Student Perceptions of Proof as Communication in an Inquiry-Based Course

Jeffrey D. Pair
Middle Tennessee State University

Sarah K. Bleiler
Middle Tennessee State University

Abstract

Research on student-centered approaches to teaching proof can identify practices that support meaningful learning of proof. Traditionally, proof has been thought of as a means to obtain conviction. De Villiers (1990) suggests that students need broader experiences with other functions of proof besides mere verification. For example, in the discipline of mathematics proof can be used to communicate mathematical knowledge. What instructional activities provide students the opportunity to use proof as a means to communicate? In our research we sought to understand how an inquiry-based environment can promote engagement in the communication role of proof as described by de Villiers. Students in an inquiry-based transition-to-proof course completed an end-of-the-semester reflective assignment in which they were asked to recall and describe the classroom activities in which they experienced proof as communication. Our analysis indicates that undergraduates perceived that activities such as discussion, presentation, and critique of peers' arguments were crucial to engaging them in the communication role of proof. These inquiry-based activities provided students the opportunity to consider the communicative nature of the proofs they wrote, get new ideas and techniques for proof writing, and learn important mathematical content.

Introduction

Mathematical proof is a key component of the discipline of mathematics, and is central to the practice of mathematicians. Yet, students and teachers across K-16 grade levels struggle with constructing and validating mathematical proofs (Bleiler, Thompson, & Krajcevski, 2014; Healy & Hoyles, 2000; Selden & Selden, 2003; Weber, 2001). Moreover, some teachers hold beliefs that proof is a topic of study that should only occur within particular courses such as high school geometry, and that proof is something that only advanced students can do (Knuth, 2002; Harel & Sowder, 2009). Part of this struggle may be because students and teachers have had few opportunities to engage with proof in meaningful ways. For example, Boyle, Bleiler, Yee, and Ko (under review) show that undergraduate mathematics majors have had limited experiences with proof aside from viewing their instructors construct proofs at the front of the classroom and then practicing similar problems on their own. In such a setting, students may see proof as a mere exercise in following steps and following a particular argumentation structure (Furinghetti & Morselli, 2011). Moreover, students may have limited conceptions of and skill with proof due to an overemphasis of the view of proof as a means to verify mathematical arguments that are presented by the instructor (de Villiers, 1990). When instructors present students with the proposition to be proved, students are stripped of the need for conviction because they trust the authority of the instructor.

In the broader mathematical community, proof has many functions that extend beyond that of verification (de Villiers, 1990), one of which is the communication of mathematical ideas and of proof techniques/strategies. De Villiers describes how mathematicians share mathematical knowledge and negotiate criteria for acceptable argumentation. Among practitioners, proof is seen as a “form of discourse” and a “human interchange based on shared meanings” (de Villiers, 1990, p. 22). We work from the assumption that if students are going to develop a broader conception of proof in ways that align with the practice of mathematicians, then it is necessary that they are provided with opportunities to engage with proof in ways that extend beyond watching and reproducing instructor-developed arguments.

In mathematics, active inquiry-based learning (IBL) (Yoshinobu & Jones, 2012) serves as a possible alternative to traditional proof instruction. IBL approaches are often constituted by a focus on discourse and student-to-student interaction in the classroom (e.g., Stein, Engle, Smith, & Hughes, 2008). Some researchers have demonstrated the benefits of approaches that allow for inquiry-based instruction in learning mathematical proof, such as students' attention to meaning rather than form upon constructing proofs (Smith, 2006). What other opportunities for learning about proof do inquiry-based courses provide for students? In particular what are the opportunities to engage in communication as a role of proof within an inquiry-based environment?

In this study, we contribute to this research area by considering student perceptions of their engagement in the communication role of proof at the end of an IBL proofs course. We seek insight into the activities/events of the IBL course that were lasting, or most memorable, to students with respect to legitimately engaging in communication as described by de Villiers (1990).

