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Design of Discrete Second Order Filters for Continuous-Discrete Models

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***Abstract*—The paper deals with state estimation of nonlinear stochastic dynamic systems. In particular, the stress is laid on development of second-order discrete-time filters for nonlinear continuous-discrete models. Two different filters are derived based on different order of nonlinear function approximation and discretization of the continuous dynamics equation. For the approximation the Taylor expansion is used and for the discretization either exact method or approximate method based on the stochastic Taylor expansion are utilized. Performance of the filter algorithms is illustrated using an example of tracking a ship in deep water.**

Keywords: state estimation, nonlinear filters, Kalman filtering, second order filter, continuous-discrete model

List of Acronyms

SDE	- stochastic differential equation
CDM	- continuous-discrete model
PDF	- probability density function
EKF	- extended Kalman filter
TE	- Taylor expansion
SOF	- second-order filter
STE	- stochastic Taylor expansion
UT	- unscented transform
RMSE	- root mean squared error
ARMSE	- time-average root mean squared error

I. Introduction

Recursive state estimation of nonlinear stochastic dynamic systems from noisy or incomplete measured data has been a subject of considerable research interest for the last five decades [1]–[3]. It plays an essential role in fields such as signal processing, navigation and especially tracking an object in motion [4], [5].

The motion of the object is largely described by a stochastic differential equation (SDE) for the object state with a noise expressing a mismatch between the mathematical model and the actual motion of the object affected by immeasurable external forces [6], [7]. The object is observed using a set of sensors providing measurements of some variables of the model. The measurements are imprecise and hence in the model they are usually assumed being corrupted by noises. As the modern sensors are often digital, they provide the measurements at discrete time instants [6], [8]. A model based on continuous-time description of the system state dynamics with measurements available at discrete-time instants only is denoted as continuous-discrete model (CDM).

The problem of estimating the state of the system given noisy measurements can efficiently be solved using the Bayesian approach that optimally combines prior information given by the model and the measured data. A conditional probability density function (PDF) of the state completely describing the state estimate can be computed by solving i) the Fokker-Planck equation [1], which propagates the conditional PDF between the time instants of the measurement, and ii) the Bayesian relation, which merges information given by the current measurement. Unfortunately, the relations can be solved analytically for a few special cases only [1], [9] and their numerical solution is computationally expensive. Hence, often the conditional PDF calculation is replaced by a simpler computation of point estimates of the state only, usually the conditional mean of the state.

A usual approach in the area of state estimation to calculate the point estimate for a CDM model is to discretize the nonlinear continuous dynamics using an approximate method to obtain a discrete-time model and to apply a nonlinear discrete-time filter such as the extended Kalman filter (EKF) [1] and various derivative-free filters, e.g., the divided difference filter [10], the unscented filter [11].

In the area of navigation system design, the approach to design a filter for a CDM is different. Rather a linearization of the nonlinear model is done first, the obtained model linear CDM is consequently discretized and the estimate is then computed using the discrete-time Kalman filter relations [12].

Having a nonlinear discrete-time model obtained by the first approach, a number of discrete filters can be used to obtain the state estimate, where the EKF represents a standard choice. It is based on approximation of the nonlinear functions using the Taylor expansion (TE) at the current estimate and ignoring the quadratic and higher terms. To improve estimate quality of the EKF, a number of second-order EKFs have been proposed in [1], [4], [13]–[15], which take the quadratic term of the TE into account. A similar idea was used to design of a second-order divided difference filter [10].

This paper focuses on second-order filters (SOFs) and the goal is to compare the two approaches to design a discrete-time SOF for a nonlinear CDM, which differ in order of the TE-based nonlinear function approximation and continuous dynamics discretization. The aim is to detail the approaches, especially the consequences of the order. The order was not discussed in the literature much and it will be shown in a numerical target tracking example that the approaches may lead to completely different estimation quality.

The rest of the paper is organized as follows. Section II provides state estimation problem formulation of the CDM; model approximation and discretization techniques are given in Section III and the design of the SOFs is presented in Section IV. A numerical illustration and concluding remarks are given in Section V and VI, respectively.

II. State estimation of CDM

Consider the following nonlinear CDM

$$d\mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t), t)dt + \sum_{i=1}^{n_\beta} \mathbf{g}^i(\mathbf{x}(t), t)d\beta^i(t) \tag{1}$$

$$\mathbf{z}_l = \mathbf{H}_l\mathbf{x}_l + \mathbf{v}_l, \tag{2}$$

where $\mathbf{x}(t) \in \mathbb{R}^{n_x}$ is the state at time t, $\beta^1(t), \dots, \beta^{n_\beta}(t)$ are independent Wiener processes with $\mathsf{E}[\beta^i(t)\beta^i(\tau)] = \delta(t-\tau)$, $\mathbf{z}_l \in \mathbb{R}^{n_z}$ is a measurement at time $t_l = l \cdot T_z$, and $\mathbf{v}_l$ is a zero-mean Gaussian white measurement noise with known covariance matrix $\Sigma_l^{\mathbf{v}}$ independent of the state noise $\beta(t) = [\beta^1(t), \dots, \beta^{n_\beta}(t)]^{\mathrm{T}}$. The symbol T_z denotes a measurement period.

The mappings $\mathbf{f} : \mathbb{R}^{n_x} \times \mathbb{R}^+ \to \mathbb{R}^{n_x}$, $\mathbf{g}^i : \mathbb{R}^{n_x} \times \mathbb{R}^+ \to \mathbb{R}^{n_x}$, and the matrix $\mathbf{H}_l \in \mathbb{R}^{n_z \times n_x}$ are supposed to be known. The initial condition $\mathbf{x}_0 = \mathbf{x}(t_0)$ of the state is independent of the state noise $\beta(t)$ and the measurement noise $\mathbf{v}_l$ and is Gaussian with known mean $\mathsf{E}[\mathbf{x}_0] = \bar{\mathbf{x}}_0$ and covariance matrix $\mathsf{cov}[\mathbf{x}_0] = \mathbf{P}_0^{\mathbf{xx}}$. The measurement equation (2) is assumed to be linear in this paper to pay attention solely to the dynamics, its approximation and discretization.

