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# Reasoning with Shapes

Pacing: 40 Days

## Topic 1: Using a Rectangular Coordinate System

Students investigate the properties of squares and use transformations of squares to construct a coordinate plane. Students prove the slope criteria for parallel and perpendicular lines. They develop strategies for determining the perimeters and areas of rectangles, triangles, parallelograms, and composite plane figures on the coordinate plane. Students also explore the effects of proportional and non-proportional changes to the dimensions of a plane figure on its perimeter and area.

**Standards:** G.CO.10, G.CO.11, G.CO.1, G.GO.5, G.CO.12, G.GPE.4, G.GPE.5, G.GPE.7, G.MG.2 **Pacing:** 13 Days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
1	<b>The Squariest Square</b> From Informal to Formal Geometric Thinking	G.CO.10 G.CO.11	1	Students investigate and recall the properties of a square by drawing squares freehand, trying to draw a perfect square, and determining the criteria needed to assess the “squareness” of a drawn square. Students are then introduced to the purpose of high school geometry—to formalize, with more precise definitions and proofs, the geometric relationships they have studied up to this point. Students consider a geometry problem involving angle measures in three squares. They measure the angles, compare their findings with their classmates, and develop a conjecture about the sum of the angles. Students list the concepts and processes they remember from past courses that may be helpful in proving the relationship shown by the angle measures in the three squares. Finally, students see that formal reasoning in mathematics often involves creative thinking, drawing new lines and seeing relationships from different perspectives.	<ul style="list-style-type: none"> <li>• Mathematicians make conjectures, test predictions, experiment with patterns, and consider arguments and different perspectives.</li> <li>• Mathematical reasoning can be used to validate a conjecture.</li> </ul>
2	<b>Hip to Be Square</b> Constructing a Coordinate Plane	G.CO.1 G.CO.5 G.CO.12	2	Students complete geometric constructions using patty paper or a compass and a straightedge. They analyze worked examples and then repeat the strategies to bisect segments, construct perpendicular bisectors, and duplicate line segments. Students apply these constructions, along with rigid motions, to construct a square and then the squares of a coordinate plane. They further describe rigid motions that can be used to create two-dimensional shapes on a coordinate plane and identify the coordinates of the vertices of these shapes. Students demonstrate their knowledge by describing a sequence of transformations of a coordinate plane square that can produce a given line. They determine the slope of the line and write its equation.	<ul style="list-style-type: none"> <li>• When you construct geometric figures, you create exact figures using only a compass and straightedge or patty paper.</li> <li>• The midpoint of a segment is a point that divides the segment into two congruent segments.</li> <li>• A segment bisector is a line, line segment, or ray that divides a line segment into two line segments of equal length.</li> <li>• A perpendicular bisector is a line, line segment, or ray that bisects a line segment and is also perpendicular to the line segment.</li> <li>• Any point on a perpendicular bisector is equidistant to the endpoints of the original segment it bisects.</li> <li>• The diagonals of a square are congruent, bisect each other, are perpendicular to one another, and bisect the angles of the square.</li> <li>• A coordinate plane can be created by constructing a square and applying rigid motion transformations to the square.</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
3	<b>Ts and Train Tracks</b> Parallel and Perpendicular Lines	G.CO.12 G.GPE.5	2	Students explore how to use translations and rotations to create parallel and perpendicular segments. They then construct parallel lines both off and on the coordinate plane. Students use rotations to prove that if two lines are perpendicular, then the slopes of the lines are negative reciprocals. Students write the equation of a line perpendicular to a given line that passes through a given point. Students write an informal paragraph proof for the slope criteria of parallel lines (that they are equal).	<ul style="list-style-type: none"> <li>The 90° rotation of a line creates a line perpendicular to the original line.</li> <li>Perpendicular lines have slopes that are negative reciprocals of each other.</li> <li>The translation of a line creates an identical line or a line parallel to the original line.</li> <li>Parallel lines have equal slopes.</li> </ul>