Theoretical Framework

Proof is an activity central to the discipline of mathematics (Hemmi, 2010), and the roles of proof articulated by de Villiers (1990) provide us with further insight into how mathematicians use proof. De Villiers identified five roles of proof: verification, explanation, systematization, discovery, and communication. We use the roles of proof as a lens to qualitatively understand the different ways in which students engage with proof in the classroom. In particular for this study, we attend to the communication role of proof, which de Villiers describes in the following way:

> ...one of the real values of proof is that it creates a forum for critical debate. According to this view proof is a unique way of communicating mathematical results between professional mathematicians, between lecturers and students, between teachers and pupils, and among students and pupils themselves. The emphasis thus falls on the **social** process of reporting and disseminating mathematical knowledge in society. Proof as a form of social interaction therefore also involves the **subjective negotiation** of not only the meanings of concepts concerned, but implicitly also of the criteria for an acceptable argument. In turn such a social filtration of a proof in various communications contributes to its refinement and the identification of errors, as well as sometimes to its rejection by the discovery of a counter-example… As Hanna (1989b:20) has furthermore pointed out, this social process is usually far more important in the acceptance of a particular result and its proof by practising mathematicians than the mere application of certain formal criteria in judgement of the logical rigour of the given argument. (pp. 22-23)

Of central importance in this study is understanding “what actually comes to the fore of [students’] attention” (Marton, Runesson, and Tsui, 2004, p. 5) with respect to the communication role of proof. In particular, we follow Marton et al. in their attention to the *space of learning*. We are interested in identifying “effective ways of arranging for learning” (p. 3) that lead students to engage in the communication role of proof. Hiebert and colleagues (1995) discuss *residue* as a way of thinking about what students take with them from classroom experiences, and explain that such residue "might be influenced by the way in which the subject is treated by the curriculum and the teacher, the kinds of tasks students complete, and the everyday rituals of the classroom" (Hiebert et al., 1996, p. 17). In this study, we attempt to understand such elements of the *space of learning* that result in *residue* for students with respect to the communication role of proof.

Methodology

Setting

Thirteen undergraduate students participated in this research study. Nine were mathematics majors, seven of whom were prospective secondary mathematics teachers, and four were mathematics minors. All were enrolled in a single 3-credit section of the transition-to-proof course at a large southeastern university. The goals of the course as articulated in the course catalog include students' learning of the language of mathematics, set theory and proof, relations and functions, number systems, and mathematical structures. The instructor, and second author of this paper, designed the course with an aim toward broadening students' understanding of proof and providing experiences for students to engage with proof in ways similar to mathematicians' practice in the discipline. More specifically, one of the course objectives as listed on the syllabus was for students to, "Gain an appreciation of the many roles of proof and reasoning in the discipline of mathematics (e.g., verification, explanation, systematization, discovery, communication)."

To illustrate the nature of the course, we describe one of the early classroom activities that set the tone for the remainder of the semester. The instructor aimed to develop a classroom community where students felt comfortable sharing and critiquing ideas. Moreover, she wanted the students to actively consider what counts as proof within their classroom community.

Therefore, in the first few days of the class, students worked together to build a proof rubric. To begin the activity, students were asked to create mathematical arguments, and the instructor selected some of these arguments for students to critique. Classroom discussions resulted in a list of criteria that they perceived as critical for constructing a “good” mathematical proof (Bleiler, Ko, Yee, and Boyle, 2015), and that translated into the beginning of a course rubric. The rubric was an evolving document that students modified throughout the semester based on their evolving understanding of what constitutes “good” in the context of mathematical proof-writing. The rubric-building activity was designed so that students could experience mathematics in a way that aligns with practicing mathematicians who historically have negotiated the “the criteria for an acceptable argument” (de Villiers, 1990, p. 22).

This IBL course was structured so that students worked individually on problem sets outside of class time, and then during class they worked collaboratively to either solve new problems or refine proofs to problems they had already worked. The instructor served as a facilitator of class activities and discussions, and used little direct instruction or lecture. The problem sets, modified from Taylor (2007), also provided few models of finished proofs, and instead presented students with a list of problems to prove or disprove.

