Preservice Teachers' Understanding of Directly and Inversely Proportional Relationships

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Abstract

This study uses hands-on and real-world problems to examine 4 preservice middle grades teachers' ability to determine directly and inversely proportional relationships, the solution strategies they used, and the difficulties they faced. I report on teachers' reasoning about ratios and proportional relationships from the knowledge-in-pieces perspective. This study makes use of the coordination classes construct to analyze middle and secondary grade teachers' responses. Although the preservice teachers accurately determined the directly and inversely proportional relationships between two covarying quantities, their determination was based on attending to the qualitative relationships. Therefore, they had difficulty distinguishing the directly and inversely proportional relationships from nonproportional relationships that consisted of a constant difference or quadratic growth.

Keywords: preservice teacher education, proportional reasoning, ratio, proportional relationships, hands-on tasks

Background

Understanding ratios, proportions, and proportional reasoning constitutes a main area of school mathematics that is critical for students to learn but difficult for teachers to teach (Lobato, Ellis, & Zbiek, 2010). Proportional reasoning plays a key role in students' mathematical development, and it as an important concept in children's elementary school arithmetic and in higher mathematics (Lesh, Post, & Behr, 1988). In middle school, students need to develop skills that are essential for the development of proportionality. Two of those skills, as reflected in the National Council of Teachers of Mathematics (NCTM; 2000)'s *Principles and Standards for School Mathematics*, are understanding and using ratios and proportions to represent quantitative relationships; and developing, analyzing, and explaining methods for solving problems involving proportions (Number and Operations Standards for Grades 6-8 section, para. 7). For Lamon (2007), "Proportional *reasoning* refers to detecting, expressing, analyzing, explaining, and providing evidence in support of assertions about proportional relationships" (Lamon, 2007, p. 647). As stated in the Common Core State Standards for Mathematics, to be able to reason proportionally, students should be able to "decide whether two quantities are in a proportional relationship (7.RP.2a)" (CCSSM; Common Core State Standards Initiative, 2010, p. 48).

Statement of the Problems

One of the problems of teaching and learning proportional relationships is that traditional proportion instruction places an emphasis on rule memorization and rote computations (Izsák & Jacobson, 2013). Hence, the most common textbook strategy for solving a missing-value problem is the cross-multiplication strategy (Fisher, 1988), which requires setting a proportion and cross-multiplying numbers within the proportion. As observed by many researchers (e.g., Fisher, 1988; Riley, 2010), teachers often use the cross-multiplication strategy when solving proportion problems. A second problem is that, according to Izsák and Jacobson (2013), mathematics education research has overlooked teachers' proportional reasoning. In particular, only a few researchers (e.g., Fisher, 1988; Izsák & Jacobson, 2013; Lim, 2009; Riley, 2010) have studied teachers' proportional reasoning regarding inverse proportions. A third problem is that teachers tend to judge nonproportional relationships as proportional (Cramer, Post, & Currier, 1993; Fisher, 1988; Izsák & Jacobson, 2013; Lim, 2009; Riley, 2010). In addition to these three problems, preservice teachers are likely to use additive strategies to solve proportion problems (Riley, 2010; Simon & Blume, 1994), and they have difficulty understanding ratio-as-measure and the invariance of a ratio (Simon & Blume, 1994).

Significance of the Study

In earlier research, researchers investigated teachers' proportional reasoning mostly using missing-value word problems, which usually involved a single proportional or nonproportional relationship. Similarly, instruction on proportions traditionally uses missing-value word problems in teaching, and cross-multiplication is the typical choice for a general solution strategy. Hence, preservice and in-service teachers usually have some experience with missing-value word problems. In this study, a combination of hands-on activities and real-world missing-value problems, which involved either single or multiple directly and inversely proportional relationships, were used. Because multiple proportion problems cannot be solved by simply forming a single proportion and applying the cross-multiplication strategy, it is expected that teachers will avoid using cross-multiplication and the additive strategies in those problems. Likewise, it is expected that the use of physical devices (e.g., plastic gears, a mini-number balance system) will provide hands-on experience and will generate a checking mechanism for teachers, which will eventually help them have well-developed understandings of directly and inversely proportional relationships.

