Solving Quadratic Equations Summary

KEY TERMS

- polynomial
- monomial
- binomial
- trinomial
- degree of a polynomial
- closed, closure
- difference of two squares
- perfect square trinomial
- Factor Theorem
- polynomial long division
- Remainder Theorem
- principal square root
- roots
- double root
- Zero Product Property
- completing the square
- Quadratic Formula
- discriminant

LESSON 1 This Time, With Polynomials

Linear and quadratic expressions are part of a larger group of expressions known as *polynomials*. A **polynomial** is an expression involving the sum of powers in one or more variables multiplied by coefficients. A polynomial in one variable is the sum of terms of the form ax^k, where a, called the coefficient, is a real number and k is a non-negative integer. A polynomial is written in general form when the terms are in descending order, starting with the term with the greatest degree and ending with the term with the least degree.

$$a_1x^k + a_2x^{k-1} + \ldots a_nx^0$$

Each product in a polynomial is called a *term*. Polynomials are named according to the number of terms: a **monomial** has exactly 1 term, a **binomial** has exactly 2 terms, and a **trinomial** has exactly 3 terms.

The exponent of a term is the degree of the term, and the greatest exponent in a polynomial is the **degree of the polynomial**.

The characteristics of the polynomial $13x^3 + 5x + 9$ are shown in the chart.

	1st term	2nd term	3rd term
Term	$13x^3$	$5x$	9
Coefficient	13	5	9
Power	x^3	x^1	x^0
Exponent	3	1	0

This trinomial has a degree of 3 because 3 is the greatest degree of the terms in the trinomial.

When an operation is performed on any of the numbers in a set and the result is a number that is also in the same set, the set is said to be **closed** (or to have **closure**) under that operation. The definition of *closure* can also be applied to polynomials.

Polynomials can be added or subtracted by identifying the like terms of the polynomial functions, using the Associative Property to group the like terms together and combining the like terms to simplify the expression.

For example, to add the polynomial expressions $(7x^2 - 2x + 12)$ and $(8x^3 + 2x^2 - 3x)$, use the Associative Property to combine like terms.

$$(7x^2 - 2x + 12) + (8x^3 + 2x^2 - 3x)$$
$$8x^3 + (7x^2 + 2x^2) + (-2x - 3x) + 12$$
$$8x^3 + 9x^2 - 5x + 12$$

The product of 2 binomials can be determined by using a multiplication table, or area model, which organizes the two terms of the binomials as factors of multiplication expressions.

$$(9x - 1)(5x + 7)$$

·	**9*x***	**−1**
5*x*	$45x^2$	$-5x$
7	$63x$	-7

$$\begin{aligned}(9x - 1)(5x + 7) &= 45x^2 - 5x + 63x - 7\\ &= 45x^2 + 58x - 7\end{aligned}$$

The Distributive Property can also be used to multiply polynomials. Depending on the number of terms in the polynomials, the Distributive Property may need to be used multiple times.

For example, to multiply the polynomials $x + 5$ and $x - 2$, first, use the Distributive Property to multiply each term of $x + 5$ by the entire binomial $x - 2$.

$$(x + 5)(x - 2) = (x)(x - 2) + (5)(x - 2)$$

Next, distribute x to each term of $x - 2$, and distribute 5 to each term of $x - 2$.

$$x^2 - 2x + 5x - 10$$

Finally, collect the like terms and write the solution in general form.

$$x^2 + 3x - 10$$

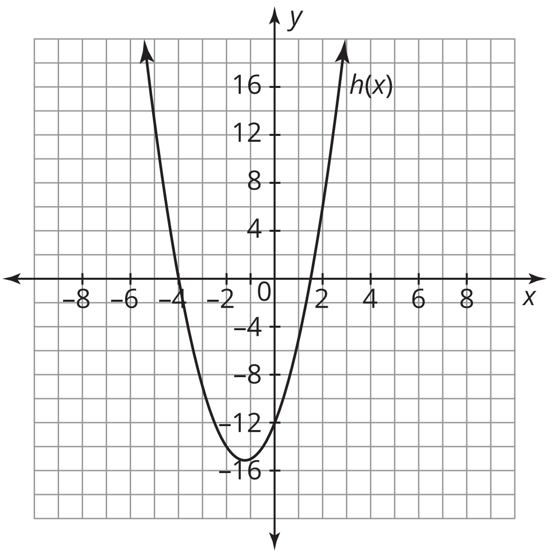
There are special products of degree 2 that have certain characteristics. The expression in the form $a^2 - b^2$ that has factors $(a + b)(a - b)$ is called the **difference of two squares**. A **perfect square trinomial** is formed by multiplying a binomial by itself. It is an expression in the form $a^2 + 2ab + b^2$, or in the form $a^2 - 2ab + b^2$. A perfect square trinomial can be written as the square of a binomial. In these cases, the factors are $(a + b)^2$ and $(a - b)^2$, respectively.