The aim of state estimation is to find the state $\mathbf{x}(t)$ using a set of previous measurements $\mathbf{z}^l \triangleq [\mathbf{z}_0^{\mathrm{T}}, \mathbf{z}_1^{\mathrm{T}}, \dots, \mathbf{z}_l^{\mathrm{T}}]^{\mathrm{T}}$. This estimate is described by the conditional PDF $p(\mathbf{x}(t)|\mathbf{z}^l)$. As has been mentioned above, the PDF can be obtained by solving the Fokker-Planck partial differential equation for $t \in (t_{l-1}, t_l)$ and Bayesian relation for $t = t_l$. Unfortunately, such solution can be obtained analytically for a few special cases such as for a linear mapping $\mathbf{f}(\mathbf{x}(t), t) = \mathbf{F} \cdot \mathbf{x}(t)$ and function $\mathbf{g}^i(\mathbf{x}(t), t) = \mathbf{g}(t)$ independent of the state. Often the conditional PDF calculation is replaced by a simpler calculation of the first moment, i.e., the conditional mean $\mathsf{E}[\mathbf{x}(t)|\mathbf{z}^l]$. Nevertheless, for nonlinear mappings $\mathbf{f}(\cdot)$ and $\mathbf{g}^i(\cdot)$ computation of the point estimate $\mathsf{E}[\mathbf{x}(t)|\mathbf{z}^l]$ requires some approximations.

In this paper, the state estimate will be calculated by discrete-time SOFs; hence the SDE (1) describing the state dynamics has to be discretized. The discretization period is denoted T and for convenience it is supposed that $T_z = a \cdot T$ with $a \in \mathbb{Z}^+$, i.e., the measurement period T_z is an integer multiple of the discretization period T. To simplify the notation, the discretized state at time instant t_k will be denoted as $\mathbf{x}_k = \mathbf{x}(t_k)$, where $t_{k+1} - t_k = T$ and the measurement will be available only at time instants for which relation $t_k = a \cdot T$ holds.

The discretized model of the state dynamics will be described by the stochastic difference equation

$$\mathbf{x}_{k+1} = \phi_k(\mathbf{x}_k, \mathbf{w}_k), \tag{3}$$

where $\mathbf{w}_k$ is white noise. The state estimate will be denoted as $\hat{\mathbf{x}}_{k|l} = \mathsf{E}[\mathbf{x}_k|\mathbf{z}^l]$, where the index l denotes time instant of the last available measurement. Hence, if $k > l$, the estimate is a prediction and if $k = l$ the estimate is filtering.

The generic algorithm of the discrete-time filters for a discrete-time model with linear measurement presented in Section IV is briefly described below.

Algorithm 1: Generic local filter

Step 1: (**initialization**) Set the time instant $k = 0$ and define initial condition by the predictive mean $\hat{\mathbf{x}}_{0|-1} = \mathsf{E}[\mathbf{x}_0] = \bar{\mathbf{x}}_0$ and the predictive covariance matrix $\mathbf{P}_{0|-1}^{\mathbf{xx}} = \mathrm{cov}[\mathbf{x}_0] = \mathbf{P}_0^{\mathbf{xx}}$.
Step 2: (**filtering**) If a measurement $\mathbf{z}_k$ is available, the filtering estimate $\hat{\mathbf{x}}_{k|k}$ and the corresponding covariance matrix of the estimate error $\mathbf{P}_{k|k}^{\mathbf{xx}}$ are obtained according to

$$\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + \mathbf{K}_k(\mathbf{z}_k - \hat{\mathbf{z}}_{k|k-1}), \tag{4}$$

$$\mathbf{P}_{k|k}^{\mathbf{xx}} = \mathbf{P}_{k|k-1}^{\mathbf{xx}} - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_{k|k-1}^{\mathbf{xx}}, \tag{5}$$

where $\mathbf{K}_k = \mathbf{P}_{k|k-1}^{\mathbf{xx}}\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}_{k|k-1}^{\mathbf{xx}}\mathbf{H}_k^{\mathrm{T}} + \Sigma_k^{\mathbf{v}})^{-1}$ is the filter gain. The index denoting time instant of the last available measurement is set to $l = k$.
Step 3: (**prediction**) The predictive statistics

$$\hat{\mathbf{x}}_{k+1|l} = \mathsf{E}[\mathbf{x}_{k+1}|\mathbf{z}^l], \tag{6}$$

$$\mathbf{P}_{k+1|l}^{\mathbf{xx}} = \mathsf{E}[(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k})(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|l})^T|\mathbf{z}^l] \tag{7}$$

are given by the relations of the respective second order filter. Let $k = k + 1$. The algorithm then continues by **Step 2**.

As the measurement equation (2) is linear, calculation of the statistics (4) and (5) is given by the Kalman filter relations. The individual SOFs differ in the relations for statistics (6) and (7), which will be detailed in Section IV. But before that, model approximation and discretization techniques used by the filters will be introduced in Section III.

III. Model approximation and discretization techniques

A. Model approximation technique

For approximation of the nonlinear mappings $\mathbf{f}$ and $\mathbf{g}^i$, $i = 1, \dots, n_\beta$, the TE is typically utilized by the SOFs, [1], [4], [13]–[15]. The TE of $\mathbf{f}$ around a point $\bar{\mathbf{x}}$ is considered in the following form

$$\mathbf{f}(\mathbf{x}, t) = \sum_{j=0}^{\infty} \tfrac{1}{j!}\nabla_{\mathbf{x}}^{[j]} \otimes \mathbf{f}(\mathbf{x}, t)|_{\mathbf{x}=\bar{\mathbf{x}}}(\mathbf{x} - \bar{\mathbf{x}})^{[j]}, \tag{8}$$

where $\nabla_{\mathbf{x}}$ is the gradient, the square brackets in $\nabla_{\mathbf{x}}^{[j]}$ are used for the Kronecker power of $\nabla_{\mathbf{x}}$ (i.e., $\nabla_{\mathbf{x}} \otimes \nabla_{\mathbf{x}} \otimes \cdots \otimes \nabla_{\mathbf{x}}$ repeated j times) and the symbol $\otimes$ stands for the Kronecker product [16]. Note that $\nabla_{\mathbf{x}}^{[0]} \otimes \mathbf{f}(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, t)$, $\nabla_{\mathbf{x}}^{[1]} \otimes \mathbf{f}(\mathbf{x}, t) = \nabla_{\mathbf{x}} \otimes \mathbf{f}(\mathbf{x}, t) \in \mathbb{R}^{n_x \times n_x}$ is the Jacobian of the vector function

$\mathbf{f}(\mathbf{x},t)$ and $\nabla_{\mathbf{x}}^{[2]} \otimes \mathbf{f}(\mathbf{x},t) \in \mathbb{R}^{n_x \times (n_x)^2}$ is the Hessian of the vector function $\mathbf{f}(\mathbf{x},t)$.

Also note that other approximation techniques can be used in principle as well such as the Stirling interpolation [10].