This study makes four contributions to the current research base in mathematics education. First, very little research has been conducted on preservice teachers' proportional reasoning. In particular, only a few researchers have studied teachers' proportional reasoning regarding inverse proportions, and even fewer researchers have studied multiple proportions. Second, the use of hands-on tasks and real-world missing-value problems together precipitates the gathering of relevant information regarding preservice teachers' proportional reasoning. Third, this study builds a bridge between mathematics education and science education by making use of science concepts—velocity, gear ratio, and balance. Fourth, this study examines the construct of coordination classes for analyzing teachers' capability of detecting and explaining directly and inversely proportional relationships in problem tasks with more complex structures and with which teachers have less experience.

Theoretical Framework

This study makes use of the construct of *coordination classes* (diSessa & Sherin, 1998), a concept established in science education as part of the knowledge-in-pieces epistemological perspective (diSessa, 1988), to analyze teachers' facility with precise identification of directly and inversely proportional relationships and multiplicative relationships. As stated by Thaden-Koch (2003), coordination classes is a fairly new concept and has not been thoroughly explored. Most recently, Izsàk and Jacobson (2014) investigated preservice middle and secondary grades teachers' facility with multiplicative relationships and identification of directly and inversely proportional relationships by utilizing coordination classes. However, the missing-value problems used by Izsàk and Jacobson (2014) involved either a single inversely proportional or nonproportional relationship. Izsàk and Jacobson (2014) suggest that future research should involve more complex cognitive structures to analyze teachers' responses to the proportion problems. In order to examine complex cognitive structures, Izsàk and Jacobson (2014) recommend using problem tasks that involve physical devices and other contexts with which teachers have less experience. Since this study uses hands-on problem tasks and multiple proportion problems to examine teachers' proportional reasoning, it extends and strengthens the knowledge-in-pieces perspective by applying core components of this perspective to understand the more complex cognitive structures used by teachers to identify directly and inversely proportional relationships and multiplicative relationships.

As explained by diSessa and Sherin (1998), to highlight the coordination class perspective, they see *coordination* as a term representing "see" or "determine information" (pp. 1171-1172). Following diSessa and Sherin (1998), I use the term *coordination* in the sense of determining and integrating information within a problem context. A coordination class contains two essential tools: readout strategies and the causal net. Readout strategies "deal with the diversity of presentations of information to determine, for example, characteristic attributes of a concept exemplar in different situations" (diSessa & Sherin, 1998, p. 1171), or more simply, they are strategies for acquiring information about the physical world. The causal net is "the general class of knowledge and reasoning strategies that determines when and how some observations are related to the information at issue" (diSessa & Sherin, 1998, p. 1176).

Research Questions

This study included the following research questions:

1. How do preservice middle school mathematics teachers determine and explain directly and inversely proportional relationships in single and multiple proportion problems, and

what types of knowledge resources do they use in the determination and explanation of directly and inversely proportional relationships?

2. What kinds of difficulties do they encounter in the process of determining and explaining directly and inversely proportional relationships?

Methods and Data Sources

An explanatory multiple-case study methodology (e.g., Yin, 2009) was used in designing this study. Because the purpose of this study was to explore preservice teachers' reasoning, each individual participant constituted a case, and a multiple-case study methodology best suited the scope of this study. In the fall semester of 2014, two female secondary grade (8-12 grades) and two female middle grade (4-8 grades) preservice teachers at one large public university in the Southeast participated in the study. To maintain confidentiality, the following pseudonyms for the secondary and middle grade preservice teachers were used: Kathy and Susan, and Carol and Helen, respectively. With the exception of Susan, who was in the third year of her program, the remaining participants were in the fourth year of their programs. They all had been attending courses with a focus on directly and inversely proportional relationships. The data were collected through semistructured clinical interviews (e.g., Bernard, 1994). Each participant was interviewed for approximately 4 to 5 hours. Two video cameras were used during the interviews: One focused on the participant's written work, and the other focused on the interview setting. I interviewed all five participants, and one graduate student helped me operate the two video cameras. All interview videos were transcribed verbatim.

The tasks used in this study are provided in Table 1. Participants' responses on the three hands-on tasks—Gear 1A, Gear 1B, and Balance—and four real-world tasks—Bakery, Speed, Fence, and Scout Camp—were examined. Fence and Scout Camp tasks were used as extras. I developed the Gear 1A, Gear 1B, and Balance tasks and adopted the Bakery, Speed, and Fence tasks from Dr. Sybilla Beckman's mathematics textbook, *Mathematics for Elementary Teachers* (2013), and adopted the Scout Camp task from Vergnaud's (1983) study. In the Gear and Balance tasks, participants were provided with plastic gears and a mini-number balance system, which was a simple version of an equal-arm beam balance scale.