LESSON 2 The Great Divide

The **Factor Theorem** states that a polynomial function $p(x)$ has $(x - r)$ as a factor if and only if the value of the function at r is 0, or $p(r) = 0$. You can use the Factor Theorem to show that a linear expression is a factor of a polynomial.

Consider the graph of the polynomial function $h(x) = 2x^2 + 5x - 12$.

The graph appears to have a zero at $(-4, 0)$, so a possible linear factor of the polynomial is $(x - (-4))$ or $(x + 4)$.



Determine the value of the polynomial at $x = -4$, or $h(-4)$.

$$h(-4) = 2(-4)^2 + 5(-4) - 12$$
$$h(-4) = 32 - 20 - 12$$
$$h(-4) = 0$$

So, $(x + 4)$ is a linear factor of the polynomial function.

Polynomial long division is an algorithm for dividing one polynomial by another of equal or lesser degree. The process is similar to integer long division. For example, if 2 is a factor of 24, then 24 can be divided by 2 without a remainder. In the same way, the factors of a polynomial divide into that polynomial without a remainder. Therefore, polynomial long division can be used to determine if a linear expression is a factor of a polynomial.

Integer Long Division	Polynomial Long Division
$3660 \div 12$ or $\frac{3660}{12}$ $\begin{array}{r} 305 \\ 12\overline{)3660} \\ -36 \\ \hline 6 \\ -0 \\ \hline 60 \\ -60 \\ \hline 0 \end{array}$	$(2x^2 + 5x - 12) \div (x + 4)$ or $\frac{2x^2 + 5x - 12}{x + 4}$ $\begin{array}{r} \text{A}\ 2x \ \text{D} -3 \\ x + 4\overline{)2x^2 + 5x - 12} \\ \text{B}\ -(2x^2 + 8x) \downarrow \ \text{C} \\ \hline -3x - 12 \\ -(-3x - 12) \\ \hline \text{Remainder } 0 \end{array}$ A. Divide $\frac{2x^2}{x} = 2x$. B. Multiply $2x(x + 4)$, and then subtract. C. Bring down -12. D. Divide $\frac{-3x}{x} = -3$. E. Multiply $-3(x + 4)$, and then subtract.

For the polynomial long division example above, the remainder is 0, which means that $(x + 4)$ is a factor of $2x^2 + 5x - 12$. The quotient is $(2x - 3)$, which is the other factor of the polynomial. Therefore, $2x^2 + 5x - 12 = (x + 4)(2x - 3)$.

Remember from your experiences with division that:

$$\frac{\text{dividend}}{\text{divisor}} = \text{quotient} + \frac{\text{remainder}}{\text{divisor}}$$

It follows that any polynomial, $p(x)$, can be written in the form:

$$\frac{p(x)}{\text{linear expression}} = \text{quotient} + \frac{\text{remainder}}{\text{linear expression}}$$

or

$p(x) = (\text{linear expression})(\text{quotient}) + \text{remainder}$.

Generally, the linear expression is written in the form $(x - r)$, the quotient is represented by $q(x)$, and the remainder is represented by R.

$p(x) = (x - r)q(x) + \text{R}$

Consider this example. Determine the quotient of $(3x^2 + 10x + 1) \div (x + 4)$.

$$\begin{array}{r l}
 & 3x - 2 \\
x + 4 & \overline{)\,3x^2 + 10x + 1} \\
 & \underline{-(3x^2 + 12x)} \\
 & -2x + 1 \\
 & \underline{-(-2x - 8)} \\
 & 9
\end{array}$$

The quotient of $(3x^2 + 10x + 1) \div (x + 4)$ is $3x - 2 + \frac{9}{x + 4}$.

$$\frac{3x^2 + 10x + 1}{x + 4} = 3x - 2 + \frac{9}{x + 4}$$

$$3x^2 + 10x + 1 = (x + 4)(3x - 2) + 9$$

The **Remainder Theorem** states that when any polynomial function, $f(\mathrm{x})$, is divided by a linear expression of the form $(x - r)$, the remainder $\mathrm{R} = f(r)$, or the value of the function when $x = r$. Remember, the Factor Theorem states that a polynomial has a linear expression as a factor if and only if the remainder is zero. Therefore, if $\mathrm{R} = 0$, then $f(r) = 0$, and $(x - r)$ is a factor of $f(x)$.