B. Model discretization technique

For approximate discretization of (1) the stochastic Taylor expansion (STE) [17] will be used. The STE is based on application of the Itô lemma to (1). The Itô lemma states that for a sufficiently smooth function $\mathbf{r}: \mathbb{R}^{n_x} \times \mathbb{R} \to \mathbb{R}^{n_x}$ it holds that

$$d\mathbf{r}(\mathbf{x}(t),t) = [\frac{\partial \mathbf{r}}{\partial t} + (\nabla_{\mathbf{x}} \otimes \mathbf{r})\mathbf{f} + \frac{1}{2}(\nabla_{\mathbf{x}}^{[2]} \otimes \mathbf{r})(\sum_{i=1}^{n_\beta} (\mathbf{g}^i)^{[2]})]dt + (\nabla_{\mathbf{x}} \otimes \mathbf{r}) \sum_{i=1}^{n_\beta} \mathbf{g}^i d\beta_i, \quad (9)$$

or equivalently

$$\mathbf{r}(\mathbf{x}(t),t) = \mathbf{r}(\mathbf{x}(\tau),\tau) + \int_\tau^t [\frac{\partial \mathbf{r}}{\partial t} + (\nabla_{\mathbf{x}} \otimes \mathbf{r})\mathbf{f} + \frac{1}{2}(\nabla_{\mathbf{x}}^{[2]} \otimes \mathbf{r})(\sum_{i=1}^{n_\beta} (\mathbf{g}^i)^{[2]})]ds + \sum_{i=1}^{n_\beta} \int_\tau^t [(\nabla_{\mathbf{x}} \otimes \mathbf{r})\mathbf{g}^i] d\beta_i(s), \quad (10)$$

where the second integral is an Itô integral [1]. Applying the relation (10) to the nonlinear functions $\mathbf{f}$ and $\mathbf{g}^i$ it is possible to obtain

$$\begin{aligned} \mathbf{x}(t_{k+1}) = {} & \mathbf{x}(t_k) + \mathbf{f}(\mathbf{x}(t_k),t_k) \int_{t_k}^{t_{k+1}} dt \\ & + \sum_{i=1}^{n_\beta} \mathbf{g}^i(\mathbf{x}(t_k),t_k) \int_{t_k}^{t_{k+1}} d\beta^i(t) + \iint_{t_k,t_k}^{t_{k+1},s} \mathcal{L}^0 \mathbf{f} dr ds \\ & + \sum_{i=1}^{n_\beta} \iint_{t_k,t_k}^{t_{k+1},s} \mathcal{L}^i \mathbf{f} d\beta^i(r) ds + \sum_{i=1}^{n_\beta} \iint_{t_k,t_k}^{t_{k+1},s} \mathcal{L}^0 \mathbf{g}^i dr d\beta^i(s) \\ & + \sum_{i=1,j=1}^{n_\beta,n_\beta} \iint_{t_k,t_k}^{t_{k+1},s} \mathcal{L}^j \mathbf{g}^i d\beta^j(r) d\beta^i(s), \end{aligned} \quad (11)$$

where the following notations are used

$$\mathcal{L}^0 \mathbf{h} \triangleq [\frac{\partial \mathbf{h}}{\partial t} + (\nabla_{\mathbf{x}} \otimes \mathbf{h})\mathbf{f} + \frac{1}{2}(\nabla_{\mathbf{x}}^{[2]} \otimes \mathbf{h})(\sum_{i=1}^{n_\beta} (\mathbf{g}^i)^{[2]})],$$

$$\mathcal{L}^i \mathbf{h} \triangleq (\nabla_{\mathbf{x}} \otimes \mathbf{h})\mathbf{g}^i.$$

Ignoring all double integrals in (11) [18] the Euler discretization can be obtained, which is in fact the first-order STE (STE1)

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{f}_k(\mathbf{x}_k) \int_{t_k}^{t_{k+1}} dt + \sum_{i=1}^{n_\beta} \mathbf{g}_k^i(\mathbf{x}_k) \int_{t_k}^{t_{k+1}} d\beta^i(t), \quad (12)$$

where $\mathbf{x}_k = \mathbf{x}(t_k)$, $\mathbf{f}_k(\mathbf{x}_k) = \mathbf{f}(\mathbf{x}_k,t_k)$, and $\mathbf{g}_k^i(\mathbf{x}_k) = \mathbf{g}^i(\mathbf{x}_k,t_k)$ are used to shorten the notation. Now, since $\int_{t_k}^{t_{k+1}} d\beta^i(s)$ has Gaussian distribution with zero mean and variance $T = t_{k+1} - t_k$, the STE1 can be written in the form

$$\mathbf{x}_{k+1} = \mathbf{x}_k + T \cdot \mathbf{f}_k(\mathbf{x}_k) + \mathbf{g}_k^{(:)}(\mathbf{x}_k)\mathbf{w}_k, \quad (13)$$

where $\mathbf{g}_k^{(:)}(\mathbf{x}_k) = [\mathbf{g}^1(\mathbf{x}(t_k),t_k), \ldots, \mathbf{g}^{n_\beta}(\mathbf{x}(t_k),t_k)]$ and $p(\mathbf{w}_k) = \mathcal{N}\{\mathbf{w}_k; \mathbf{0}, T \cdot \mathbf{I}_{n_\beta}\}$, with $\mathcal{N}\{\mathbf{x}; \mathbf{a}, \Sigma\}$ denoting Gaussian distribution of $\mathbf{x}$ with mean $\mathbf{a}$ and covariance matrix Σ and $\mathbf{I}_m$ denotes the $m \times m$ dimensional identity matrix. Repeated application of (10) to functions $\mathcal{L}^0\mathbf{f}$, $\mathcal{L}^i\mathbf{f}$, $\mathcal{L}^0\mathbf{g}^i$, and $\mathcal{L}^j\mathbf{g}^i$ in (11) and ignoring triple integrals leads to the second-order STE (STE2).

IV. Second order filters

Here, two basic approaches to design of SOFs for a nonlinear CDM will be presented. The first approach is based on exact discretization of a linear SDE, which is obtained by an approximation of the nonlinear functions $\mathbf{f}$ and $\mathbf{g}^i$ in (1). The second approach starts with an approximate discretization of (1) by the STE and then uses approximation of the functions appearing in the discretized model.

A. Filters based on exact discretization

In this subsection two filters will be introduced based on exact discretization of an SDE with functions $\mathbf{f}$ and $\mathbf{g}^i$ replaced by their TE-based approximations.