Table 1

Description of the Tasks

Name of the task	Brief descriptions
Gear 1A	This task involved determining a directly proportional relationship between the size of a gear and the number of notches it possessed. For example, in one of the questions, participants calculated the number of notches of a gear with a 2-cm radius, given that the second gear had a 3-cm radius and 12 notches.
Gear 1B	This task involved determining an inversely proportional relationship between the number of revolutions that a gear makes and its size. For instance, in one of the questions, participants calculated the number of revolutions of a gear with a 3-cm radius, given that the second gear had a 4-cm radius and revolved 6 times.
Bakery	In this task, participants explored one inversely and two directly proportional relationships among the number of people, the number of cupcakes, and the number of minutes. The task involved single and multiple proportion questions. For example, in one of the questions, participants calculated how many cupcakes could be frosted by 2 people in T minutes, considering that 3 people frosted N cupcakes in T minutes.
Balance	In this task, I provided participants with a mini-number balance system with which they balanced the system through hanging weights, which I also provided, on hooks that were placed on both sides of the system. They explored an inversely proportional relationship between the distance (how far from center a weight hung) and the number of weights that were hung.
Speed	This task involved one inversely and two directly proportional relationships among the distance, speed, and time. The participants worked on questions similar to this one: If you covered the distance between two markers in 90 seconds driving at 60 mph and if you want to cover the same distance in 60 seconds, then what must be your speed?
	Extra Tasks
Fence	This task involved identifying one inversely and two directly proportional relationships among the number of workers, the number of days, and the number of fences painted. The participants worked on questions similar to the following one: If 3 people take 2 days to paint 5 fences, how long will it take 2 people to paint 1 fence?

Scout Camp	This task involved three inversely proportional relationships among the number of people, the amount of cereal each person eats per day, and the number of days they stayed in the camp. Participants worked on questions to calculate the number of people, the amount of cereal each person ate per day, or the number of days they stayed in the camp.

In the following pages, I present the analysis of these four cases. The case analysis begins with a brief summary, and a cross-tasks analysis of the participants' responses follows it. I analyze the participants' responses to the problem tasks in two categories: constant ratio relationships and constant product relationships. I employ the knowledge-in-pieces epistemological perspective theoretical frameworks to make sense of those responses. There are no deletions in the transcripts that I provided. I show pauses with ellipses, and actions are described within square brackets.

Data Analysis

Secondary Grade Preservice Teachers

Summary

Kathy attended to the multiplicative relationships between measure spaces to infer a constant ratio relationship between two covarying quantities. She recognized the constancy of the products in the Gear 1B and Balance tasks, but she did not recognize constancy in the Bakery and Speed tasks. Kathy usually preferred reasoning within measure spaces when solving single and multiple proportion questions. She also reasoned in a variety of ways about proportional relationships. Susan attended to the constancy of the rate of change and linearity to infer relationship between two covarying quantities as directly proportional. On the other hand, she attended to the discrete structures (e.g., numbers, points, graphs, etc.) presented in the tasks to infer constant product relationships.

Cross-Tasks Analysis

Constant Ratio Relationships. In the Speed task, Kathy successfully calculated the speed of a car, given that it covered 2 miles in 100 seconds, to be 72 mph using a scientific unit conversion method, which she said she used in chemistry and physics. When asked if there was a relationship between the distance and time, Kathy correctly inferred a proportional relationship by forming multiplicative relationships between measure spaces. When asked why she determined the relationship between the distance and time to be proportional, Kathy drew the ratio table in Figure 1 and demonstrated that there was a constant $\frac{1 \text{ mile}}{50 \text{ seconds}}$ ratio relationship between the distance and time. The following exchanges show Kathy's explicit statement of this constant ratio relationship:

Kathy: I mean so you are going at a constant speed, okay so then miles and seconds [drawing a ratio table], miles is here and seconds and then we know this relationship 2 and 100 and we know 1 is 50 and 3 is 150 and 4 is 200 and so on.
Int: So then you think it is proportional.
Kathy: Yeah because all these are all these have to same like ratio this 1 over 50, and 2 over 100 is going to be 1 over 50…3 over 150 is 1 over 50, it keeps going.

Therefore, the exchanges and Figure 1 confirmed that Kathy's causal net was sufficient to see that driving at a constant speed was yielding a constant ratio relationship between the distance and time.