In the previous example, using polynomial long division we determined $\mathrm{R} = 9$; therefore, by the Factor Theorem, $(x + 4)$ is not a factor of $3x^2 + 10x + 1$. This can be confirmed using the Remainder Theorem.

If $f(x) = 3x^2 + 10\mathrm{x} + 1$, to determine if $(x + 4)$ is a factor of $f(x)$, we must determine the value of $f(-4)$.

$$f(-4) = 3(-4)^2 + 10(-4) + 1$$
$$f(-4) = 9$$

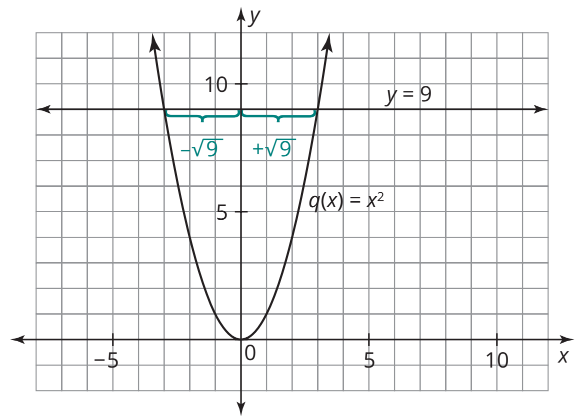
This means the remainder of $f(x) = 9$, not 0; therefore, $(x + 4)$ is not a factor of $f(x) = 3x^2 + 10x + 1$.

LESSON 3 Solutions, More or Less

A quadratic function is a function of degree 2 because the greatest power for any of its terms is 2. This means that it has at most 2 zeros, or at most 2 solutions, at $y = 0$.

The two solutions of a quadratic function can be represented as square roots of numbers. Every positive number has two square roots, a positive square root, which is also called the **principal square root**, and a negative square root. To solve the equation $x^2 = 9$, take the square root of both sides of the equation.

$$\sqrt{x^2} = \sqrt{9}$$
$$x = \pm 3$$



You can solve $x^2 = 9$ on a graph by finding the points of intersection between $y = x^2$ and $y = 9$. The solutions are both 3 units from the axis of symmetry, $x = 0$.

The *x*-intercepts of a graph of a quadratic function are called the *zeros* of the quadratic function. The zeros are called the **roots** of the quadratic equation.

The quadratic function $q(x) = x^2$ has two solutions at $y = 0$; therefore, it has 2 zeros: $x = +\sqrt{0}$ and $x = -\sqrt{0}$. These two zeros of the function, or roots of the equation, are the same number, 0, so the function $q(x) = x^2$ is said to have a **double root**.

When you encounter solutions that are not perfect squares, you can either determine the approximate value of the radical or rewrite it in an equivalent radical form.

To approximate a square root, determine the perfect square that is closest to, but less than, the given value and the perfect square that is closest to, but greater than, the given value. You can use these square roots to approximate the square root of the given number.

For example, the approximate value of $\sqrt{40}$ falls between $\sqrt{36}$, or 6, and $\sqrt{49}$, or 7. Since $6.3^2 = 39.69$ and $6.4^2 = 40.96$, the approximate value of $\sqrt{40}$ is 6.3.

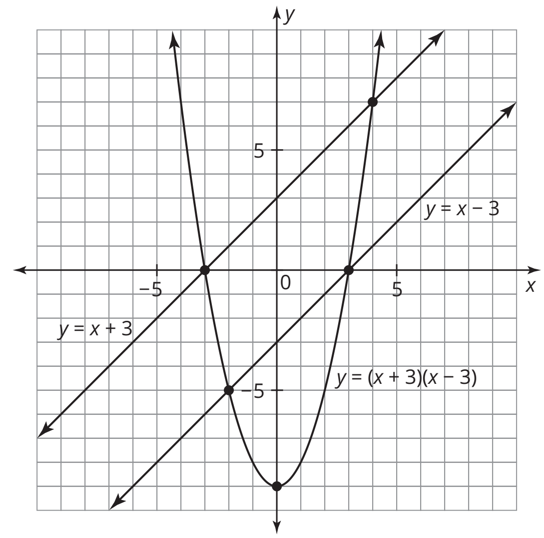
To rewrite a square root in equivalent radical form, first rewrite the product of the radicand to include any perfect square factors. Then, extract the square roots of those perfect squares.