4	<b>Where Has Polly Gone?</b> Classifying Shapes on the Coordinate Plane	G.GPE.4 G.GPE.5	2	Students sort triangles and quadrilaterals based on properties. They are introduced to the Distance Formula and use it to calculate the lengths of sides of triangles and quadrilaterals on the coordinate plane. Students also use the slope formula to determine whether opposite sides of a quadrilateral are parallel and whether consecutive sides of a quadrilateral are perpendicular. They use these skills to classify triangles and quadrilaterals that lie on a coordinate plane and to determine the fourth point of a quadrilateral when given three points. Students are then introduced to the Midpoint Formula and use it to classify the secondary figures formed by connecting the midpoints of consecutive sides of quadrilaterals. Finally, students consider translations as a strategy for identifying the coordinates that create quadrilaterals with parallel sides.	<ul style="list-style-type: none"> <li>The Distance Formula states that the distance <math>d</math> between points <math>(x_1, y_1)</math> and <math>(x_2, y_2)</math> on a coordinate plane is given by the equation <math>d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}</math>.</li> <li>The Distance Formula can be used to classify triangles and quadrilaterals based on side lengths.</li> <li>The slope formula can be used to determine whether opposite sides are parallel or consecutive sides are perpendicular in a quadrilateral on the coordinate plane.</li> <li>The Midpoint Formula states that if <math>(x_1, y_1)</math> and <math>(x_2, y_2)</math> are two points on the coordinate plane, then the midpoint of the line segment that joins these two points is <math>((x_1 + x_2)/2, (y_1 + y_2)/2)</math>.</li> <li>The use of translations is an effective strategy when determining endpoints of parallel segments on a coordinate plane.</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
5	<b>In and Out and All About</b> Area and Perimeter on the Coordinate Plane	G.GPE.5 G.GPE.7 G.MG.2	3	Students calculate the perimeter and area of rectangles and triangles on the coordinate plane. They double dimensions of figures and explain how this affects the area of the figure and also translate figures on the coordinate plane to more efficiently determine their perimeter and area. Students algebraically determine the non-vertical height of a triangle as they treat each side as the base; they then use the height to calculate the area of the triangle. They conclude that the area of a triangle remains the same regardless of the side considered as the base and the height determined by that base. Next, the term <i>composite figure</i> is defined, and students divide a composite figure into various known polygons to compute its area. They then consider real-world situations requiring them to calculate the perimeter and area of polygons that lie on a coordinate plane using the Distance Formula and decomposing the polygons into triangles and rectangles. A velocity-time graph is used to model a real-world scenario. Students determine distances represented as the area under the curve of these graphs.	<ul style="list-style-type: none"> <li>Rigid motion transformations (translations, rotations, and reflections) can be used to change the position of figures on the coordinate plane.</li> <li>Performing translations on figures can help to compute perimeter and area more efficiently.</li> <li>Non-vertical heights of a figure can be calculated algebraically using formulas, writing equations and solving a system of equations.</li> <li>The area of a triangle is the same regardless of what base and height of the triangle are used in the calculation.</li> <li>A composite figure is a figure that is formed by combining different shapes.</li> <li>Polygons can be divided into a combination of triangles and rectangles to help determine their area.</li> <li>The area of a composite figure is determined by dividing the figure into familiar shapes and using the area formulas associated with those shapes.</li> <li>The Distance Formula, slope formula, and the Pythagorean Theorem can be used to determine the area of polygons and composite figures on the coordinate plane.</li> <li>A velocity-time graph can model acceleration, and distance can be determined by calculating the area under a curve.</li> </ul>
	<b>Learning Individually with MATHia or Skills Practice</b>	G.CO.1 G.GPE.5 G.GPE.7	3	Students practice identifying geometric entities from their names, writing names for various geometric entities, and identifying when an entity has multiple possible names. They practice writing measure statements for segments and angles using appropriate notation. Students answer questions related to an animation demonstrating that the rotation of a point $(x, y)$ $90^\circ$ counterclockwise on the coordinate plane is given by the coordinates $(-y, x)$ . They then answer questions to discover that the slopes of perpendicular lines are negative reciprocals of each other. Students use graphs of functions to understand that the slopes of parallel lines are equal. Finally, students use their knowledge of parallel and perpendicular lines as graphs of functions to solve problems in a real-world context. Students answer questions related to an animation demonstrating how the Distance Formula is derived using the Pythagorean Theorem and then use interactive Explore Tools and the Distance Formula to solve mathematical problems about the distances between two points on the coordinate plane. Students use the Distance Formula to determine perimeters and areas of different shapes.	