Data Collection

Students in this class had a two-part final exam. The first part was an in-class exam and the second part was an at-home reading/reflection assignment. The at-home assignment constitutes the primary data source for this study. For this assignment, students read de Villiers (1990) paper, ranked the five roles of proof according to perceived level of engagement, and described events related to the class in which they recalled engaging in each of the five roles. In particular, students responded to the prompt, “Think back on your experience in this course and identify a different time when you believe you were engaged in each of the five roles/functions of proof. Describe clearly and completely your recollection of this event. Also, please rank order the roles/functions of proof according to which you believe you engaged in the most (5) to that which you believe you engaged in the least (1) throughout this semester."

Data Analysis

As described above, we collected data from students related to all five roles of proof discussed by de Villiers. In this section, we describe the qualitative analysis procedures that we conducted in the larger study, for all five roles of proof. However, the results and discussion in this article will be focused primarily on the communication role of proof.

Sixty-five descriptions of recollected events (coming from 13 students for five roles of proof) form our units of analysis and were each about one paragraph in length. The analysis occurred as a four-step process. In step 1, two researchers independently read the collection of students' written responses and engaged in open process coding (Saldaña, 2009). In step 2, we used our individual lists to help us compile one list of process codes that could be used to describe the activities that students perceived engaged them in the five roles of proof. Through comparison of our individual lists and reference back to students' written reflections, we identified six broad activities to which students referred: *presenting, discussing, conjecturing, working on problem sets, critiquing, and constructing/developing proofs.* In Step 3, we returned to the data to conduct a more precise second-cycle coding (Saldaña, 2009), individually assigning each unit of analysis to the relevant codes from Step 2. Finally, in Step 4, we focused on the role and activity pairings for which each of us coded four or more student responses (see Table 1). For these pairings, we returned to the original data and identified themes across the relevant student responses in terms of how those students perceived the activity (e.g., critiquing) engaged them in the role (e.g., communication).

Table 1. Types and frequency of activities students recalled when reflecting on engagement in the five roles of proof. The number inside each cell represents the number of student responses (out of 13) that both researchers coded for a particular activity/role pairing. Shaded cells represent the activity/role pairing where both researchers coded four or more student responses.

	Verification	Explanation	Systematization	Discovery	Communication
Presenting	2	2	1	0	5
Discussing	0	5	1	1	7
Conjecturing	4	1	1	7	0
Working on Problem Sets	6	7	5	6	8
Critiquing	0	0	2	0	4
Constructing/ Developing	5	5	8	4	2

Notice that with respect to the communication role of proof, four of the activities had four or more student responses, namely, *presenting*, *discussing*, *working on problem sets*, and *critiquing*. In what follows, we focus our attention on these four activities.

Results

In their written reflections, many students wrote of constant engagement in the communication role of proof:

"I experience this everyday in class" -Carla

"This happened all the time in class" -Cody

"The communication role of proof was used every day in class." -Jerry

Moreover, when students rank-ordered their engagement in the five roles of proof, communication was the highest ranked role across the five (see Figure 1 for mean rankings).