$$\begin{aligned}\sqrt{27} &= \sqrt{9 \cdot 3}\\ &= \sqrt{9} \cdot \sqrt{3}\\ &= 3\sqrt{3}\end{aligned}$$

A quadratic function written in factored form is in the form $f(x) = a(x - r_1)(x - r_2)$, where $a \neq 0$. In factored form, r_1 and r_2 represent the *x*-intercepts of the graph of the function. The *x*-intercepts of the graph of the quadratic function $f(x) = ax^2 + bx + c$ and the zeros of the function are the same as the roots of the equation $ax^2 + bx + c = 0$.

To write a quadratic function in factored form, first determine the zeros of the function $f(x) = x^2 - 9$, set the trinomial expression equal to 0, and solve for x.

$$\begin{aligned} 0 &= x^2 - 9 \\ 9 &= x^2 \\ \sqrt{9} &= \sqrt{x^2} \\ \pm 3 &= x \end{aligned}$$



You can then use the zeros to write the function in factored form, $f(x) = (x + 3)(x - 3)$.

The **Zero Product Property** states that if the product of two or more factors is equal to zero, then at least one factor must be equal to zero. You can see from the graph that the zeros of the function $f(x) = x^2 - 9$ occur where either $y = x + 3$ or $y = x - 3$ are zero.

LESSON **4**

Transforming Solutions

The solutions to any quadratic equation are located on the parabola, equidistant from the axis of symmetry.

A quadratic function in vertex form $f(x) = a(x - h)^2 + k$ is translated horizontally h units, dilated vertically by the factor a, and translated vertically k units.

For the equation $y = (x - c)^2$, the solutions can be represented by $c \pm \sqrt{y}$. For the equation $y = a(x - c)^2$, the solutions can be represented by $c \pm \sqrt{\frac{y}{a}}$. For the equation $y = a(x - c)^2 + d$, the solutions can be represented by $c \pm \sqrt{\frac{y - d}{a}}$.

For example, consider the equation $2(x - 1)^2 + 2 = 20$.

$$\begin{aligned} x &= 1 \pm \sqrt{\frac{20 - 2}{2}} \\ &= 1 \pm \sqrt{9} \\ &= 1 \pm 3 \end{aligned}$$

The solutions to the equation are 3 units away from the axis of symmetry, $x = 1$. The solutions are $x = -2$ and $x = 4$.

LESSON 5

The Missing Link

You can factor trinomials by rewriting them as the product of two linear expressions.

For example, to factor the trinomial $x^2 + 10x + 16$, determine the factor pairs of the constant term. The factors of 16 are (1)(16), (2)(8), and (4)(4). Then, determine the pair whose sum is the coefficient of the middle term, 10.

·	**x**	**8**
x	x^2	$8x$
2	$2x$	16

The sum of $2x$ and $8x$ is $10x$. So, $x^2 + 10x + 16 = (x + 2)(x + 8)$.

You can use factoring and the Zero Product Property to solve quadratics in the form $y = ax^2 + bx + c$.

For example, you can solve the quadratic equation $x^2 - 4x = -3$.

$$\begin{aligned} x^2 - 4x &= -3 \\ x^2 - 4x + 3 &= -3 + 3 \\ x^2 - 4x + 3 &= 0 \\ (x - 3)(x - 1) &= 0 \end{aligned}$$

$$\begin{array}{rcl} (x - 3) = 0 & \text{or} & (x - 1) = 0 \\ x - 3 + 3 = 0 + 3 & \text{or} & x - 1 + 1 = 0 + 1 \\ x = 3 & \text{or} & x = 1 \end{array}$$

For a quadratic function that has zeros but cannot be factored, there is another method for solving the quadratic equation. **Completing the square** is a process for writing a quadratic expression in vertex form, which then allows you to solve for the zeros.

For example, you can calculate the roots of the equation $x^2 - 4x + 2 = 0$.