1) First-order-TE based filter: First, consider (1) with TE of the functions

$$d\mathbf{x}(t) = (\sum_{j=0}^{\infty} \tfrac{1}{j!} \nabla_{\mathbf{x}}^{[j]} \otimes \mathbf{f}(\mathbf{x},t)|_{\mathbf{x}=\bar{\mathbf{x}}} (\mathbf{x} - \bar{\mathbf{x}})^{[j]}) dt \quad (14)$$

$$+ \sum_{i=1}^{n_\beta} (\sum_{j=0}^{\infty} \tfrac{1}{j!} \nabla_{\mathbf{x}}^{[j]} \otimes \mathbf{g}^i(\mathbf{x},t)|_{\mathbf{x}=\bar{\mathbf{x}}} (\mathbf{x} - \bar{\mathbf{x}})^{[j]}) d\beta^i(t) \quad (15)$$

For exact discretization of (14) with arbitrary nonlinear functions $\mathbf{f}$ and $\mathbf{g}^i$, only two terms of the TE of $\mathbf{f}$ and a single term of $\mathbf{g}^i$ have to be retained, and the remaining terms are ignored:

$$d\mathbf{x}(t) \approx (\mathbf{f}(\bar{\mathbf{x}},t) + \mathbf{F}_1(t)(\mathbf{x} - \bar{\mathbf{x}}))dt + \sum_{i=1}^{n_\beta} \mathbf{g}^i(\bar{\mathbf{x}},t) d\beta^i(t), \quad (16)$$

where $\mathbf{F}_1(t) \triangleq \nabla_{\mathbf{x}} \otimes \mathbf{f}(\mathbf{x},t)|_{\mathbf{x}=\bar{\mathbf{x}}}$ Suppose that $\mathbf{f}(\bar{\mathbf{x}},t)$, $\mathbf{F}_1(t)$, and $\mathbf{g}^i(\bar{\mathbf{x}},t)$ are constant over the period T of discretization. Then the difference equation obtained by solving (16) is given by

$$\mathbf{x}_{k+1} = \mathbf{F}_{1D}\mathbf{x}_k + (\mathbf{F}_{1D} - \mathbf{I}_{n_x})\mathbf{F}_1^{-1}(t_k)\,[\mathbf{f}_k(\bar{\mathbf{x}}) - \mathbf{F}_1(t_k)\bar{\mathbf{x}}] + \sum_{i=1}^{n_\beta} \int_{t_k}^{t_{k+1}} e^{\mathbf{F}_1(t_k)\cdot(t_{k+1}-t)} \mathbf{g}_k^i(\bar{\mathbf{x}}) d\beta^i(t), \quad (17)$$

where $\mathbf{F}_{1D} \triangleq e^{\mathbf{F}_1(t_k)T}$. Now, **Algorithm 1** requires specification of the moments $\hat{\mathbf{x}}_{k+1|l}$ and $\mathbf{P}_{k+1|l}^{\mathbf{xx}}$, which can be calculated from (17) as

$$\hat{\mathbf{x}}_{k+1|l} = \mathbf{F}_{1D}\hat{\mathbf{x}}_{k|l} + (\mathbf{F}_{1D} - \mathbf{I}_{n_x})\mathbf{F}_1^{-1}(t_k)\left[\mathbf{f}_k(\bar{\mathbf{x}}) - \mathbf{F}_1(t_k)\bar{\mathbf{x}}\right] \tag{18}$$

$$\mathbf{P}^{\mathbf{xx}}_{k+1|l} = \mathbf{F}_{1D}\mathbf{P}^{\mathbf{xx}}_{k|l}\mathbf{F}^{\mathrm{T}}_{1D} + \sum_{i=1}^{n_\beta}\int_{t_k}^{t_{k+1}}[e^{\mathbf{F}_1(t_k)\cdot(t_{k+1}-t)}\mathbf{g}^i_k(\bar{\mathbf{x}})][\cdot]^{\mathrm{T}}dt, \tag{19}$$

where $[\cdot]$ represents the term before it. Now, the point $\bar{\mathbf{x}}$ at which the TEs (14) and (15) are done should naturally be picked as the best estimate of $\mathbf{x}_k$ available, i.e., $\bar{\mathbf{x}} = \hat{\mathbf{x}}_{k|l}$. Additionally, the discretization period T is supposed to be small, hence $e^{\mathbf{F}_1(t_k)\cdot(t_{k+1}-t)} \approx \mathbf{I}_{n_x}$. Then, the relations for the moments are

$$\hat{\mathbf{x}}_{k+1|l} = \hat{\mathbf{x}}_{k|l} + (\mathbf{F}_{1D} - \mathbf{I}_{n_x})\mathbf{F}_1^{-1}(t_k)\mathbf{f}(\hat{\mathbf{x}}_{k|l}, t_k) \tag{20}$$

$$\mathbf{P}^{\mathbf{xx}}_{k+1|l} = \mathbf{F}_{1D}\mathbf{P}^{\mathbf{xx}}_{k|l}\mathbf{F}^{\mathrm{T}}_{1D} + T\sum_{i=1}^{n_\beta}[\mathbf{g}^i(\hat{\mathbf{x}}_{k|l}, t_k)][\cdot]^{\mathrm{T}}. \tag{21}$$

Note that the term $(\mathbf{F}_{1D} - \mathbf{I}_{n_x})\mathbf{F}_1^{-1}(t_k)$ may be calculated without computing the inverse of $\mathbf{F}_1(t_k)$ by using a TE expansion of $\mathbf{F}_{1D}$ around zero.

Algorithm 1 with state predictive moments given by (20) and (21) is in fact a first order discrete filter for the original CDM. Such filter is frequently used for the state estimation of CDMs [12], which is the reason why its relations are specified here and why it will be used for reference purposes in the numerical example. As the relations (20) and (21) are based on approximation of $\mathbf{f}$ by the first order TE and exact discretization, the algorithm will be denoted as FOF-TE1-ExD. Note that this algorithm corresponds to the EKF algorithm.

2) Second-order-TE based filter: To design a SOF, three elements of the TE of $\mathbf{f}$ (14), i.e., including a quadratic term, will be used. For the TE of $\mathbf{g}^i$ (15) two elements will be considered. The SDE (1) now becomes

$$d\mathbf{x}(t) = \left[\mathbf{f}(\bar{\mathbf{x}}, t) + \mathbf{F}_1(t)(\mathbf{x} - \bar{\mathbf{x}}) + \mathbf{F}_2(t)(\mathbf{x} - \bar{\mathbf{x}})^{[2]}\right]dt + \sum_{i=1}^{n_\beta}\left[\mathbf{g}^i(\bar{\mathbf{x}}, t) + \mathbf{G}^i_1(t)(\mathbf{x} - \bar{\mathbf{x}})\right]d\beta^i(t), \tag{22}$$

where $\mathbf{F}_2(t) \triangleq \nabla^{[2]}_{\mathbf{x}}\otimes\mathbf{f}(\mathbf{x}, t)|_{\mathbf{x}=\bar{\mathbf{x}}}$ and $\mathbf{G}^i_1(t) \triangleq \nabla_{\mathbf{x}}\otimes\mathbf{g}^i(\mathbf{x}, t)|_{\mathbf{x}=\bar{\mathbf{x}}}$. Adding the terms to the TEs of $\mathbf{f}$ and $\mathbf{g}^i$ makes solution to (22) for generally nonlinear functions impossible. As a solution to this problem [1] suggests replacing the third term the TE of $\mathbf{f}$ and the second term of the TE of $\mathbf{g}^i$ by their expectations, i.e.,

$$d\mathbf{x}(t) = \left[\mathbf{f}(\bar{\mathbf{x}}, t) + \mathbf{F}_1(t)(\mathbf{x} - \bar{\mathbf{x}}) + \mathbf{F}_2(t)\mathsf{E}[(\mathbf{x} - \bar{\mathbf{x}})^{[2]}]\right]dt + \sum_{i=1}^{n_\beta}\left[\mathbf{g}^i(\bar{\mathbf{x}}, t) + \mathbf{G}^i_1(t)\mathsf{E}[(\mathbf{x} - \bar{\mathbf{x}})]\right]d\beta^i(t), \tag{23}$$