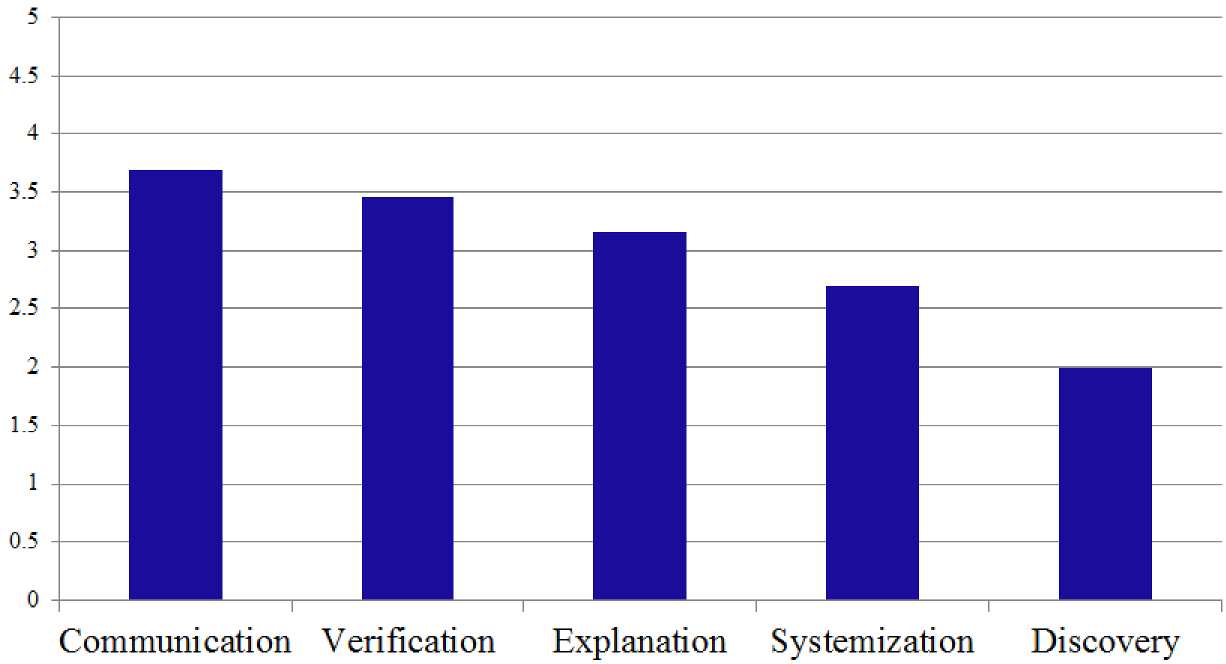

Figure 1: Students' engagement rankings in roles of proof.

Our qualitative analysis revealed that students perceived they engaged in communication when *discussing*, *presenting*, *working on problem sets*, and *critiquing*. These activities did not always occur in isolation. For example, the following response from Stephanie was coded as *discussing* and *critiquing*: "In our groups, we were given proofs already worked out and we would critique those proofs using our rubric. There would be a lot of discussion/debate within our groups as well as in the class that involved noticing errors in the proof or the good aspects of that proof. It helped with giving a better understanding of how proofs work." Our discussion of each of these activities will necessarily overlap with each other. At the conclusion of this section, we will identify some of the key themes that run throughout students' reflections on engaging in the communication role of proof.

Discussing

Students' reflections often referred to instances when they were able to verbally *discuss* ideas with peers during class time. This verbal exchange of ideas often led to the development of shared understandings among students in the classroom community. In particular, students reflected on instances when they would bring their individual proof arguments to class and discuss the ideas from those arguments within their small group. The instructor frequently asked students to discuss their individual ideas and attempt to build a "group proof" that best communicated their mathematical argument. Several students specifically mentioned this activity as meaningful to their engagement in the communication role of proof.

For example, Jerry recalled, "In our groups when we would talk to each other about how we proved something in a problem set a certain way, then come up with a 'group proof.' That was one form of communication through proof that we used in class." Likewise, Millie explained, "All semester, we have been sharing our proofs with our classmates and receiving questions and comments on our work. We have been working together to read the arguments of each member of our group and present ideas through those arguments. We then formed arguments in our groups that we could present to the class." Millie's reflection provides further insight into how the group proof activity, and in particular the related questions and comments from peers, allowed students to develop meaningful shared understandings that could then be presented to member of the larger classroom community.

Students also reflected on the activity of *discussing* as influential to developing their understanding of language and notation used in mathematical discourse. For example, Tina wrote, "The most specific and most memorable moment for me is when we were learning about the set builder notation. I understood the actual notation, but how to read it was a bit trickier.

However, with the help of my classmates and [teacher], I was able to get it down." Similarly, Cody reflected, "Always finding the best mathematical language, definitions, and such was always a challenge in this class. It was always fun to debate and communicate our ideas. This class was all about communication and the best and most efficient way to do it." For students, *discussing* within the context of proof-related topics helped them to better understand and become proficient in the use of established mathematical language and notation, and also to consider and negotiate the best ways to communicate mathematically.

