Introduction to Fractals and Scaling
Homework for Unit 5:

Empirical Power Laws

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Beginner

1. Suppose a quantity x is distributed according to a power law with exponent $\alpha = 2$. What is the exponent for the cumulative distribution function?

2. Consider the rank/frequency plot shown in Fig. 1.

   (a) How many data points are equal to or larger than 1?
   (b) How many data points are equal to or larger than 5?
   (c) How many data points are equal to or larger than 15?

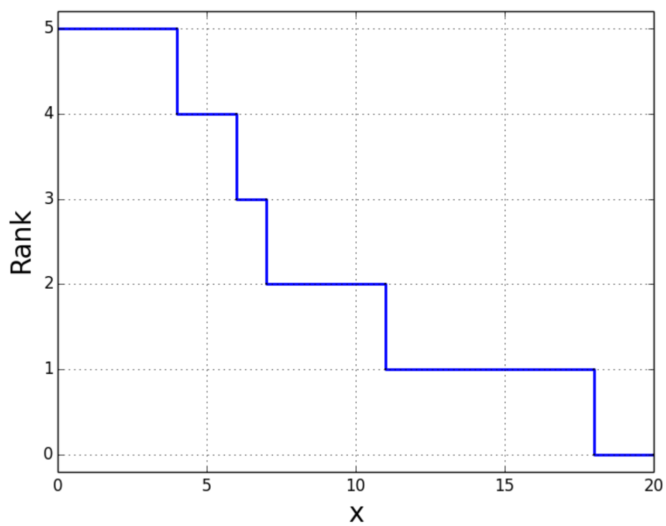

Figure 1: A rank-frequency plot.

3. Suppose you have the following data: $3, 5, 9, 12, 13, 25, 31, 41, 43, 58$. Let $P(x)$ be the cumulative distribution function for these data. What is the value of:

   (a) $P(1)$?
   (b) $P(10)$?
   (c) $P(25)$?
   (d) $P(72)$?

Intermediate

1. Suppose you found the best power-law fit for a set of data. You then estimated the p-value for this fit and obtained a value of 0.85. Does this provide evidence for or against the proposition that the data is well-described by a power law? What if the p-value had been 0.04?

Advanced

1. In this problem we will work with a continuous power law distribution,

$$p(x) \; = \; Ax^{-\alpha} \; , \tag{1}$$

where x can take on any value between 1 and ∞. However, the power-law behavior does not extend for all x. Instead, it only applies for $x \geq x_{\min}$. Thus, the normalization condition is

$$\int_{x_{\min}}^{\infty} p(x)\, dx \; = \; 1 \; . \tag{2}$$

   (a) Evaluate this integral and solve for A .

   (b) The show that one can write Eq. (1) in the following form:

$$p(x) \; = \; \frac{\alpha - 1}{x_{\min}} \left(\frac{x}{x_{\min}} \right)^{-\alpha} \; . \tag{3}$$

2. Suppose you have n data points: $x_1, x_2, \ldots x_n$ and you wish to fit a power-law $p(x)$ to this data. Derive the maximum likelihood estimator for α. Use the form for $p(x)$ given in Eq. (3). You should end up with:

$$\hat{\alpha} \; = \; 1 + n \left(\sum_{i=1}^{n} \ln \frac{x_i}{x_{\min}} \right)^{-1} \; . \tag{4}$$

To do so, it will be convenient to maximize the logarithm of the likelihood function instead of the likelihood function. Some intermediate steps in the derivation can be found in Aaron Clauset, Cosma Rohilla Shalizi, and Mark E.J. Newman, Power-law distributions in empirical data, *S*IAM review. 51.4 (2009): 661-703.