Solutions

Introduction to Dynamical Systems and Chaos
Homework for Unit 2: Differential Equations

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http://www.complexityexplorer.org

Beginner

1. (a) The phase line is shown in Fig. 1.

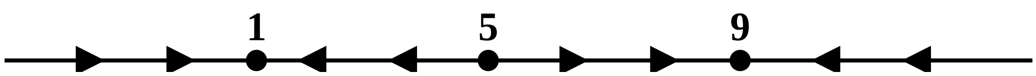

Figure 1: The solution to problem 1(a).

(b) There is a stable fixed point at 1, an unstable fixed point at 5, and a stable fixed point at 9.

(c) If the initial condition is $X = 8$, then the solution will increase and approach the stable fixed point at $X = 9$.

(d) If the initial condition is $X = 6$, then the solution will increase and approach the stable fixed point at $X = 9$.

(e) If the initial condition is $X = 2$, then the solution will decrease and approach the stable fixed point at $X = 1$.

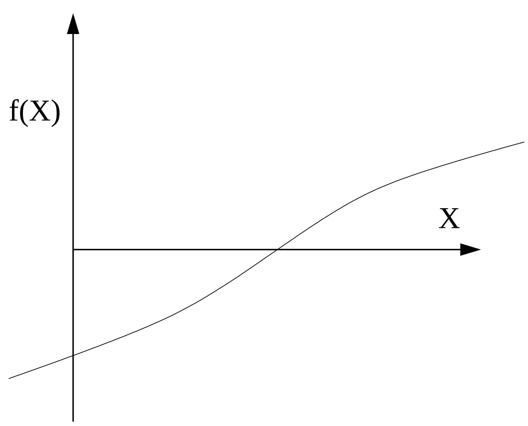

Figure 2: A solution to problem 2(a).

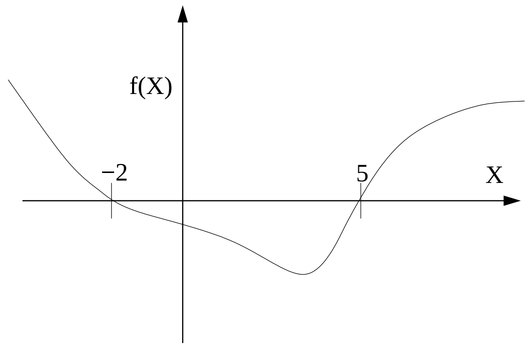

Figure 3: A solution to problem 2(b).

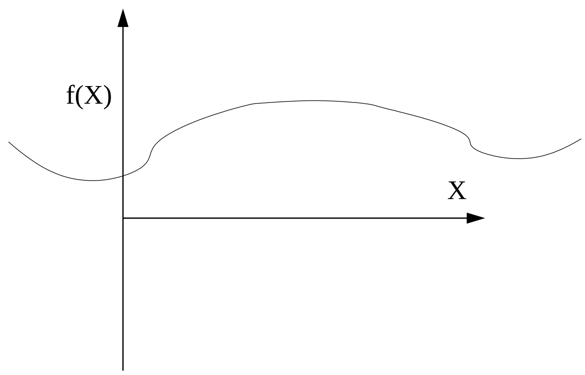

Figure 4: A solution to problem 2(c).

2. (a) There are many possible solutions. To have one unstable fixed point, $f(X)$ must cross the zero-axis only once. And this crossing must be from below, as one looks left-to-right. This will lead to a fixed point that is unstable, because to the left of the fixed point $f(X)$ is negative (and so $X(t)$ decreases), and to the right of the fixed point $f(X)$ is positive (and so $X(t)$ increases). One possible $f(X)$ is shown in Fig.2.

   (b) There are many possible solutions. The function $f(X)$ must cross the zero-axis only twice. Once at -2 and once at 5. In order for the fixed point at -2 to be stable, the crossing at -2 must be from above. And in order for the fixed point at 5 to be unstable, the crossing at 5 must be from below. One such $f(X)$ is shown in Fig. 3.