\*Pacing listed in 45-minute days

## Topic 2: Composing and Decomposing Shapes

Students investigate and conjecture about geometric figures. They use circles and their defining characteristics as the template upon which to construct lines, angles, triangles, and quadrilaterals. Students use reasoning to conjecture about the relationships they notice, preparing them for formal proof in future topics.

**Standards:** G.CO.9, G.CO.10, G.CO.11, G.CO.12, G.CO.13, G.C.3 **Pacing:** 13 Days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
1	<b>Running Circles Around Geometry</b> Using Circles to Make Conjectures	G.CO.9	2	Circles are used to make conjectures about line and angle relationships. Students construct a circle, a perpendicular bisector of a diameter, and a chord to identify circle parts. Next, they conjecture about angle relationships given parallel lines intersected by a transversal. Students make conjectures related to inscribed angles and angles formed at the point of tangency when two lines intersect at a point outside the circle.	<ul style="list-style-type: none"> <li>When you conjecture, you use what you know through experience and reasoning to presume that something is true. The statement of a conjecture, once proven, is then called a theorem.</li> <li>Circles can be helpful in constructing geometric figures in order to make conjectures about line and angle relationships.</li> </ul>
2	<b>The Quad Squad</b> Conjectures About Quadrilaterals	G.CO.11	2	Students use circles to investigate and conjecture about the properties of quadrilaterals. Students construct several quadrilaterals from the diameters of concentric circles. Using the measurements of sides and angles, they are able to name the quadrilaterals. Students make conjectures about the diagonals and angle relationships of kites and isosceles trapezoids. They identify quadrilaterals with given properties and then describe how to construct various quadrilaterals given only one diagonal. Students conjecture about the figure formed by adjacent midsegments of quadrilaterals and the measure of the midsegment of a trapezoid in relation to its bases. Finally, they conjecture about the sum of the measures of opposite angles of different cyclic quadrilaterals.	<ul style="list-style-type: none"> <li>The diagonals of any convex quadrilateral create two pairs of vertical angles and four linear pairs of angles.</li> <li>Parallelograms, rhombi, and kites have diagonals that are not congruent.</li> <li>Rectangles, squares, and isosceles trapezoids have congruent diagonals.</li> <li>Circles can be helpful in understanding that the diagonals of parallelograms bisect each other, the diagonals of rectangles are congruent, and the diagonals of kites are perpendicular.</li> <li>The measure and relationship of the diagonals of quadrilaterals can be used to make conjectures about quadrilaterals.</li> <li>The relationship of the interior angles of quadrilaterals can be used to make conjectures about quadrilaterals.</li> <li>The midsegment of a quadrilateral is any line segment that connects two midpoints of the sides of the quadrilateral.</li> <li>A quadrilateral whose vertices all lie on a single circle is a cyclic quadrilateral.</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
3	<b>Into the Ring</b> Constructing an Inscribed Regular Polygon	G.CO.12 G.CO.13	4	Students construct a regular hexagon inscribed in a circle using two methods: duplicating $60^\circ$ central angles to create six equilateral triangles sharing the center of the circle as a vertex and then constructing and connecting six adjacent congruent chords the same length as the radius of the circle around the circumference of the circle. Students then construct a square inscribed in a circle using two different methods: rotating a right triangle to create four congruent right triangles sharing the center of the circles and then constructing and connecting perpendicular diameters of a circle. Students analyze worked examples that demonstrate how to bisect an angle using patty paper and a compass and straightedge. They then construct both an equilateral triangle given a side length and an equilateral triangle inscribed in a circle. Finally, they construct an analyze inscribed angles, a $75^\circ$ angle, and a regular octagon inscribed in a circle.	<ul style="list-style-type: none"> <li>• Constructions can be used to duplicate a given angle.</li> <li>• A <math>60^\circ</math> angle can be constructed by creating an equilateral triangle with a circle.</li> <li>• A regular hexagon can be inscribed in a circle by duplicating <math>60^\circ</math> angles to create six equilateral triangles sharing the center of the circle as a vertex.</li> <li>• A regular hexagon can be inscribed in of a circle by constructing six adjacent congruent chords the same length as the radius of the circle around the circumference of the circle.</li> <li>• When a square is inscribed in a circle, a segment that is a diagonal of the square is also a diameter of the circle.</li> <li>• An angle bisector is a line, segment, or ray that is drawn through the vertex of an angle and divides the angle into two congruent angles. Angle bisectors can be constructed using patty paper or a compass and straightedge.</li> <li>• Both an equilateral triangle with a given side length and an equilateral triangle inscribed in a circle can be created using construction tools.</li> <li>• The central angle of a circle is twice the measure of an inscribed angle which intercepts the same arc of the circle.</li> <li>• Constructions can be used to verify geometric theorems.</li> </ul>