Presenting

The group-proof activity required students to present their arguments to other members of their small groups, and then co-create proofs that could then be presented to the whole class. Presentations, both in small groups and as a whole class activity, allowed for feedback and identification of errors. Savannah wrote, "I remember when working through the problem set that first involved proof by contradiction, and … ended up contriving a proof by creating assumptions that shouldn't have been made. Without trying to explain to my group why it was right, it might have not become clear why the way I had proved it wasn't a logical proof". Presentations, and the feedback students received after them, allowed students to identify mathematical errors. As we see above, Savannah's presentation, and the feedback she received, were crucial to her emerging understanding of proof by contradiction.

Presenting was a normal classroom activity. There was an expectation that students would be required to share their arguments with their peers. For the students, this expectation required increasing their attention to the social dimension of mathematical proof. In order for their arguments to be well-understood and accepted by their peers, students had to organize their thinking, and use the most appropriate mathematical language. Solomon's reflection captures and builds on this idea:

> I would say the whole course had communication as a primary goal, always emphasizing on great clarity, completeness and usage of the right terminology and vocabulary, which are vital, given that the readers of the proof are other human beings and that there is a social aspect to writing mathematical documents. Whenever we were involved in presenting our proofs in class, we were conscious of this fact and tried to make them vivid and communicative.

Savannah's and Solomon's comments demonstrate that *presenting*, and the work that went into preparing presentations, provided opportunities for students to see proof as a form of communication.

Working on problem sets

The course was structured so that students *worked on problem sets* individually before being engaged in collaborative work. After this collaborative work, students received feedback on their individual problem submissions from the instructor, and then had the opportunity to revise their work after which more instructor feedback was received. Susan wrote, "We used proof in all of our problem sets and tests to communicate our understanding of the material to the teacher. And then to develop a greater understanding through the feedback and revisions." For Susan and others, problem sets provided an important medium through which communication took place between student and teacher.

Even when *working on their problem sets* individually, students were aware that they needed to communicate their ideas to the reader of the proof. Sometimes students perceived that this reader was the instructor, and in other reflections, students merely referred to "the reader" of a proof. For instance, Jeb wrote about his efforts to formally communicate his intuitive understanding of a theorem to a reader:

> I found that much of the communication role of proofs came up during set theory. This was due to the fact that sets could be related in several ways, and you had to do a lot of explaining about and describing the idea you were proving. An example that stood out to me was Problem 4.24. For each part, you had to put a lot of thought in trying to describe why this set was a subset of that set or why the union of these two sets was a subset of this set. In effect, you were trying to communicate the idea that $A \cup \emptyset = A$ or $A \cup \bar{A} = U$ using more descriptive terms about the sets and how they were related. Plus, you were trying to show/communicate simple ideas like $A \cap \bar{A} = \emptyset$. Looking at a Venn diagram, one can easily see why this is the case, but in the proof we take a lot more time trying to make the reader understand why this is true.

Jeb's comments reveal a desire to communicate mathematical understanding and insight to the reader. He saw the role of his proof to be helping the reader "understand why this is true". Another student, David, wrote that he was on the receiving end of this communication as he worked on an exercise from the problem set:

> In problem set 5, we played the receiver of the communication in the proof of Theorem 2.10. We had to go line by line in the following exercises to justify why the author did what he did and what he was trying to communicate to us. This was treating the proof as a form of dialogue rather than a math problem.

Students had the opportunity to experience proof as communication as they *worked on problem sets*. They gained critical feedback from their instructor and were required to think deeply about how to communicate their mathematical understanding to a reader. These comments reveal that mathematical proof is a two-way communication between the author and a reader.

