Introduction to Fractals and Scaling
Solutions to Unit 1 Homework

Introduction to Fractals and the Self-Similarity Dimension

David P. Feldman
http://www.complexityexplorer.org

Beginner

1. There are 8 small copies in the large copy, and each copy is scaled down by 1/3. So

$$\text{No. of small copies} \ = \ (\text{magnification factor})^D \ . \tag{1}$$

Plugging in and solving for D:

$$8 \ = \ 3^D \ , \tag{2}$$

$$\log(8) \ = \ \log(3^D) \ , \tag{3}$$

$$\log(8) \ = \ D \log(3) \ , \tag{4}$$

$$D \ = \ \frac{\log(8)}{\log(3)} \approx 1.892 \ . \tag{5}$$

2. A step $n = 3$ in the construction of the Sierpiński triangle:

   (a) There are $3^3 = 27$ small triangles.

   (b) At each step the triangles are 1/4 the area than at the previous step. So, taking the initial area to be 1, then the area at $n = 3$ is $(1/4)^3 = 1/64$.

   (c) There are 27 triangles, each with area 1/64. So the total area is 27/64.

   (d) Let's assume that the side of the triangle at step $n = 0$ is 1. Then at $n = 1$, the perimeter of a single triangle is 3/2, since the triangles have three sides, each with a length of 1/2. So at step $n = 2$, the perimeter of each triangle is 3/4 (again each side is halved), and at $n = 3$, the perimeter is 3/8.

   (e) The total perimeter of the shape at $n = 3$ is $27 \times (3/8) = 3(3/2)^3 = 10.125$.

3. (a) At $n = 2$ each line segment has a length of 1/9, since at each step the line segments are 1/3 the size as the previous.

   (b) At $n = 2$ there are $2^2 = 4$ line segments. At every step, the number of line segments is doubled.

   (c) At $n = 2$ the total length of all the line segments is $4(1/9) = 4/9$.

   (d) At step n, the length of the line segments is $(1/3)^n$

   (e) At step n, the number of line segments is 2^n.

   (f) At step n, the total length of the line segments is $(2/3)^n$.

(g) As n gets larger and larger, the number of line segments gets larger and larger.

(h) As n gets larger and larger, the total length of the line segments approach zero.

4. There are two small copies of the set in the large set, and each small copy must be scaled up by a factor of 3. So,

$$\text{No. of small copies} = (\text{magnification factor})^D \; . \tag{6}$$

Plugging in,

$$2 = 3^D \; . \tag{7}$$

Solving for D, we find

$$D = \frac{\log(2)}{\log(3)} \approx 0.631 \; . \tag{8}$$

5. If a circle is scaled up by a factor of 3, its area increased by $3^2 = 9$, because it is two dimensional.

6. If a tomato is scaled up by a factor of 2.5, then its volume is scaled up by $2.5^3 \approx 15.625$, because it is three dimensional.

7. If an object with a dimension of 1.81 is scaled up by a factor of 2, then the object's size has increased by $2^{1.81} \approx 3.51$.

Intermediate

1. There are 5 small copies, each of which is a copy of the big pyramid scaled down by 1/2. So,

$$\text{No. of small copies} = (\text{magnification factor})^D \; , \tag{9}$$

$$5 = 2^D \; . \tag{10}$$

Solving for D, we find

$$D = \frac{\log(5)}{\log(2)} \approx 2.32 \; . \tag{11}$$

2. There are 20 small copies of the shape in the full shape. To see this, note that in the first step of the construction, 7 cubes are missing: one from each of the 6 faces and one from the middle. The magnification factor is 3, since each small cube's side is 1/3 that of the big cube. Thus,

$$\text{No. of small copies} = (\text{magnification factor})^D \; , \tag{12}$$

$$20 = 3^D \; . \tag{13}$$

Solving for D, we find:

$$D = \frac{\log(20)}{\log(3)} \approx 2.73 \; . \tag{14}$$

3. (a) The next step in the construction is shown in Fig. 1.
   (b) There are 5 small copies of the shape, each of which is 1/3 the size of the full copy. So,

