are true ... until you find the exception? D: I guess—until you learn the exception. You just have to have an open mind that there can be exceptions.	I still like Ivy's more [than Heather's]. I just think that it shows that Ivy has a more open mind about it. When you have set in your mind that things are a certain way, it's harder to change those ways. So as a teacher, I would rather have a student who has an open mind about it and is more willing to hear how there can be exceptions to certain things. Whereas Heather is just dead set in her mind that it works. "I've done it a million times. It has always worked; it is always going to work." That, I feel, would be harder to change—not change but introduce more information to—if it might conflict with their old information.

An Argument's Substance

PTs showed evidence of three different viewpoints regarding the substance of an argument. As suggested by their preference for arguments that used standard mathematical notation, some initially took a *symbol-centered* view, preferring arguments like Andy's and Flavia's, those that focused on mathematical symbols and procedures for manipulating them. In response to the instructor's expectation that PTs use diagrams to explain their solutions, some showed evidence of a *picto-symbolic* viewpoint, replacing symbols with diagrams but preserving relationships from standard mathematical notation (see Figure 2 for examples). By the end of the semester, however, the interviewees generally approached arguments from a more *quantitative* viewpoint, attempting to make sense of the arguments by analyzing relationships among the quantities involved.

(a) Grace's diagram



(b) Diana's diagram

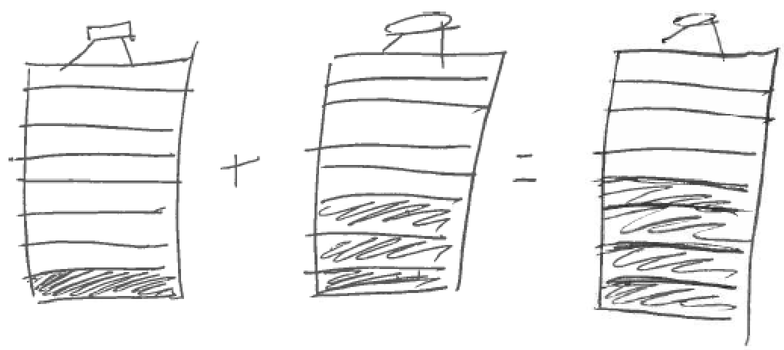


Figure 2. Grace and Diana revisions of Georgia's diagram.

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Appendix A: Focal Problems for the Survey and Interviews

Thinking about Students' Explanations

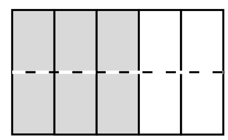
In the situations described below, elementary students answer questions and explain their answers. As you read each explanation, consider whether it convinces you that the answer must be correct; is it a valid explanation? Then write your responses to the questions below.

1. A teacher gave her class the following problem: *True or false:* $\frac{3}{5}=\frac{6}{10}$. *Explain your answer.*

   Two students gave the responses below.

   Andy: It's true. I cross-multiplied and got $3\times10=30$ and $5\times6=30$. If you get the same number when you cross-multiply, the fractions are equal, so $\frac{3}{5}=\frac{6}{10}$.

   Beth: It's true, they are equal. I drew a picture and shaded three-fifths (see the picture at right). If you cut each of the fifths into two parts, you get ten parts altogether, so each fifth is two tenths. Three of the fifths are shaded, and six of the tenths are shaded. So six-tenths is the same amount as three-fifths.



   Choose one of the following:

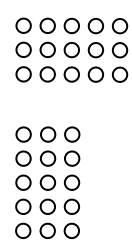
   (a) I think Andy's explanation is more convincing than Beth's.
   (b) I think Beth's explanation is more convincing than Andy's.
   (c) I think both explanations are equally convincing.
   (d) I think neither explanation is convincing.

   Explain why your choice makes sense to you.

2. A fourth grade class has noticed that 45×32 and 32×45 both have the same answer, 1440. The teacher asks them if every multiplication problem can be done in either order. Will the results always be the same both ways? If so, explain why. Three students gave the responses below.

   Caitlyn: Yes, they will always be the same. I saw it in a math book. It's called the commutative property of multiplication. It says $a\times b=b\times a$. That means you can multiply in either order, and the answers will be equal.

   Dawn: Yes, that will always work. Look at our multiplication table. When you switch the order there, you get the same answer every time. $2\times3=6$ and $3\times2=6$. $5\times7=35$ and $7\times5=35$. I looked at every one, and it always works. For any multiplication problem, you get the same answer if you switch the numbers.



   Evan: Yes, multiplying both ways will always give the same result. Look at 3×5, for example. That's like counting three rows of dots with five in each row (see pictures at right). If you make it 5×3, that's five rows with three dots in each row. The second is just like the first, but turned sideways. Both are going to have the same number of dots altogether. That will always happen when you change the order of the numbers you multiply. It's just flipping the picture sideways. It won't change the answer.

Choose one of the following:

(a) I think ________________'s explanation is the most convincing.

(b) I think ________________'s and ________________'s explanations are equally convincing.

(c) I think all three explanations are equally convincing.

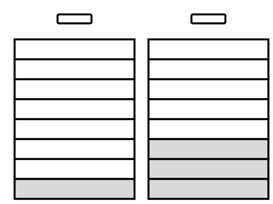
(d) I think none of the three explanations are convincing.

Explain why your choice makes sense to you.

3. The teacher gave the following problem to a fifth-grade class: *Two one-gallon jugs are filled with liquids that are a mix of grape-juice and water. The first is one-eighth grape juice. The second is three-eighths grape juice. If the two gallons are combined, what fraction of the combined mixture will be grape juice? Explain your answer.* Two students gave the responses below.

Flavia: Combined means added together. So I added $\frac{1}{8}+\frac{3}{8}=\frac{4}{8}$. I divided by 4 and reduced it to $\frac{1}{2}$. So the combined mix is one-half grape juice.

Georgia: I drew pictures of two gallons, and I shaded one-eighth of one and three-eighths of the other (see picture at right). When the two pictures are combined together, there are 16 equal parts and 4 are shaded. So the mix would be $\frac{4}{16}$ grape juice.



Choose one of the following:

(a) I think Flavia's explanation is more convincing than Georgia's.
(b) I think Georgia's explanation is more convincing than Flavia's.
(c) I think both explanations are equally convincing.
(d) I think neither explanation is convincing.

Explain why your choice makes sense to you.

4. A second-grade class has been talking about what happens when you add or subtract whole numbers. Several students have said that adding makes the numbers bigger, and subtracting makes them smaller. They have found lots of examples that illustrate these ideas. The teacher asks them if they think this will always happen when you add or subtract whole numbers. She also asks them to explain why they think so.

Heather: Yes, that always works. I can show you a hundred problems—or even a thousand—where adding makes the numbers bigger and subtracting makes them smaller. It always works.

Ivy: I don't think it *always* works. Just because we tried it and it worked for some problems, how do we know it will work for the next problem we try? Maybe it works for some numbers and not for others.

Choose one of the following:

(a) I think Heather's explanation is more convincing than Ivy's.
(b) I think Ivy's explanation is more convincing than Heather's.
(c) I think both explanations are equally convincing.
(d) I think neither explanation is convincing.

Explain why your choice makes sense to you.