4	<b>Tri- Tri- Tri- and Separate Them</b> Conjectures About Triangles	G.CO.10	2	Students decompose quadrilaterals to form the triangles they investigate in this lesson. They write the converses of conditional statements and then explore the converse of their base angles conjecture for isosceles triangles. Students construct an equilateral triangle and conjecture about the sum of the interior and exterior angle measures of a triangle. Students use a circle diagram to make conjectures about triangle inequality and triangle midsegments.	<ul style="list-style-type: none"> <li>• Circles can be helpful in constructing geometric figures to make conjectures about triangles.</li> <li>• A convex quadrilateral can be divided by any one of its diagonals into two triangles.</li> <li>• The converse of a statement is different from the original statement and is formed by interchanging the hypothesis and conclusion of the original statement.</li> <li>• The truth value of a conditional statement and its converse are not necessarily the same.</li> <li>• The base angles of an isosceles triangle are congruent.</li> <li>• A point that lies on a perpendicular bisector of a line segment is equidistant from the endpoints of the line segment.</li> <li>• The measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.</li> <li>• The sum of the measures of the interior angles of a triangle is <math>180^\circ</math>.</li> <li>• The length of the third side of a triangle cannot be equal to or greater than the sum of the measures of the other two sides.</li> <li>• The midsegment of a triangle is one-half the measure and parallel to the third side.</li> <li>• A conjecture is a statement believed to be true based on observations. A conjecture must be proved with definitions and theorems to be fully accepted.</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
5	<b>What's the Point?</b> Points of Concurrency	G.CO.10 G.C.3	2	Students construct the four points of concurrency—the incenter, circumcenter, centroid, and orthocenter. They construct perpendicular bisectors, angle bisectors, medians, and altitudes to locate these points in acute, obtuse, right, and equilateral triangles. They use the circumcenter to circumscribe a circle about a triangle and the incenter to inscribe a circle in a triangle. Students use their constructions to make conjectures.	<ul style="list-style-type: none"> <li>• A point of concurrency is a point at which three lines, rays, or line segments intersect.</li> <li>• The circumcenter is the point of concurrency of the three perpendicular bisectors of the sides of a triangle, and it is equidistant from each vertex of the triangle.</li> <li>• The circumcenter can be used to circumscribe a circle about a triangle.</li> <li>• The incenter is the point at which the three angle bisectors of a triangle are concurrent and it is equidistant from each side of the triangle.</li> <li>• The incenter can be used to construct a circle inscribed in a triangle.</li> <li>• The median of a triangle is a line segment formed by connecting a vertex of a triangle to the midpoint of the opposite side of the triangle.</li> <li>• The centroid is the point at which the three medians of a triangle are concurrent.</li> <li>• The distance from the centroid to the vertex is twice the distance from the centroid to the midpoint of the opposite side.</li> <li>• The orthocenter is the point at which the three altitudes of a triangle are concurrent.</li> </ul>
	<b>Learning Individually with MATHia or Skills Practice</b>	G.C.1 G.C.2 G.C.3	1	Students watch an animation defining some of the terminology of circle parts. They then identify chords, tangents, points of tangency, and secants of circles. Next, students sort inscribed and central angles and classify minor and major arcs as well as semicircles. They then calculate the measure of an arc or an angle using the definition of a central angle, the Arc Addition Postulate, or the Inscribed Angle Theorem. Students are shown an inscribed quadrilateral, prove the Inscribed Quadrilateral-Opposite Angles Conjecture, and use the theorem to determine the measure of an angle in an inscribed quadrilateral given the measure of the opposite angle.	