$$\text{No. of small copies} \;=\; (\text{magnification factor})^D \;, \tag{15}$$

$$5 \;=\; 3^D \;. \tag{16}$$

   Thus,

$$D \;=\; \frac{\log(5)}{\log(3)} \;\approx\; 1.465 \;. \tag{17}$$

Advanced

1. (a) The first several steps in the construction of the middle-fifths Cantor set are shown in Fig. 2.
   (b) There are two small copies of the shape in the full shape. Each small copy is (2/5) the size of the full copy. (To see this, note that the gap in the middle is one fifth. The two remaining pieces are each two fifths.) So, the magnification factor is (5/2) or 2.5. We can now calculate the dimension, as usual:

$$\text{No. of small copies} \;=\; (\text{magnification factor})^D \;, \tag{18}$$

$$2 \;=\; 2.5^D \;. \tag{19}$$

   Thus,

$$D \;=\; \frac{\log(2)}{\log(2.5)} \;\approx\; 0.756 \;. \tag{20}$$

2. The first step in the construction of the Peano curve is shown in Fig. 3.
   (a) The next step in the construction of the curve is shown in Fig. 3.
   (b) See Fig. 3. The curve is approaching a solid square. Such a curve is said to be *space filling.*
   (c) To figure out the dimension, it is easiest to examine what happens from step $n = 0$ to $n = 1$. One line segment is replaced by 9 line segments, each of which is a third as long as the original one. So,

$$\text{No. of small copies} \;=\; (\text{magnification factor})^D \;, \tag{21}$$

$$9 \;=\; 3^D \;, \tag{22}$$

   and we see that the dimension D is 2. This seems reasonable. The curve is so windy that it fills up space—it takes up area. Hence, the dimension of the curve is 2.

3. (a) The equation $5^x = -10$ does not have any solutions. There are a number of ways to see this. We could take the logarithm of both sides of the equation in an attempt to isolate solve for x. However, $\log(-10)$ does not exist.[1] One could also make a plot of 5^x and note that it is always positive, again confirming that there is no value of x for which $5^x = -10$.

   (b) At first glance, the equation $5^x = \frac{1}{2}x$ looks harmless. Let's take the logarithm of both sides and see what happens:

$$\log(5^x) = \log(0.5x) \; . \tag{23}$$

   Applying log properties, we get,

$$x \log(5) = \log(0.5x) \; . \tag{24}$$

   The good news is that the x on the left-hand side of the equation is now downstairs. The bad news is that the x on the right-hand side is inside a log. What to do? Well, we can exponentiate both sides of the equation; this will free the right-hand x from the log:

$$10^{x \log(5)} = 10^{\log(0.5x)} \; , \tag{25}$$

$$10^{\log(5^x)} = 10^{\log(0.5x)} \; , \tag{26}$$

$$5^x = 0.5x \; . \tag{27}$$

   Good news: the right-hand side x is no longer inside the log. Bad news: the left-hand x is back up in the exponent. We've gone in a circle and are right back where we started.

   It turns out that this equation cannot be solved using algebra. Instead, numerical or graphical methods are needed. A graph of the functions 5^x and $0.5x$ are shown in Fig. 4. We see that the two curves to not intersect. This means that there is no x value for which 5^x equals $0.5x$. Thus, the equation does not have a solution.

   (c) The equation $5^x = 10x$ is similar to the one analyzed above; it cannot be solved using algebra. So again we turn to a plot of the left- and right-hand sides; see Fig. 5. Here we can see that there are two solutions to the equation, since there are two points where the curves intersect. The x values where the curves intersect appear to be around 0.1 and 1.8. We could zoom in on the graph if we desired more precise numbers or use a program to determine a numerical answer. Doing so on wolframalpha.com via the command `solve(56x = 10x)`, I get $x \approx 0.121621$ and $x \approx 1.79372$.

[1] Why? Because there is no number a such that $10^a = -10$.