Critiquing

Student reflections suggested that the consideration and *critique* of peers' work allowed them to gain strategies that they could use in future proof attempts and also provided them with insight into what counts as proof. There were several instances throughout the semester when students engaged in a particular critiquing activity wherein the instructor would select several samples of students' individual arguments, rewrite the arguments in her own handwriting to maintain anonymity of the student authors, and ask students to make individual and small-group decisions about whether each argument should count as proof and why. Several students mentioned this particular activity as influential to their engagement in the communication role of proof. For example, Krissy shared:

> I found the most significant examples of engagement with the "communication" role of proof as the times in class that we evaluate multiple arguments for the same proof. The most recent example of the exercise came in reviewing four different arguments for Problem 4.27. Every time we have engaged in this exercise, I have found new ideas and techniques for proof-writing that I eagerly attempted to use in my own proofs. The elegance and efficiency of Argument 3 confirmed my view that there must be a simpler way to construct proofs for some of our problems rather than creating essentially two separate proofs any time we needed to indicate equality. This exercise also created the forum for discussion and criticism that De Villiers also referred to as a function of the "communication" role. Our inability to come to a consensus among three people when evaluating a particular argument also demonstrated how difficult it might be for the

> global community of mathematicians to achieve agreement when it comes to proof style and validity.

Similarly, Carla reflected on the same day of instruction and the same critiquing activity, although she was in a different small-group than Krissy:

> One time [that I engaged in the communication role of proof] was when we were looking at proofs and saying whether or not the arguments were proofs. I don't recall the exact problem, but I know it used the biconditional statement/symbol <=> to prove something vs. writing out a whole page. Since I had seen that before I knew it was valid, but my team members all disagreed. We then began expressing why or why not we thought it was a proof. I tried to ask them questions that would broaden their view on why they felt it wasn't a proof in comparison to the ones they felt were proofs. This opened up a learning environment, as well as room for healthy mathematical debate!

We see from both Krissy and Carla that engaging in active *critique* of their peers' work allowed them to learn and discuss new strategies that they could use in future proof attempts. Moreover, they both recognized how this activity allowed a "forum for debate," that is a central role of communication through proof in the work of practicing mathematicians. This forum for debate allowed students to make sense of the criteria for acceptable mathematical argumentation and determine what counts as proof.

Summary

We found that each of the four activities (i.e., *discussing*, *presenting*, *working on problem sets*, and *critiquing*) were influential in engaging students in different aspects of the communication role of proof. Through *discussing*, students were able to develop shared understandings about proof and to develop their understanding of mathematical notation and language. Through *presenting*, students received feedback, were able to identify errors, and attended to the clarity and communicative aspects of their arguments. Through *working on problem sets*, students perceived that they were able to communicate with and receive feedback from their instructor, and recognized the two-way correspondence that occurs between reader and writer of a proof. Through *critiquing*, students were able to see proof as a forum for critical debate and developed their understanding of what counts as proof.

Through this analysis, we noticed that students repeatedly reflected on two specific instructional activities. The first instructional activity was the "group proof" activity, where students shared their individual arguments with group members and then the group worked together to create a new product that best communicates their argument. The second instructional activity was the "critiquing activity" where the instructor pre-selected samples of students' arguments and students came to a consensus as a group as to their validity. Students perceived that these were influential to their engagement in the communication role of proof.

In summary, the student reflections highlighted de Villiers' (1990) notion that proof "creates a forum for critical debate" (p. 22) and for the "refinement and the identification of errors" (p. 22). As students shared their arguments and worked together, for example, to form group proofs, they were required to "build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments" (NCTM, 2014, p. 29). Class activities also supported students in negotiating "the criteria for an acceptable argument" (de Villiers, 1990, p. 22). Student comments suggest that using proof as a means to communicate can be a powerful way for students not only to understand important mathematics, but also to understand the process of proving.

Conclusions and Implications

Because of the inquiry-based nature of the class, students had frequent opportunities to carefully consider their arguments before they presented them to classmates. The social environment, characterized by activities such as presenting, discussing, and critiquing, necessitated that they pay attention to important communicative aspects of their proofs. It also provided them opportunities to correct their errors and learn new techniques for proving. Student discussion, presentation, and critique of peers' arguments are not staples of traditional instruction in university-level proof courses (e.g., Weber, 2004). However, we found that students perceived their engagement in such activities as critical for supporting their understanding of mathematical notation, developing a sense of and negotiating validity, understanding the different approaches to writing proof, and encouraging precision in mathematical communication. Moreover, student reflections suggested that for them the communication role of proof goes above and beyond its role for mathematicians. As learners of mathematics, they saw their engagement in the communication role of proof as critical to their developing understanding of proof itself.