\*Pacing listed in 45-minute days

### Topic 3: Rigid Motions on a Plane

Using the intuitive understandings of rigid motions built in middle school, students learn the formal definitions of translations, reflections, and rotations. They define translations in terms of equal distances along directed line segments, reflections in terms of perpendicular lines, and rotations in terms of equal arcs around concentric circles. They use rigid motions to solve problems and identify a sequence of rigid motions that maps a given figure onto another. Finally, students consider reflectional and rotational symmetry—the set of rotations and reflections that map a plane figure onto itself. They identify the lines of reflection and angles of rotation for given plane figures.

**Standards:** G.CO.1, G.CO.2, G.CO.3, G.CO.4, G.CO.5 **Pacing:** 14 Days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
1	<b>Put Your Input In, Take Your Output Out</b>  Geometric Components of Rigid Motions	G.CO.1 G.CO.4	3	Students use a transformation machine and patty paper to translate shapes along line segments, rotate shapes on figures around points, and reflect shapes across lines of reflection. Students then analyze the component parts of the transformation machine throughout the rest of the lesson, including constructing parallel line segments and identifying and defining lines, angles, and rotation angles. Students recall that an image transformed by rigid motions such as translations, reflections, and/or rotations is congruent to its pre-image.	<ul style="list-style-type: none"> <li>Pre-images transformed by rigid motions such as translations, reflections, and rotations are congruent to their images.</li> <li>Two lines perpendicular to a third line are parallel to each other.</li> <li>If two lines intersected by a transversal have corresponding angles, alternate interior angles and alternate exterior angles congruent, then the lines are parallel.</li> <li>A line is a geometric object such that if any part of the line is translated to another part of the line so that the two parts have two points in common, then the first part will lie exactly on top of the second part.</li> <li>Translations can be described using lines and line segments. Reflections can be described using lines. Rotations can be described using rotation angles.</li> </ul>
2	<b>Bow Thai</b>  Translations as Functions	G.CO.2 G.CO.4	2	Students recall that they used lines and line segments as “transformation machines” to translate plane figures. They identify different ways of drawing the same transformation machine and investigate a transformation machine created with multiple line segments. Students use the context of designing an animated website banner to investigate translations as functions. Students learn that parallel lines can be used for translations. They distinguish between rigid motions, or isometries, and transformations, such as dilations, which are not isometries.	<ul style="list-style-type: none"> <li>Translations along parallel lines are rigid motions and always produce images that are congruent to the pre-image.</li> <li>A translation is a function, represented as <math>T_{AB}(P) = P'</math> which takes as its input the location of a point <math>P</math> and translates it a distance <math>AB</math> in the direction <math>AB</math>.</li> <li>Isometries are rigid motion transformations that preserve size and shape.</li> </ul>
3	<b>Staring Back at Me</b>  Reflections as Functions	G.CO.2 G.CO.4 G.CO.5	2	Students construct a perpendicular bisector of a line segment and then use patty paper to recognize that the perpendicular bisector is a line of symmetry for the line segment, which allows a reflection across the line to match up its endpoints. Students are then encouraged to attempt the impossible task of drawing two points in the plane that cannot be reflected one onto the other. Students investigate reflections as functions using the context from the previous lesson of animating objects on a website, and they describe the points of reflection as equidistant from the line of reflection, which is the perpendicular bisector of the segment connecting the points. Students prove the Perpendicular Bisector Theorem and its converse and then use sequences of isometries to demonstrate that two plane figures are congruent.	<ul style="list-style-type: none"> <li>The perpendicular bisector of a line segment is a line of reflection between the two endpoints of the segment.</li> <li>Reflections are isometries.</li> <li>A reflection is a function, <math>R_\ell</math>, which takes as its input, <math>P</math>, the location of a point with respect to some line of reflection, <math>\ell</math>, and outputs <math>R_\ell(P)</math> or the opposite of the location of <math>P</math> with respect to the line of reflection.</li> <li>The Perpendicular Bisector Theorem states: “If two points are equidistant from a third point, the third point lies on the perpendicular bisector of the segment connecting the two points.”</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