   (c) There are many possible solutions. Any $f(X)$ that is always positive will lead to a solution curve that has no fixed points and which always increases. One such $f(X)$ is shown in Fig. 4.

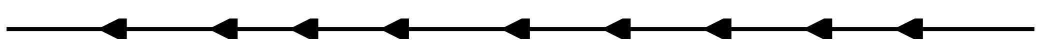

Figure 5: The solution to problem 3(a).

3. (a) The phase line is shown in Fig. 5.

   (b) This function has no fixed points. There are no X values for which $\frac{dX}{dt} = 0$.

4. Initially, $T = 10$. We find the rate of change from the differential equation:

$$\frac{dT}{dt} = 0.2(20 - 10) = 0.2(10) = 2\,\mathrm{C/min}\ . \tag{1}$$

So, we pretend that this rate of change is constant for 2 minutes to find the temperature at $t = 2$:

$$T(2) = 10\,\mathrm{C} + 2\,\mathrm{min}\,(2\,\mathrm{C/min}) = 14\,\mathrm{C}\ . \tag{2}$$

So at $t = 2$, the temperature T is 14. To determine the rate of change of the temperature at $t = 2$, plug 14 in to the differential equation:

$$\frac{dT}{dt} = 0.2(20 - 14) = 0.2(6) = 1.2\,\mathrm{C/min}\ . \tag{3}$$

We then pretend that this rate of change is constant from $t = 2$ to $t = 4$ to find the temperature T at $t = 4$:

$$T(2) = 14\,\mathrm{C} + 2\,\mathrm{min}\,(1.2\,\mathrm{C/min}) = 16.4\,\mathrm{C}\ . \tag{4}$$

Intermediate

1. (a) The phase line is shown in Fig. 6. Note that $X(t)$ is increasing everywhere except for the fixed point at 2.

   (b) This fixed point is stable from the left and unstable from the right. That is, if one is at the fixed point at 2 and moves a bit to the left, one returns to the fixed point. So this is stable. However, if one is at 2 and moves a bit to the right, one gets pushed away. So this is unstable. A fixed point that is stable on one side and unstable on the other is sometimes called semi-stable or half-stable. In my experience these fixed points do not frequently occur, which is why I didn't mention them in the lectures.

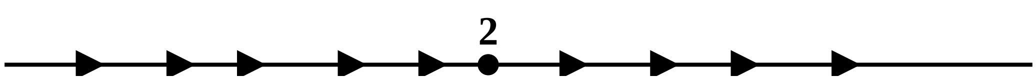

Figure 6: The phase line for intermediate problem 1.

2. We are given that $Y = 100$ at $t = 0$. We can figure out the initial rate of change via the differential equation:

$$\frac{dY}{dt} = -\frac{1}{2}100 = -50\ . \tag{5}$$

We pretend that this rate of change is constant from $t = 0$ to $t = 2$ to get the value of Y at $t = 2$:

$$Y(2) = 100 + (-50)(2) = 0\ . \tag{6}$$

So at $t = 2$, $Y = 0$. The differential equation then tells us the rate of change at $t = 2$. Plugging in $Y = 0$, we obtain:

$$\frac{dY}{dt} = -\frac{1}{2}0 = 0 . \tag{7}$$

So the rate of change is 0. Thus the value of Y is not changing, and so $Y(4) = 0$.

This example illustrates that Euler's method with a too large Δt can give potentially misleading results. In this case the solution to this differential equation should exponentially decay, approaching a stable fixed point at $Y = 0$. However, since Δt is so large, the approximate Euler solution happens to land exactly on the fixed point after two steps, so we wouldn't observe the exponential decay.

In other cases even more dramatically wrong behavior can be observed. If Δt is too large it is possible to "step over" a stable fixed point and then get pushed away by a repelling fixed point. In practice, though, this isn't too big a problem, as long as one remembers that Euler's method is only approximate at that it is important to experiment with smaller and smaller Δt values. Also, often a numerical method like Euler's would be used in combination with the qualitative techniques. That is, one can find the fixed points and the phase line first to get a qualitative picture of what the solutions have to look like and then use this as a guide when evaluating numerical solutions.