Now, suppose that $\mathbf{f}(\bar{\mathbf{x}}, t)$, $\mathbf{F}_1(t)$, $\mathbf{F}_2(t)$, $\mathbf{g}^i(\bar{\mathbf{x}}, t)$, and $\mathbf{G}^i_1(t)$, are constant over the period T of discretization. Then the difference equation obtained by solving (22) is given by

$$\mathbf{x}_{k+1} = \mathbf{F}_{1D}\mathbf{x}_k + (\mathbf{F}_{1D} - \mathbf{I}_{n_x})\mathbf{F}_1^{-1}(t_k)\Big(\mathbf{f}_k(\bar{\mathbf{x}}) - \mathbf{F}_1(t_k)\bar{\mathbf{x}} + \mathbf{F}_2(t_k)\mathsf{E}[(\mathbf{x}_k - \bar{\mathbf{x}})^{[2]}]\Big) + \sum_{i=1}^{n_\beta}\int_{t_k}^{t_{k+1}}e^{\mathbf{F}_1(t_k)\cdot(t_{k+1}-t)}\bar{\mathbf{g}}^i_k d\beta^i(t), \tag{24}$$

where $\bar{\mathbf{g}}^i_k = \mathbf{g}^i(\bar{\mathbf{x}}, t_k) + \mathbf{G}^i_1(t)\mathsf{E}[(\mathbf{x}_k - \bar{\mathbf{x}})]$. Now, **Algorithm 1** requires specification of the moments $\hat{\mathbf{x}}_{k+1|l}$ and $\mathbf{P}^{\mathbf{xx}}_{k+1|l}$ which can be calculated from (24) as

$$\hat{\mathbf{x}}_{k+1|l} = \mathbf{F}_{1D}\hat{\mathbf{x}}_{k|l} + (\mathbf{F}_{1D} - \mathbf{I}_{n_x})\mathbf{F}_1^{-1}(t_k)\Big(\mathbf{f}_k(\bar{\mathbf{x}}) - \mathbf{F}_1(t_k)\bar{\mathbf{x}} + \mathbf{F}_2(t_k)\mathsf{E}[(\mathbf{x}_k - \bar{\mathbf{x}})^{[2]}]\Big) \tag{25}$$

$$\mathbf{P}^{\mathbf{xx}}_{k+1|l} = \mathbf{F}_{1D}\mathbf{P}^{\mathbf{xx}}_{k|l}\mathbf{F}^{\mathrm{T}}_{1D} + \sum_{i=1}^{n_\beta}\int_{t_k}^{t_{k+1}}[e^{\mathbf{F}_1(t_k)\cdot(t_{k+1}-t)}\bar{\mathbf{g}}^i_k][\cdot]^{\mathrm{T}}dt, \tag{26}$$

As $\bar{\mathbf{x}} = \hat{\mathbf{x}}_{k|l}$, $\mathsf{E}[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|l})^{[2]}|\mathbf{z}^l] = \mathrm{vec}(\mathbf{P}^{\mathbf{xx}}_{k|l})$, where $\mathrm{vec}(\mathbf{A})$ denotes a column-wise stacking of $\mathbf{A}$, $\mathsf{E}[(\mathbf{x}_k - \hat{\mathbf{x}}_{k|l})|\mathbf{z}^l] = \mathbf{0}$. T is again supposed to be small, the relations for the moments are

$$\hat{\mathbf{x}}_{k+1|l} = \hat{\mathbf{x}}_{k|l} + (\mathbf{F}_{1D} - \mathbf{I}_{n_x})\mathbf{F}_1^{-1}(t_k)\Big[\mathbf{f}(\hat{\mathbf{x}}_{k|l}, t_k) + \mathbf{F}_2(t_k)\,\mathrm{vec}(\mathbf{P}_{k|l})\Big] \tag{27}$$

$$\mathbf{P}^{\mathbf{xx}}_{k+1|l} = \mathbf{F}_{1D}\mathbf{P}^{\mathbf{xx}}_{k|l}\mathbf{F}^{\mathrm{T}}_{1D} + T\sum_{i=1}^{n_\beta}[\mathbf{g}^i_k(\hat{\mathbf{x}}_{k|l})][\cdot]^{\mathrm{T}}. \tag{28}$$

Note that the second-order information about $\mathbf{f}_k$ given by $\mathbf{F}_2(t_k)$ appears in the relation for the mean $\hat{\mathbf{x}}_{k+1|l}$ (27) only. The SOF given by **Algorithm 1** with (27) and (28) will be denoted as SOF-TE2-ExD.

Both, the FOF-TE1-ExD and SOF-TE2-ExD follow the approach where first the nonlinear CDM is approximated to obtain a linear CDM that is consequently linearized.

B. Filter based on approximate discretization

To discretize the nonlinear SDE (1) directly, an approximate technique must be used, such as the STE. It will be done by applying STE (10) twice, first to the nonlinear functions $\mathbf{f}$ and $\mathbf{g}^i$ and consequently to $\mathcal{L}^j\mathbf{f}$ and $\mathcal{L}^j\mathbf{g}^i$, $j = 0, 1, \cdots n_\beta$. After a few rearrangements, the discretization has the form

$$
\begin{aligned}
\mathbf{x}_{k+1} = \mathbf{x}_k &+ T \cdot \mathbf{f}_k(\mathbf{x}_k) \\
&+ \sum_{i=1}^{n_\beta} \mathbf{g}_k^i(\mathbf{x}_k) \int_{t_k}^{t_{k+1}} d\beta^i(t) + \mathcal{L}^0 \mathbf{f}_k(\mathbf{x}_k) \iint_{t_k,t_k}^{t_{k+1},s} dr ds \\
&+ \sum_{i=1}^{n_\beta} \mathcal{L}^i \mathbf{f}_k(\mathbf{x}_k) \iint_{t_k,t_k}^{t_{k+1},s} d\beta^i(r) ds \\
&+ \sum_{i=1}^{n_\beta} \mathcal{L}^0 \mathbf{g}_k^i(\mathbf{x}_k) \iint_{t_k,t_k}^{t_{k+1},s} dr d\beta^i(s) \\
&+ \sum_{i=1,j=1}^{n_\beta,n_\beta} \mathcal{L}^j \mathbf{g}_k^i(\mathbf{x}_k) \iint_{t_k,t_k}^{t_{k+1},s} d\beta^j(r) d\beta^i(s), \qquad (29)
\end{aligned}
$$