4	<b>Turn Yourself Around</b> Rotations as Functions	G.CO.1 G.CO.2 G.CO.4 G.CO.5	3	Students draw concentric circles and build a rotation of a triangle of $75^\circ$ . They investigate the rotation and determine that it is an isometry, producing a figure with congruent corresponding sides. Students formally define a rotation as a function which takes as input the location of a point with respect to a center of rotation and outputs the rotation of the point about the center through a rotation angle. Students draw rotation transformations given functions. Students then use what they know about the Perpendicular Bisector Theorem to determine the center of rotation and angle of rotation given only the pre-image and image figures. Students complete the lesson by describing sequences of translations, reflections, and rotations which map congruent figures onto each other.	<ul style="list-style-type: none"> <li>Rotations are isometries.</li> <li>A rotation is a function, <math>R_{E,t}(P) = P'</math> that maps its input, a point <math>P</math>, to another location, <math>P'</math>. This movement to a new location is defined by a center of rotation, <math>E</math>, and a rotation angle, <math>t</math>.</li> <li>The center of rotation lies on the perpendicular bisector of each pair of corresponding points of a pre-image and its rotated image. For this reason, the center of rotation is the point of intersection of any two of these perpendicular bisectors.</li> </ul>
5	<b>OKEECHOBEE</b> Reflectional and Rotational Symmetry	G.CO.3	1	Students use patty paper to create and fold shapes into two matching parts and rotate shapes so that they match exactly to their starting position. They then investigate given cutout shapes to see if they have these folding and rotating properties. Students define <i>reflectional symmetry</i> and <i>rotational symmetry</i> and identify shapes with these properties. They determine the number of lines of symmetry for a given shape and the angles a shape can be rotated through to match the original shape. Students identify and draw the sequences of reflections and rotations that carry a figure onto itself and investigate how a figure's lines of symmetry relate to these properties.	<ul style="list-style-type: none"> <li>A plane figure has reflectional symmetry if you can draw a line so that the figure to one side of the line is a reflection of the figure on the other side.</li> <li>A plane figure has rotational symmetry if you can rotate the figure more than <math>0^\circ</math> and less than <math>360^\circ</math> and the resulting figure is the same as the original figure.</li> <li>An individual figure may have horizontal symmetry, vertical symmetry, and/or rotational symmetry.</li> <li>A regular polygon of <math>n</math>-sides has <math>n</math> lines of symmetry.</li> <li>The measure of the angle of rotation of a regular polygon with <math>n</math> sides is <math>360^\circ/n</math>, which is the supplement of the measure of each of its interior angles.</li> </ul>
<b>Learning Individually with MATHia or Skills Practice</b>		G.CO.3 G.CO.4 G.CO.5	3	Students learn the formal definitions for translation, reflection, and rotation as rigid motions. They apply the formal definitions to identify rigid motions that carry figures onto other figures or onto themselves. Student learn that figures that can be reflected or rotated onto themselves have reflectional or rotational symmetry. They then select multiple transformations to map a pre-image to a target image given a reference point.	

\*Pacing listed in 45-minute days

# 2

## Establishing Congruence

Pacing: 38 Days

### Topic 1: Congruence Through Transformations

This topic builds on the work of the previous topic and the work with triangles in middle school. Students use the definitions of congruence through rigid motions to determine the minimum criteria for triangle congruence. First, they are grounded in formal geometric reasoning. Students consider counterexamples, conditional statements, truth values, and truth tables. They consider terms used in formal geometry proof, postulate and theorem, and investigate the Linear Pair Postulate and the Segment Addition Postulate. This preparation readies students to prove by construction the Side-Side-Side, Side-Angle-Side, and Angle-Side-Angle Congruence Theorems. Students close the topic by solving problems using these theorems.