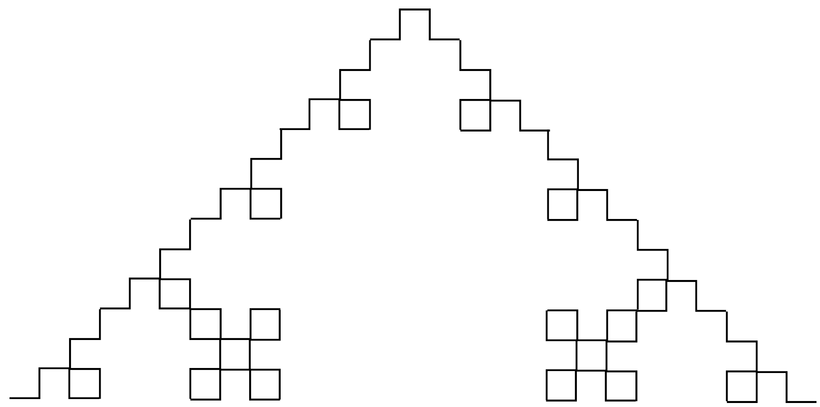

Figure 1: The next step in the construction of quadratic Koch curve.

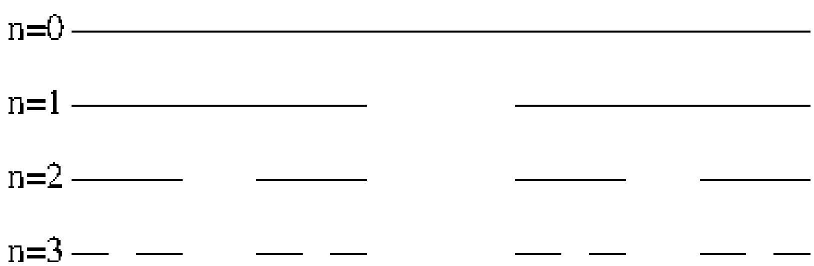

Figure 2: The next step in the construction of the Peano curve.

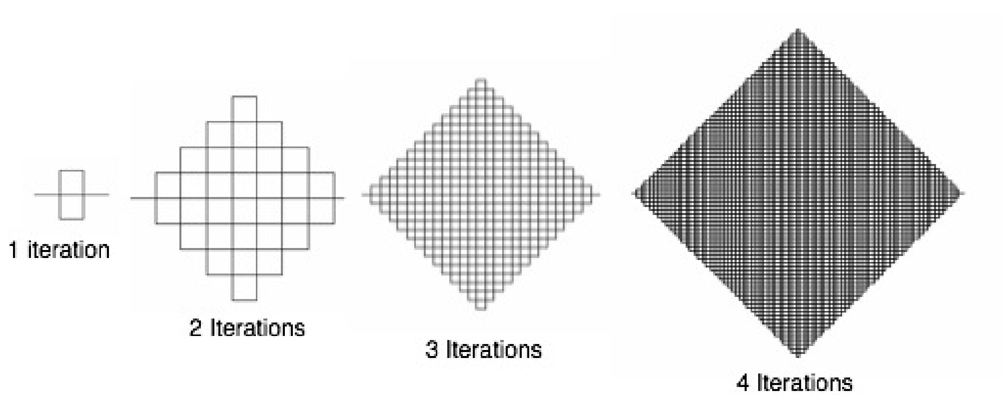

Figure 3: The next steps in the construction of the Peano curve. Figure source `http://scienceblogs.com/goodmath/2007/07/25/fractal-pathology-peanos-space/`.