where all terms involving triple integrals were ignored. Evaluating all the double integrals in (29), the following STE2 discretization can be obtained

$$
\mathbf{x}_{k+1} = \mathbf{x}_k + T \cdot \mathbf{f}_k(\mathbf{x}_k) + \frac{T^2}{2} \mathcal{L}^0 \mathbf{f}_k(\mathbf{x}_k) + \mathcal{G}_k(\mathbf{x}_k)\mathbf{w}_k, \qquad (30)
$$

where

$$
\mathcal{G}_k(\mathbf{x}_k) = \left[\mathcal{L}^{(:)} \mathbf{f}_k - \mathcal{L}^0 \mathbf{g}_k^{(:)},\ \mathbf{g}_k^{(:)} + T \cdot \mathcal{L}^0 \mathbf{g}_k^{(:)},\ \mathcal{L}^{(:)} \mathbf{g}_k^{(:)} \right], \qquad (31)
$$

$$
\mathcal{L}^{(:)} \mathbf{f}_k = \left[\mathcal{L}^1 \mathbf{f}_k(\mathbf{x}_k) \cdots \mathcal{L}^{n_\beta} \mathbf{f}_k(\mathbf{x}_k) \right], \qquad (32)
$$

$$
\mathcal{L}^0 \mathbf{g}_k^{(:)} = \left[\mathcal{L}^0 \mathbf{g}_k^1(\mathbf{x}_k) \cdots \mathcal{L}^0 \mathbf{g}_k^{n_\beta}(\mathbf{x}_k) \right], \qquad (33)
$$

$$
\mathcal{L}^{(:)} \mathbf{g}_k^{(:)} = \left[\mathcal{L}^1 \mathbf{g}_k^1(\mathbf{x}_k) \cdots \mathcal{L}^{n_\beta} \mathbf{g}_k^{n_\beta}(\mathbf{x}_k) \right] \qquad (34)
$$

and $p(\mathbf{w}_k) = \mathcal{N}\{\mathbf{w}_k; \mathbf{0}, \mathbf{D} \otimes \mathbf{I}_{n_\beta}\}$, with

$$
\mathbf{D} = \begin{bmatrix} T^3/3 & T^2/2 & 0 \\ T^2/2 & T & 0 \\ 0 & 0 & T^2/2 \end{bmatrix}. \qquad (35)
$$

Note that the noise appearing in the discretization obtained from (29) is not Gaussian as the component due to the last double integral in (29) is Chi-square distributed. The noise $\mathbf{w}_k$ used in (30) is its moment-base Gaussian approximation.

For the filter algorithm the moments $\hat{\mathbf{x}}_{k+1|l}$ and $\mathbf{P}_{k+1|l}$ have to be calculated based on (29). For a TE of the nonlinear function $\gamma_k(\mathbf{x}_k) = \mathbf{x}_k + T \cdot \mathbf{f}_k(\mathbf{x}_k) + \frac{T^2}{2} \mathcal{L}^0(\mathbf{f}_k(\mathbf{x}_k))$ with at least two or three terms to be calculated, all third and fourth, respectively, derivatives of the function $\mathbf{f}$ are required. This becomes unbearable, especially for a higher dimension n_x. Thus, to calculate the $\hat{\mathbf{x}}_{k+1|l}$ and $\mathbf{P}_{k+1|l}^{\mathbf{xx}}$ from (30), a derivative-free technique will be used such as the unscented transform (UT) used in the unscented filter [19], which is also a second order filter in terms of its accuracy [20].

Calculation of the moments $\hat{\mathbf{x}}_{k+1|l}$ and $\mathbf{P}_{k+1|l}^{\mathbf{xx}}$ using the UT will now be briefly introduced. As the noise $\mathbf{w}_k$ in (30) is not additive, it will augment the state $\mathbf{x}_k$ to provide $\mathbf{x}_k^A \triangleq [\mathbf{x}_k^{\mathrm{T}}, \mathbf{w}_k^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^N$, where $N = n_x + 3n_\beta$. Then, the following steps must be taken:

1) Calculation of the sigma points $\{\mathcal{X}^{(i)}\}_{i=0}^{2N}$ and corresponding weights $\{\mathcal{W}^{(i)}\}_{i=0}^{2N}$ according to

$$
\mathcal{X}^{(0)} = [\hat{\mathbf{x}}_{k|l}^{\mathrm{T}}, \mathbf{0}_{3n_\beta \times 1}^{\mathrm{T}}]^{\mathrm{T}}, \qquad (36)
$$

$$
\mathcal{X}^{(i)} = \mathcal{X}^{(0)} + \left(\sqrt{(N+\kappa)\mathbf{P}_{k|l}^A} \right)_i, \ i = 1, \cdots, N, \qquad (37)
$$

$$
\mathcal{X}^{(N+i)} = \mathcal{X}^{(0)} - \left(\sqrt{(N+\kappa)\mathbf{P}_{k|l}^A} \right)_i, \ i = 1, \cdots, N, \qquad (38)
$$

where $\mathbf{P}_{k|l}^A = \mathrm{diag}[\mathbf{P}_{k|l}^{\mathbf{xx}}, \mathcal{G}_k(\mathbf{x}_k)\mathbf{D} \otimes \mathbf{I}_{n_\beta} \mathcal{G}_k(\mathbf{x}_k)^{\mathrm{T}}]$, $\sqrt{\mathbf{A}}$ is a square root of $\mathbf{A}$ such that $\sqrt{\mathbf{A}}(\sqrt{\mathbf{A}})^{\mathrm{T}} = \mathbf{A}$ and $(\mathbf{A})_i$ is the i-th column of $\mathbf{A}$. The weights $\{\mathcal{W}^{(i)}\}_{i=0}^{2N}$ are given as

$$
\mathcal{W}^{(0,\cdots,2N)} = \tfrac{1}{N+\kappa}[\kappa, \tfrac{1}{2}, \cdots \tfrac{1}{2}]. \qquad (39)
$$

2) Propagation of the sigma points $\{\mathcal{X}^{(i)}\}_{i=0}^{2N}$ through (30) as

$$
\begin{aligned}
\mathcal{Y}^{(i)} = \mathcal{X}^{(i),\mathbf{x}} &+ T \cdot \mathbf{f}_k(\mathcal{X}^{(i),\mathbf{x}}) + \frac{T^2}{2} \mathcal{L}^0 \mathbf{f}_k(\mathcal{X}^{(i),\mathbf{x}}) \\
&+ \mathcal{G}_k(\mathcal{X}^{(i),\mathbf{x}})\mathcal{X}^{(i),\mathbf{w}},\ i = 0, \cdots, 2N \qquad (40)
\end{aligned}
$$

where $\mathcal{X}^{(i),\mathbf{x}}$ represents the first n_x elements of $\mathcal{X}^{(i)}$ corresponding to the original state $\mathbf{x}_k$ and $\mathcal{X}^{(i),\mathbf{w}}$ represents the remaining $3n_\beta$ elements corresponding to the noise $\mathbf{w}_k$.