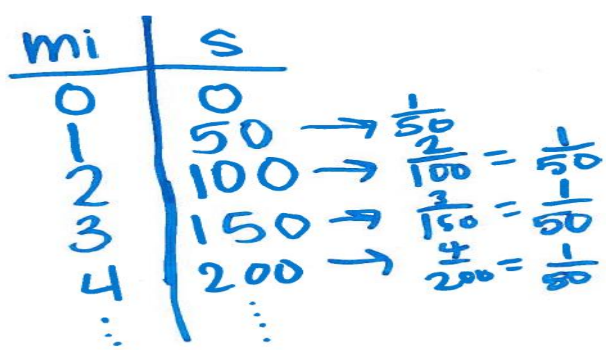

Figure 1. Kathy's expression of the constant ratio relationship between the distance and time.

In the Bakery task, when asked what the relationship between the number of people and cupcakes was, Susan used the information 3 people frost 12 cupcakes in T minutes to draw a linear graph (Figure 2). She determined that this relationship was proportional:

Susan: They are proportional.
Int: The reason is you have the graph or something…what was your main idea to graph it? Like, when I ask you to identify the relationship between these two, like the number of cupcakes and people, you said I can graph it. What was the reason for graphing to identify relationship?
Susan: So, I could show there was a linear relationship. So, that the…the ratio, there is a constant ratio between the people and the cupcakes.

Susan explained that her reason for drawing the graph was to show that it was linear and that there was a constant ratio relationship between the number of people and cupcakes. For Susan, the linearity of the graph was the reason for her inference of the proportional relationship. She used the terms *proportional relationship* and *linear relationship* interchangeably and that indicated a misunderstanding on her side about the proportional relationships.

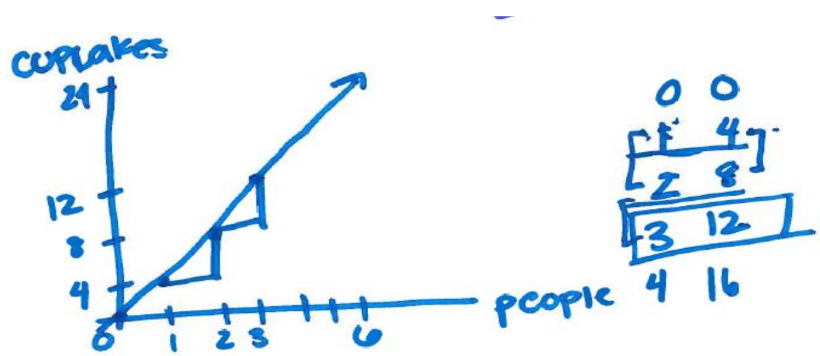

Figure 2. Susan's expression of the constant ratio relationship between the number of people and cupcakes.

Constant Product Relationships. In the Gear 1B task, when asked to calculate the number of revolutions of Gear L, with 8 notches, given that Gear M, with 14 notches, revolved 4 times, Kathy used a "total notches moved" strategy. In this strategy, Kathy multiplied 14 notches by 4 revolutions and explained that the product, 56, was the total number of notches moved on

Gear M in 4 revolutions. She then divided 56 by 8 notches and correctly calculated the number of revolutions to be 7. When asked if she could use generate a ratio table to solve the same question, Kathy generated the table in Figure 3. She recognized that the product of all rows was equal to 56:

Kathy: Okay so well that has to be 56, I mean this is 56 here. I just know, I just kind of know that like all of these, like these two [notches and revolutions] have to multiply to give me 56 like every single time. So, I am saying what times two is 56 and that is, I do not know, 28. And then $\frac{56}{3}$, I do not what that is.
Int: You can leave like that. Knowing that 56, you said 56 is the?
Kathy: is the product of notches and revolutions.

Kathy explicitly stated that 56 was "the product of notches and revolutions." Therefore, these data suggested that she coordinated a constant product relationship between the number of notches and revolutions.

Figure 3. Kathy's ratio table to express the constant product relationship.

In the Balance task, Susan described the inverse qualitative relationship between the number of weights and distance by saying, "The amount of weights is increasing as you are decreasing the distance." Some exchanges later, Susan inferred an inversely proportional relationship between the number of weights and distance:

Susan: Yeah. So, they are inversely proportional.
Int: Why do you think that is, they are inversely…?
…….

Susan: Because the 2 times the 6 equals 12, the 6 times the 2 equals 12, 4 times 3 equals 12. They are always…the distance times the amount of weights like for that distance always multiply to 12.

In the particular number of weights and distance relationship that Susan inferred, the product of the number of weights and the distance was always equal to 12. Susan seemed to focus on the numbers instead of the reciprocal relationship between the number of weights and the distance presented. Therefore, the exchanges demonstrated that Susan's inference of a constant product relationship was based on her attention to the numbers presented in the task.