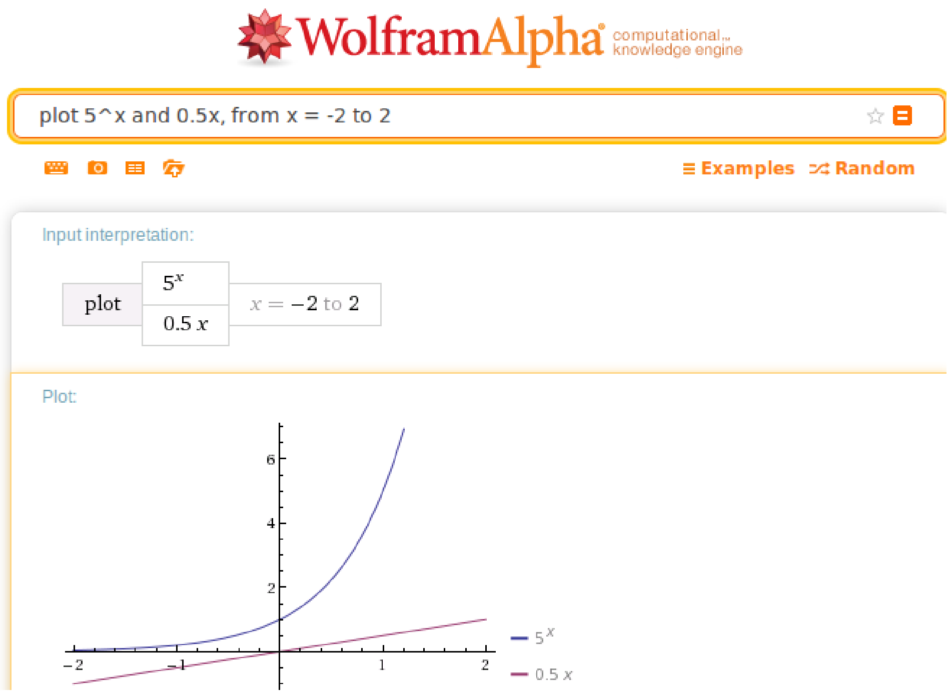

Figure 4: A graph of 5^x and $0.5x$ generated by `www.wolframalpha.com`.

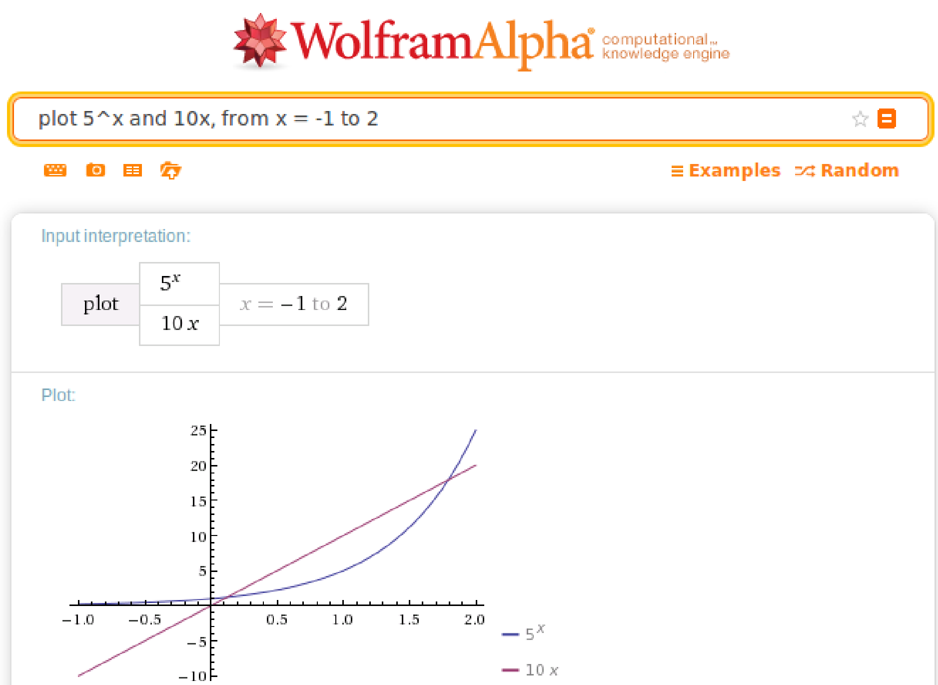

Figure 5: A graph of 5^x and $10x$ generated by `www.wolframalpha.com`.