3) The predictive mean $\hat{\mathbf{x}}_{k+1|l}$ is then calculated as

$$
\hat{\mathbf{x}}_{k+1|l} = \sum_{i=0}^{2N} \mathcal{W}^{(i)} \mathcal{Y}^{(i)} \qquad (41)
$$

and the corresponding covariance matrix $\mathbf{P}_{k+1|l}$ as

$$
\mathbf{P}_{k+1|l}^{\mathbf{xx}} = \sum_{i=0}^{2N} \mathcal{W}^{(i)} \big(\mathcal{Y}^{(i)} - \hat{\mathbf{x}}_{k+1|l}\big)\big(\cdot\big)^{\mathrm{T}}. \qquad (42)
$$

The filter given by **Algorithm 1** with the UT used to obtain state prediction mean and covariance matrix will be denoted as SOF-STE2-UT.

To summarize, the following filters given by **Algorithm 1** have been introduced:

- Approximation (linearization) first, then discretization (exact)
  - **FOF-TE1-ExD** – approximation of nonlinear functions in the SDE (1) by the first order TE followed by exact discretization (20), (21),
  - **SOF-TE2-ExD** – approximation of nonlinear functions in the SDE (1) by the second order TE, replacement of some terms by their expectations, and exact discretization (27), (28),
- Discretization (approximate) first, then approximation
  - **SOF-STE2-UT** – discretization of the SDE (1) by the second order STE followed by calculation of the state prediction moments from (30) by the UT.

V. NUMERICAL ILLUSTRATION

Performance diagnosis of the SOFs will be illustrated using a model describing ship motion in deep water [21]. The ship motion is described by the following model:

$$dx(t) = [K_v v_0 \sin(\psi(t))]\, dt, \tag{43}$$

$$dy(t) = [K_v v_0 \cos(\psi(t))]\, dt, \tag{44}$$

$$d\psi(t) = [K_v \omega(t)]\, dt, \tag{45}$$

$$d\omega(t) = \left[-q\omega(t)\frac{v_0}{2pLK_v} - \frac{s_{31}v_0^2}{2pL^2}\delta(t)\right]dt + \left[\frac{s_{31}v_0^2}{2pL^2}(a + b\omega(t))\right]d\beta(t). \tag{46}$$

The state vector $\mathbf{x}(t) = [x(t), y(t), \psi(t), \omega(t)]$ consists of the ship coordinates $x(t)$ and $y(t)$, heading $\psi(t)$ and turn-rate $\omega(t)$. The variable $\delta(t)$ denotes the control rudder angle deviation, L is the ship length, $K_v = \left(1 + 1.9\omega^2 L^2 v_0\right)^{-1}$, v_0 is the ship speed at zero turn-rate, $q = q_{21}r_{31} - q_{31}r_{21}$, $p = 0.5(q_{21} + r_{31})$ with q_{21}, r_{21}, q_{31}, r_{31}, and s_{31} being hydrodynamic coefficients depending on the ship geometry and length L. The simulation was performed with the following realistic (for a stable ship) constants $q_{21} = 1.227$, $r_{21} = -0.629$, $q_{31} = -4.64$, $r_{31} = 3.88$, $s_{31} = -1.019$, $v_0 = 30$ [m/s], and $L = 99$ [m] From (43)–(46) it follows that the position and the heading are described by ordinary differential equations while only the turn-rate is described by an SDE. The noise term given by $\left[\frac{s_{31}v_0^2}{2pL^2}(a + b\omega(t))\right]d\beta(t)$ reflects uncertainty in the control rudder angle deviation which is proportional to the turn-rate with $a = 0.5$ and $b = 5$.

The control rudder angle deviation was simulated as

$$\delta(t) = \begin{cases} 30 \cdot \sin(0.01 \cdot t)\ [^\circ] & \text{for } t \in (50, 250) \text{ [s]}, \\ 0 & \text{for } t \notin (50, 250) \text{ [s]}. \end{cases} \tag{47}$$

The model (43)–(46) was simulated using the stochastic Runge-Kutta method [17] with sampling time $T_s = 0.01$ [s] with $M = 10^3$ simulations. The state $\mathbf{x}(t)$ was observed with measurement period $T_z = 50$ [s] as

$$\mathbf{z}_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{x}_k + \mathbf{v}_k, \tag{48}$$

which means that the position of the ship was observed directly in Cartesian coordinates. The observation was corrupted by the white zero-mean Gaussian noise $\mathbf{v}_k$ with covariance matrix $\Sigma_k^{\mathbf{v}} = \mathrm{diag}([4, 4])$ [m^2].

The motion was simulated for 300 seconds with initial conditions $\mathbf{x}_0 = [10\text{ [m]}, 10\text{ [m]}, 45\ [^\circ], 0\ [^\circ/\text{s}]]$. The state was estimated by **FOF-TE1-ExD**, **SOF-TE2-ExD**, and **SOF-STE2-UT** with discretization periods $T = \{0.1, 0.5, 1\}$ [s]. The filters were initialized with $\hat{\mathbf{x}}_{0|-1} = \bar{\mathbf{x}}_0$ and $\mathbf{P}_0^{\mathbf{xx}} = \mathrm{diag}([10^4\ [\text{m}^2], 10^4\ [\text{m}^2], (1.5)^2\ [(^\circ)^2], 1\ [(^\circ/\text{s})^2]])$.

Performance of the filters was measured using the root mean squared error (RMSE)

$$\mathrm{RMSE}_{x,k} = \sqrt{\frac{1}{M}\sum_{m=1}^{M}[x_k(m) - \hat{x}_k(m)]^2} \tag{49}$$

Table I
TIME AVERAGE MSE OF THE SECOND ORDER FILTERS.

