Summary and Overview of Unit 2: Differential Equations

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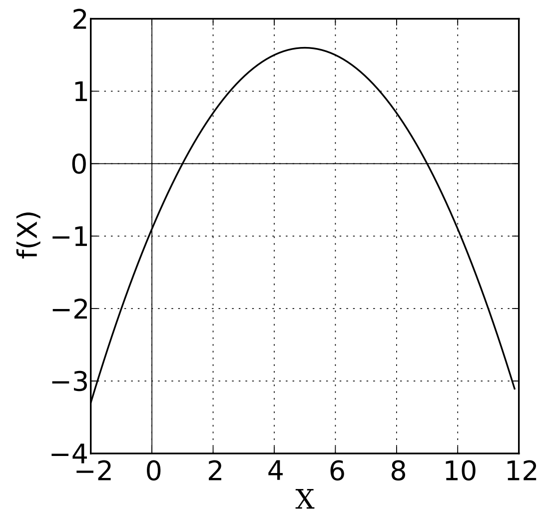
Differential Equations: A Type of Dynamical System

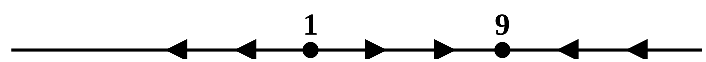
We have looked at differential equations (DEs) of the form

$$\frac{dX}{dt} = f(X) \; .$$

- A dynamical system is a system that changes in time according to a well-specified, unchanging rule.
- A DE is a dynamical system. The DE specifies the derivative of X as a function of X.
- The derivative $\frac{dX}{dt}$ is the instantaneous rate of change of X.

Solution Method 1: Qualitative





- Use a graph of $f(X)$ to find fixed points and their stability.
- Can draw phase line and sketch solutions $X(t)$.
- Can't get an exact $X(t)$.
- Gives overall feel for long-term behavior of solutions.

Solution Method 2: Computational

$$\frac{dX}{dt} = f(X) \ .$$

- Use Euler's method (or something similar) to figure out $X(t)$ step by step.
- Gets at the heart of what a differential equation is.
- Reliable. Works for all (well-posed) equations.
- Requires use of software.
- The result is called numerical, because it is a table of numbers, not a formula.

Solution Method 3: Analytic

$$\frac{dX}{dt} = f(X) \; .$$

- Finding a formula for the solution $X(t)$ using calculus.
- Most non-linear equations cannot be solved analytically.
- Even for equations that can be solved analytically, doing so does not always lead to intuition or understanding.
- In my view, many differential equations textbooks place too much emphasis on analytic techniques.
- This course will focus on qualitative and computational solutions. These methods are much better suited for dynamical systems and chaos.

Fixed Points for Differential Equations

$$\frac{dX}{dt} = f(X) \; .$$

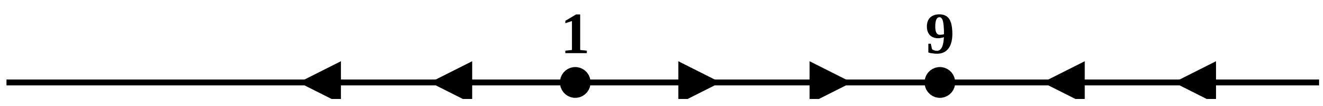
- Just like iterated functions, differential equations have fixed points.
- A fixed point is often referred to as an equilibrium point.
- A point X is fixed if it does not change.
- A point X is fixed if its derivative is zero: $\frac{dX}{dt} = 0$.

Stability

- A fixed point is **stable** if nearby points move closer to the fixed point when they are iterated.
- A stable fixed point is also called an **attracting** fixed point or an **attractor** or a **sink**.
- A fixed point is **unstable** if nearby points move further away from the fixed point when they are iterated.
- An unstable fixed point is also called a **repelling** fixed point or a **repellor** or a **source**.

Phase Lines

- The **phase line** lets us see all at once the long-term behavior of all initial conditions.
- Time information is not included in the phase line. We can tell what direction the solutions go, but not how fast.
- Example: the differential equation whose phase line is shown below has an attractor at 9 and a repellor at 1.



Dynamical Systems

- One of the goals of the study of dynamical systems is to classify and characterize the sorts of behaviors seen in classes of dynamical systems.
- So far, for differential equations, we have seen:
  - Fixed points (stable and unstable).
  - Orbits can approach a fixed point.
  - Orbits can tend toward infinity
  - Orbits can tend toward negative infinity
  - An orbit cannot increase and then decrease. Cycles or oscillations are not possible.