Isolate $x^2 - 4x$.	$x^2 - 4x + 2 - 2 = 0 - 2$ $x^2 - 4x = -2$
Complete the square and rewrite this as a perfect square trinomial.	
Determine the constant term that would complete the square.	$x^2 - 4x + ? = -2 + ?$
Add this term to both sides of the equation.	$x^2 - 4x + 4 = -2 + 4$ $x^2 - 4x + 4 = 2$
Factor the left side of the equation.	$(x - 2)^2 = 2$
Determine the square root of each side of the equation.	$\sqrt{(x - 2)^2} = \sqrt{2}$ $(x - 2) = \pm\sqrt{2}$
Set the factor of the perfect square trinomial equal to each square root of the constant and solve for x.	$x - 2 = \sqrt{2}$ or $x - 2 = -\sqrt{2}$ $x = 2 + \sqrt{2}$ or $x = 2 - \sqrt{2}$ $x \approx 3.41$ or $x \approx 0.59$

The roots are approximately 3.41 and 0.59.

Completing the square can also be used to identify the axis of symmetry and the vertex of any quadratic function written in standard form.

When a function is written in standard form, $ax^2 + bx + c$, the axis of symmetry is $x = -\frac{b}{2a}$.

Given a quadratic equation in the form $y = ax^2 + bx + c$, the vertex of the function is located at $x = -\frac{b}{2a}$ and $y = c - \frac{b^2}{4a}$.

LESSON 6

Ladies and Gents, Please Welcome the Quadratic Formula!

The **Quadratic Formula**, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, can be used to calculate the solutions to any quadratic equation of the form $ax^2 + bx + c = 0$, where a, b and c represent real numbers and $a \neq 0$.

For example, given the function $f(x) = 2x^2 - 4x - 3$ we can identify the values of a, b, and c.

$$a = 2;\ b = -4;\ c = -3$$

Then, we use the quadratic formula to solve.

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{4 \pm \sqrt{16 + 24}}{4}$$

$$x = \frac{4 \pm \sqrt{40}}{4}$$

$$x \approx \frac{4 + 6.325}{4} \approx 2.581 \quad \text{or} \quad x \approx \frac{4 - 6.325}{4} \approx -0.581$$

The roots are approximately 2.581 and −0.581.

A quadratic function can have one real zero, two real zeros, or at times, no real zeros.

You can use the part of the Quadratic Formula underneath the square root symbol to identify the number of real zeros or roots. Because this portion of the formula “discriminates” the number of real zeros or roots, it is called the **discriminant**.

If the discriminant is positive, the quadratic has two real roots. If the discriminant is negative, the quadratic has no real roots. If the discriminant is 0, the quadratic has a double real root.

You can also use the discriminant to describe the nature of the roots. If the discriminant is a perfect square, then the roots are rational. If the discriminant is not a perfect square, then the roots are irrational.

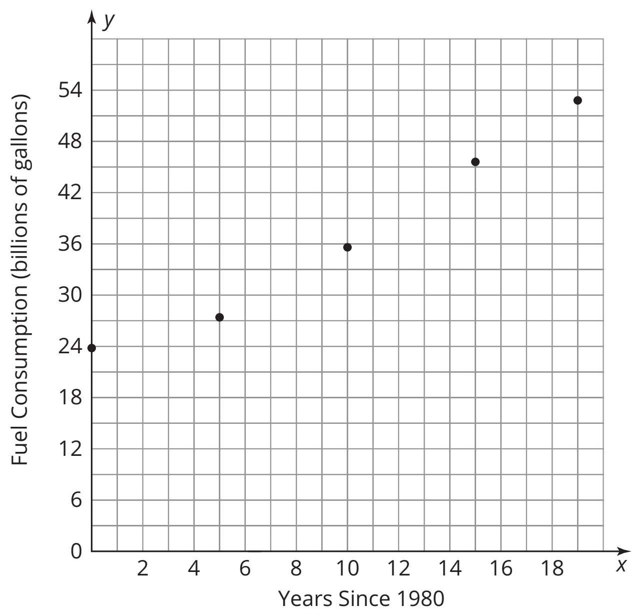
LESSON 7

Fit this Model

Quadratic regression equations can be used to model real-world situations and make predictions.

For example, as vans, trucks, and SUVs have increased in popularity, the fuel consumption of these types of vehicles has also increased.

Years Since 1980	Fuel Consumption (billions of gallons)
0	23.8
5	27.4
10	35.6
15	45.6
19	52.8



The quadratic regression equation that best fits the data is $y = 0.0407x^2 + 0.809x + 23.3$. The r^2-value for the quadratic regression fit is 0.996. Just as with linear and exponential regressions, the equation can be used to make predictions for the data.

For example, you can predict the fuel consumption in the year 2020 by substituting $x = 40$ into the regression equation.

$$y = 0.0407(40)^2 + 0.809(40) + 23.3$$
$$y \approx 121$$

In 2020, fuel consumption will be about 121 billion gallons.