Advanced

1. I will post in the forum a copy the Euler code that I wrote in python to make the figures for this chapter.

2. X is 1 at $t = 0$ and so according to the graph of $f(X)$ shown in Fig. 3 of the assignment, the rate of change of X is approximately 1 at $t = 0$. So X is initially increasing at a rate of 1. For Euler's method we would assume that this rate of change is constant for the time interval from $t = 0$ to $t = 1$. This yields a value of $X = 2$ at $t = 1$. (X started at 1 and increased at a rate of 1 for 1 time unit: $1 + (1 \times 1) = 1$.)

   However, the rate of change of X is actually not constant during this time interval. We can see from Fig. 3 of the assignment that the rate of change decreases as X increases from 1. Thus, Euler's method will *overestimate* the value of X at $t = 1$.

3. Euler's method is approximate because we are pretending that the rate of change of X (or whatever is the variable of interest) is constant when it is not. However, if we had a differential equation for which the rate of change *was* constant, then Euler's method would yield exact solutions. Thus, equations of the form

$$\frac{dX}{dt} = k , \tag{8}$$

   where k is a constant, are exactly solvable by Euler's method for any Δt. In practice, however, one wouldn't need Euler's method to solve for $X(t)$. A function $X(t)$ whose rate of change is constant (this is what Eq.(8) is asking for) is just a straight line. So solutions to Eq.(8) are:

$$X(t) = X(0) + kt , \qquad (9)$$

where $X(0)$ is the value of X at time $t = 0$.

4. It is **not** possible for a differential equation of the form $\frac{dX}{dt} = f(X)$ to have only two fixed points, both of which are attracting, unless $f(X)$ is discontinuous. I think this is easiest to see graphically. Consider the $f(X)$ shown in Fig. 7. This differential equation has two stable fixed points, one at A and one at C. As we've seen, a stable fixed point occurs when the function $f(X)$ crosses the zero-axis from above. So we are to have two stable fixed points, $f(X)$ has to cross the zero-axis twice, both times from above. This is not possible if $f(X)$ is continuous. The $f(X)$ curve has to move from below the axis to above the axis. Thus, it must cross the zero-axis from below, giving rise to an unstable fixed point. This occurs at point B in the figure.

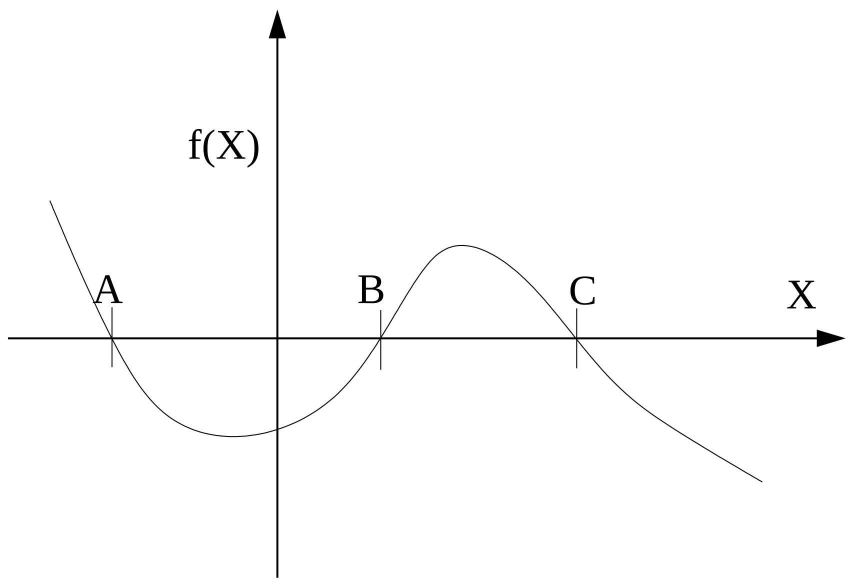

Figure 7: The figure for advanced problem 4.