Homework Problems for the Complexity Explorer Course on Random Walks

1. `Displacement of a random walk.` Consider the Pearson random walk in any spatial dimension in which the length of each step has the fixed value a, but the direction is arbitrary. Compute the fourth moment of the displacement after N steps

$$M_4 \equiv \left\langle \Big(\sum_{i=1}^{N} \mathbf{x}_i\Big)^4 \right\rangle .$$

2. `Probability distribution of a biased random walk.` Consider a random walk in one dimension in which a step to the right occurs with probability p and a step to the left occurs with probability $q = 1 - p$.

   (a) Determine the probability $P(r, \ell, t)$ that a walk of t total steps has taken r steps to the right and ℓ steps to the left.

   (b) Transform the above expression for $P(r, \ell, t)$ to $P(x, t)$, the probability that the walk is at x at time t.

   (c) Use Stirling's approximation to compute the long-time limit of the probability distribution $P(x, t)$.

3. `Diffusion equation.` Consider a random walk that steps to the right with probability 1/3, to the left with probability 1/3, and remains in place with probability 1/3.

   (a) Write the Master equation that determines the evolution of $P(x, t)$.

   (b) Taylor expand the master equation and determine the partial differential equation that is satisfied by $P(x, t)$. Determine the diffusion coefficient D of this process and compare it with the diffusion coefficient of the nearest-neighbor random walk.

4. `Kinetic theory.` The density of a typical room-temperature gas is $\rho \approx 10^{20}$ molecules/cc and each molecule has a typical speed that is roughly 30000 cm/sec. Estimate the number of collisions that the ambient air makes with your body per second.

5. `Extreme value statistics.` The distribution of velocities of an ideal gas of molecules of mass m at temperature T is given by the Maxwell-Boltzmann distribution

$$P(\mathbf{v})d\mathbf{v} = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT}\, d\mathbf{v}\,,$$

   where kT, the thermal energy of each molecule is roughly $\frac{1}{40}$ eV at room temperature ($\approx 300\,\mathrm{K}$). Thus $P(\mathbf{v})d\mathbf{v}$ is the probability that a molecule as a velocity that is in a range $d\mathbf{v}$ about $\mathbf{v}$. Using the reasoning given in the discussion of the failure of the central limit theorem, estimate the energy of the most energetic molecule in a gas at room temperature in a room of volume of 1000 cubic meters.

6. `First-passage in a finite interval.` Consider a biased random walk in the finite interval $[0, L]$ in which the walk hops to the right with probability p and to the left with probability $q = 1 - p$. The walk is immediately absorbed when it reaches either 0 or L.

(a) Determine the probability E_n that the walk is absorbed at L when it starts at site n.

(b) Determine the average time t_n for the walk to exit the interval either at 0 or at L when it starts at site n. Find the starting location that maximizes the exit time.

7. `First-passage on the semi-infinite interval.` Consider a biased nearest-neighbor random walk with hopping probabilities p and q to the right and left, respectively, on the semi-infinite interval $[0, \infty]$. The walk is absorbed if it reaches $x = 0$.

   (a) Using the backward Kolmogorov approach, determine the probability E_n that the walk is eventually absorbed at the origin when it starts at site n. Separately consider the cases $p > q$, $p < q$, and $p = q$ (symmetric walk).

   (b) For the case where $p < q$ (bias to the left), compute the average time t_n for the walk to reach the origin when it starts at site n. What is the limiting behavior of this exit time in the limit $p = q$?