The nature of this work is exploratory. We have identified several activities that students perceive as meaningful to their engagement in the communication role of proof, and these can give instructors a starting place for considering the types of activities that produce residue and are "lasting" for students. To extend this work, we believe it is important to move beyond student perceptions and to also investigate classroom interactions. This can provide researchers and practitioners with a clearer picture of why and how certain instructional activities provide opportunities for students to engage legitimately in the roles of proof. We also believe it is important to move beyond the consideration of activities that engage students in roles of proof, and consider aspects of the *space of learning* that offer students opportunities to learn to construct proofs, deepen content knowledge through proving, and improve dispositions toward mathematics and proof.

References

Bleiler, S. K., Ko, Y.-Y., Boyle, J. D., & Yee, S. P. (2015). Communal development and evolution of a course rubric for proof writing. In C. Suurtamm (Ed.), *Annual perspectives in mathematics education: Assessment to enhance learning and teaching*. Reston, VA: National Council of Teachers of Mathematics.

Bleiler, S. K., Thompson, D. R., & Krajcevski, M. (2014). Providing written feedback on students' mathematical arguments: Proof validations of prospective secondary mathematics teachers. *Journal of Mathematics Teacher Education, 17,* 105-127.

Boyle, J. D., Bleiler, S. K., Yee, S. P., & Ko, Y.-Y. (under review)Transforming perspective teachers perceptions of proof: A three part instructional sequence.

de Villiers, M. D. (1990). The role and function of proof in mathematics. *Pythagoras, 24*, 17–24.

Furinghetti, F. & Morselli, F. (2011). Beliefs and beyond: Hows and whys in the teaching of proof. *ZDM, The International Journal on Mathematics Education.* DOI: 10.1007/s11858-011-0316-7.

Harel, G., & Sowder, L. (2009). College Instructors' Views of Students Vis-á-Vis Proof. In D. A. Stylianou, M. L. Blanton, & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. 275-289). New York, NY: Routledge.

Healy, L., & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education, 31,* 396-428.

Hemmi, K. (2010). Three styles characterizing mathematicians' pedagogical perspectives on proof. *Educational Studies in Mathematics, 75,* 271-291.

Hiebert, J., Carpenter, T. P., Fennema, E., Fuson, K., Human, P., Murray, H., & Wearne, D. (1996). Problem solving as a basis for reform in curriculum and instruction: The case of mathematics. *Educational researcher*, *25*, 12-21.

Knuth, E. J. (2002). Secondary school mathematics teachers' conceptions of proof. *Journal for research in mathematics education, 33,* 379-405.

Marton, F., Runnesson, U., & Tsui, A. B. M. (2004). The space of learning. In F. Marton and A. B. M. Tsui (Eds.), *Classroom discourse and the space of learning* (pp. 3-40). Mahwah, NJ: Lawrence Erlbaum Associates.

National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all*. Reston, VA: Author.

Saldaña, J. (2009). *The coding manual for qualitative researchers.* Los Angeles, CA: Sage Publications.

Smith, J. C. (2006). A sense-making approach to proof: Strategies of students in traditional and problem-based number theory courses. *Journal of Mathematical Behavior, 25,* 73-90.

Stein, M. K., Engle, R. A., Smith, M. s., Hughes, E. K. (2008). Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell. *Mathematical Thinking and Learning, 10,* 313-340.

Selden, A. & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education, 34,* 4-36.

Taylor, R. (2007). Introduction to proof. *Journal of Inquiry-Based Learning in Mathematics, 4,* 1-37.

Weber, K. (2001). Student difficulty in constructing proofs: The need for strategic knowledge. *Educational Studies in Mathematics, 48*, 101-119.

Weber, K. (2004). Traditional instruction in advanced mathematics courses: A case study of one professor's lectures and proofs in an introductory real analysis course. *The Journal of Mathematical Behavior*, *23*, 115-133.

Yoshinobu, S., & Jones, M. G. (2012). The coverage issue. *PRIMUS: Problems, Resources, and Issues in Mathematics Undergraduate Studies, 22,* 303-316.