HMMT Online 2020: Algebra and Number Theory

Special instructions for submitting answers

The problems are on the next page. As the web server only accepts integer answers, you need to take some extra steps to format your answer appropriately. The instructions are included below. Note: Negative integer answers are possible.

1. The answer is an integer. Submit the integer.
2. The answer is an ordered pair of the form (a, b), where a and b are positive integers. Submit the concatenation of a followed by b.
3. The answer is of the form p^a, where p is a prime number and a is a positive integer. Submit the concatenation of p followed by a.
4. The answer is an integer. Submit the integer.
5. The answer is a fraction of the form $\frac{a}{b}$, where a and b are relatively prime positive integers. Submit the concatenation of a followed by b.
6. The answer is an integer. Submit the integer.
7. The answer is an integer. Submit the integer.
8. The answer is a negative integer x which has large absolute value. Submit the remainder when $|x|$ is divided by the prime number 2017.
9. The answer is of the form $p^a \cdot b$, where p is a prime number such that a is maximized, and a and b are positive integers. Submit the concatenation of p followed by a followed by b.
10. The answer is of the form p^a, where p is a prime number and a is a positive integer. Submit the concatenation of p followed by a.

HMMT February 2020

February 15, 2020

Algebra and Number Theory

1. Let $P(x) = x^3 + x^2 - r^2x - 2020$ be a polynomial with roots r, s, t. What is $P(1)$?

2. Find the unique pair of positive integers (a, b) with $a < b$ for which

$$\frac{2020 - a}{a} \cdot \frac{2020 - b}{b} = 2.$$

3. Let $a = 256$. Find the unique real number $x > a^2$ such that

$$\log_a \log_a \log_a x = \log_{a^2} \log_{a^2} \log_{a^2} x.$$

4. For positive integers n and k, let $\mho(n, k)$ be the number of distinct prime divisors of n that are at least k. For example, $\mho(90, 3) = 2$, since the only prime factors of 90 that are at least 3 are 3 and 5. Find the closest integer to

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{\mho(n, k)}{3^{n+k-7}}.$$

5. A positive integer N is *piquant* if there exists a positive integer m such that if n_i denotes the number of digits in m^i (in base 10), then $n_1 + n_2 + \cdots + n_{10} = N$. Let p_M denote the fraction of the first M positive integers that are piquant. Find $\lim_{M \to \infty} p_M$.

6. A polynomial $P(x)$ is a *base-n polynomial* if it is of the form $a_dx^d + a_{d-1}x^{d-1} + \cdots + a_1x + a_0$, where each a_i is an integer between 0 and $n - 1$ inclusive and $a_d > 0$. Find the largest positive integer n such that for any real number c, there exists at most one base-n polynomial $P(x)$ for which $P(\sqrt{2} + \sqrt{3}) = c$.

7. Find the sum of all positive integers n for which

$$\frac{15 \cdot n!^2 + 1}{2n - 3}$$

is an integer.

8. Let $P(x)$ be the unique polynomial of degree at most 2020 satisfying $P(k^2) = k$ for $k = 0, 1, 2, \ldots, 2020$. Compute $P(2021^2)$.

9. Let $P(x) = x^{2020} + x + 2$, which has 2020 distinct roots. Let $Q(x)$ be the monic polynomial of degree $\binom{2020}{2}$ whose roots are the pairwise products of the roots of $P(x)$. Let α satisfy $P(\alpha) = 4$. Compute the sum of all possible values of $Q(\alpha^2)^2$.

10. We define $\mathbb{F}_{101}[x]$ as the set of all polynomials in x with coefficients in $\mathbb{F}_{101}$ (the integers modulo 101 with usual addition and subtraction), so that two polynomials are equal if and only if the coefficients of x^k are equal in $\mathbb{F}_{101}$ for each nonnegative integer k. For example, $(x+3)(100x+5) = 100x^2 + 2x + 15$ in $\mathbb{F}_{101}[x]$ because the corresponding coefficients are equal modulo 101.

    We say that $f(x) \in \mathbb{F}_{101}[x]$ is *lucky* if it has degree at most 1000 and there exist $g(x), h(x) \in \mathbb{F}_{101}[x]$ such that

$$f(x) = g(x)(x^{1001} - 1) + h(x)^{101} - h(x)$$

    in $\mathbb{F}_{101}[x]$. Find the number of lucky polynomials.