Middle Grade Preservice Teachers

Summary

Carol successfully inferred constant ratio relationships by determining multiplicative relationships between and within measure spaces. She discussed constant product relationships in the Gear 1B and Balance tasks, but she did not discuss constancy in the Bakery and Speed tasks. Even though Helen had difficulty determining multiplicative relationships between and within measure spaces in the absence of numbers, she accurately determined the directly and inversely proportional relationships in the given questions because her main knowledge resource for determining the given relationships was attending to the qualitative relationships between two covarying quantities.

Cross-Tasks Analysis

Constant Ratio Relationships. In the Gear 1B task, reasoning between measure spaces, Carol calculated the number of notches of Gear B, with a $\frac{6}{5}$ cm radius, given that Gear A, with a 3-cm radius and 10 notches, to be 4. When asked if she could use another strategy, Carol said she could make a table (Figure 4). Carol attended to repeated addition of batches as indicated by her approach. First, knowing that there were 10 notches for a 3-cm radius, she determined a 1 notch to 0.3-cm radius relationship as one batch. Then by repeatedly adding this batch, she calculated a 1.2-cm-radius-to-4-notches relationship. When the second interviewer asked how she was making sense of her table from the meaning of proportional relationships, Carol responded as follows.

Carol: It just shows that the proportions…all of these relationships, all these ratios [pointed out 1 notch and 0.3 cm] are equivalent. The proportions are equal, they stay consistent throughout the table. So, but if this was 3 and this [pointed 0.9 cm] was 0.8, it wouldn't be proportional. If this was anything but 0.9 for 3 notches it would not be proportional.

For Carol, replacing the 0.9-cm radius with a 0.8-cm radius would disrupt the 1-notch-to-0.3-cm-radius relationship. Therefore, Carol's reasoning provided evidence of her coordination of a constant ratio relationship between the number of notches and radii.

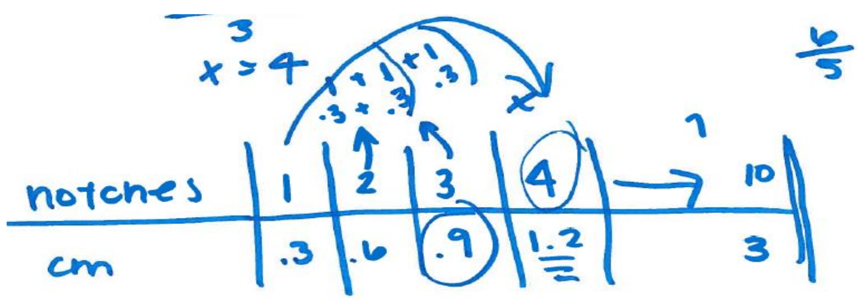

Figure 4. Carol's ratio table strategy to express the relationship between the number of notches and radii.

In the Gear 1A task, Helen successfully calculated the number of notches of Gear B, with a 6-cm radius, given that Gear A had a 3-cm radius and *m* notches, to be 2*m*. In the previous questions of this task, Helen was given numbers, and therefore she was able to identify multiplicative relationship between measure spaces. In the current question, because she was given a letter that was representing the number of notches, Helen seemed to have difficulty identifying the multiplicative relationship between measure spaces. For that reason, she identified the multiplicative relationship within radii, which seemed to be relatively much easier than identifying a multiplicative relationship between measure spaces because it just involved doubling the sizes and the number of notches. When reminded she talked about unit rates in the previous questions and asked what the unit rate was in this question, Helen explained:

Helen: Actually I do not know because if we have 1*m*. I was just saying it is like 1 but that does not make sense…We do not know it. I am…just like *m*, I am just saying if we have 1*m*, if it is *m* and then we know that for every *m* there is 3 cm that since we already know that three times like doubling 3 is 6. We doubled what he had here, so that's way we would say there is 2*m* because that would be as just we have 3 m plus 3 m is 6 and 2m. Like this [pointing 3 and *m*] is be 6 and 2m. Like I was saying that is how I think about it.

Helen's reasoning in the exchange indicated that she seemed to treat *m* as if it was an unknown amount such as the generic use of the letter *x*. For that reason, she could not recognize that the unit rate between the radii and number of notches could be stated as either there is $\frac{m}{3}$ notches per 1 cm radius or $\frac{3}{m}$ cm radius per 1 notch. In the exchange above, Helen expressed her definition of a constant ratio by saying "…for every *m* there is 3 cm." Therefore, this definition suggested that although Helen had difficulty in identifying multiplicative relationships between measure spaces in the absence of numbers, she was aware of a constant ratio relationship between the radii and number of notches.