	FOF-TE1-ExD	SOF-TE2-ExD	SOF-STE2-UT
		$T = 0.1$ [s]	
ARMSE$_x$	49.89	46.73	44.95
ARMSE$_y$	73.23	71.96	69.87
ARMSE$_\psi$	0.197	0.189	0.183
ARMSE$_\omega$	0.0085	0.0085	0.0085
		$T = 0.5$ [s]	
ARMSE$_x$	50.03	46.94	44.32
ARMSE$_y$	73.44	71.61	68.13
ARMSE$_\psi$	0.196	0.186	0.173
ARMSE$_\omega$	0.0084	0.0084	0.0084
		$T = 1$ [s]	
ARMSE$_x$	53.42	49.89	46.49
ARMSE$_y$	75.34	74.16	69.64
ARMSE$_\psi$	0.209	0.197	0.180
ARMSE$_\omega$	0.0087	0.0087	0.0086

for the position in the x-coordinate (and analogously for other state elements), where $x_k(m)$ is the true x-coordinate of the ship at time $t = k \cdot T$ and m-th simulation and $\hat{x}_k(m)$ is its estimate. The RMSE is depicted in Figs. 1, 2, 3, 4 and its time-average (ARMSE) values defined by

$$\mathrm{ARMSE}_x = \frac{1}{K}\sum_{k=1}^{K}\mathrm{RMSE}_{x,k} \tag{50}$$

are given in Table I.

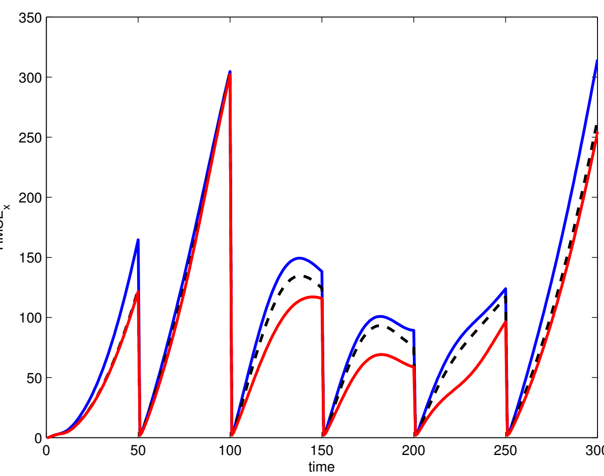

Figure 1. RMSE for the position in x coordinate for $T = 1$ [s].

From the figures and the table it is clear that both SOFs achieve higher estimate quality than the first order filter (FOF-TE1-ExD)[1], which was expected. Among the two approaches differing in the order of discretization and approximation, the approach represented by the SOF-STE2-UT, i.e., discretization followed by approximation leads to a higher estimate quality than the reverse order of the operations represented by the SOF-TE2-ExD. The difference is apparent especially in both

[1]As has been mentioned above, this algorithm corresponds to the EKF.

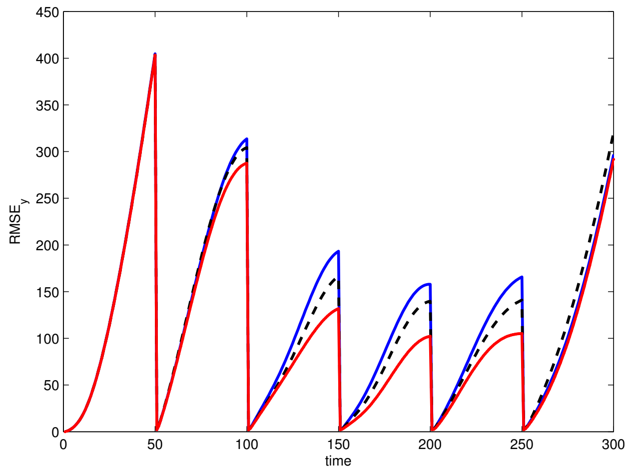

Figure 2. RMSE for the position in y coordinate for $T = 1$ [s].

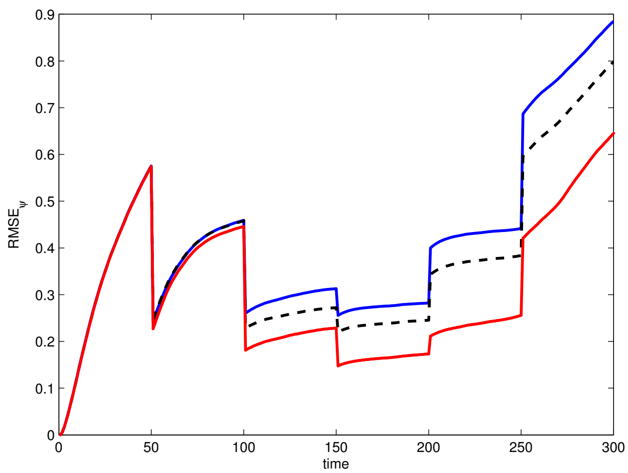

Figure 3. RMSE for the heading ψ for $T = 1$ [s].

positions and heading. Quality of the turn-rate estimates is equal for all three filters.

VI. Concluding Remarks

The paper focused on state estimation of nonlinear continuous-discrete models by means of discrete nonlinear filters. Design of such filters requires discretization of the state dynamics described by a stochastic differential equation and approximation of nonlinear functions. For the discretization the paper considers either exact discretization or the discretization by means of stochastic Taylor expansion. The nonlinear functions appearing in the dynamics are approximated by the Taylor expansion. Second order filters have been proposed for the original problem differing in order of the approximation and discretization techniques. Numerical example illustrated that the approach based on discretization followed by an approximation leads to a higher estimate quality than the approach based on approximation followed by discretization.

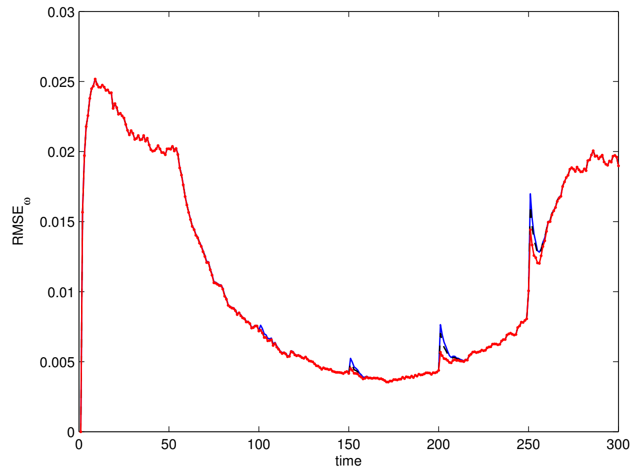

Figure 4. RMSE for the turn-rate ω for $T = 1$ [s].

Although this fact cannot be generalized, it follows from the numerical example that the order of the approximation and discretization techniques should be analyzed when applying a discrete filter in state estimation of a CDM, especially with respect to estimate quality.

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