Constant Product Relationships. In the Gear 1B task, Carol calculated the number of revolutions of Gear Z, with n_2 notches, to be $r_2 = \frac{n_1 r_1}{n_2}$ (Figure 5), given that Gear T, with n_1 notches, revolved r_1 times. When asked what $n_1 r_1$was, she explained as follows:

Carol: n1r1 is the number of notches times the number of rotations from Gear T. And then n2 is the number of notches for Gear Z and so through this, through that, realizing that relationship and then setting up like the proportions and cross-multiplying and dividing to find x, I just don't know why it works.

As appears in Figure 5, Carol set an inverse proportion based on the idea of a numerical inverse proportional relationship. The exchange showed Carol's explicit statement of $n_1 r_1$ as a product of the number of notches and revolutions; however, she accepted that she did not know why setting up a proportion and cross-multiplying worked. It seemed that her reasoning was proficient yet, she did not know why setting up a proportion and cross-multiplying worked. Similarly, when asked what 56 meant, she explained in the following manner:

Int: What was 56 in your head? I'm asking what that means.
Carol: 56 is the number of notches times the number of rotations.

Although these data suggested that Carol explicitly attended to the constant product of the number of notches times the number of rotations, it seemed like she did not understand the significance of what she had done. Therefore, Carol exhibited a limited coordination in attempting to make sense of the relationship between the constancy of the products and inverse proportionality.

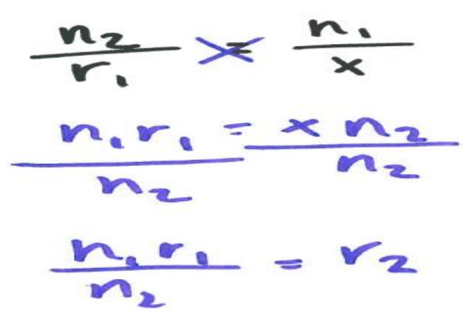

Figure 5. Carol's inverse proportion in expressing the number of notches and revolutions relationship.

In the Balance task, when asked if she could generate a ratio table from the values of quantities that she needed to use to balance the system on one side, given that on the other side 6 weights were hung on a 4 cm distance from the center, Helen multiplied 6 by 4 and got 24 and explained that she needed combinations of 24 on the other side. She generated the ratio table in Figure 6 and explained:

Helen: Well all these [circled pairs of weights and distances] values here like if I multiply these together they have to equal 24 for to balance.
Int: What is that 24?
Helen: Twenty four is the amount of distance and weight of the first side. Yeah like I showed here that this is 4 weights on the distance of 6 from the center, so that is why it has to be 24.

These exchanges provided evidence for Helen's explicit attention to the constant product relationship between the number of weights and distance. Helen did not recognize the constant product relationships in the Gear 1B, Bakery, and Speed tasks. Therefore, the context of balancing seemed to be helpful in Helen's recognition of the constant product relationship.

Figure 6. Helen's ratio table to express the number of weights and distance relationship.

Conclusion

Four knowledge resources of the preservice teachers in determining directly and inversely proportional relationships were detected in the data: (a) attention to the multiplicative relationships between and within measure spaces; (b) attention to the qualitative relationships; (c) facility with the multiplicative relationships between numbers; and (d) attention to the constancy of the rate of change and linearity of the graphs. Identifying multiplicative relationships within measure spaces seemed to be much easier for preservice teachers than identifying multiplicative relationships between measure spaces. The contexts of the Gear 1B and Balance tasks facilitated preservice teachers' inference of constant product relationships more than the contexts of the Bakery and Speed tasks. The preservice teachers appeared to prefer reasoning multiplicatively rather than additively, and this can be credited to the inclusion of the hands-on tasks. One preservice teacher had difficulty determining multiplicative relationships in the absence of numbers, and another preservice teacher attended to the discrete structures rather than the covariation to infer constant product relationships. Some preservice teachers tended to use the terms *inverse* and *inversely proportional*, and *linear* and *directly proportional* interchangeably, and this suggested possible constraints in the preservice teachers' understanding of proportional and nonproportional relationships.

Acknowledgements

I thank Dr. Andrew Izsàk for helpful comments on earlier drafts of this paper.

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