**Standards:** G.CO.6, G.CO.7, G.CO.8, G.CO.9 **Pacing:** 9 Days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
1	<p><b>The Elements</b></p> <p>Formal Reasoning in Euclidean Geometry</p>	G.CO.9	2	<p>Conditional statements are analyzed and associated with truth values. Truth tables are used to help students organize information. Euclid's first five postulates and Euclid's Elements are mentioned. The terms <i>postulate</i> and <i>theorem</i> are defined, and students use the Linear Pair Postulate, the Segment Addition Postulate, and the Angle Addition Postulate to answer related questions.</p>	<ul style="list-style-type: none"> <li>The two reasons why a conclusion may be false is either the assumed information is false or the conclusion does not follow from the hypothesis.</li> <li>A counterexample is used to show a general statement is not true.</li> <li>A conditional statement is a statement that can be written in the form "If <math>p</math>, then <math>q</math>." The variable <math>p</math> represents the hypothesis and the variable <math>q</math> represents the conclusion.</li> <li>A truth value is whether or not a conditional statement is true or false; it is true if the conditional statement could be true, and it is false if the conditional statement could not be true.</li> <li>Truth tables are used to organize truth values of conditional statements.</li> <li>A postulate is a statement that is accepted without proof. A theorem is a statement that can be proven.</li> <li>The Linear Pair Postulate states: "If two angles form a linear pair, then the angles are supplementary."</li> <li>The Segment Addition Postulate states: "If point <math>B</math> is on line <math>\overline{AC}</math> and between points <math>A</math> and <math>C</math>, then <math>AB + BC = AC</math>."</li> <li>The Angle Addition Postulate states: "If point <math>D</math> lies in the interior of <math>\angle ABC</math>, then <math>m\angle ABD + m\angle DBC = m\angle ABC</math>."</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
2	<b>ASA, SAS, and SSS</b> Proving Triangle Congruence Theorems	G.CO.7 G.CO.8	3	Students use what they have learned in the previous topic: (1) isometries preserve distances and angle measures, (2) any point in the plane can be reflected across a line to map to another point in the plane, and (3) a point is equidistant from two other points if and only if it lies on their perpendicular bisector. They use these facts to create and verify proofs of the SSS, SAS, and ASA Congruence Theorems using rigid motion transformations. Students then explore some non-examples of congruence theorems (AAA and SSA). Students explore a problem at the beginning of the lesson which can be solved by creating congruent triangles at the end of the lesson.	<ul style="list-style-type: none"> <li>A proof is a series of statements and corresponding reasons forming a valid argument that starts with a hypothesis and arrives at a conclusion.</li> <li>The Side-Side-Side (SSS) Congruence Theorem states: "If three sides of one triangle are congruent to the corresponding sides of another triangle, then the triangles are congruent."</li> <li>Corresponding parts of congruent triangles are congruent, abbreviated as CPCTC, is often used as a reason for stating congruences in geometric proofs after triangles have been proven congruent.</li> <li>The Side-Angle-Side (SAS) Congruence Theorem states: "If two sides and the included angle of one triangle are congruent to the corresponding sides and included angle of another triangle, then the triangles are congruent."</li> <li>The Angle-Side-Angle (ASA) Congruence Theorem states: "If two angles and the included side of one triangle are congruent to the corresponding angles and included side of another triangle, then the triangles are congruent."</li> <li>Triangle congruence theorems such as SSS, SAS, and ASA can be proven using rigid motion transformations.</li> </ul>
3	<b>I Never Forget a Face</b> Using Triangle Congruence to Solve Problems	G.CO.6	3	Students use the criteria for triangle congruence they proved in the previous lesson—Side-Side-Side, Side-Angle-Side, and Angle-Side-Angle—to solve real-world and mathematical problems. Students learn that triangle congruence has been an important factor in developing computer face recognition techniques. They apply these techniques to a few problems and then use the triangle congruence criteria to determine whether two triangles are congruent—both for triangles presented on a coordinate plane and for triangles not on a coordinate plane.	<ul style="list-style-type: none"> <li>The SSS, SAS, and ASA Congruence Theorems can be applied to solve real-world and mathematical problems.</li> <li>Congruent parts of triangles can be depicted from a diagram rather than stated. These can be instances where two triangles share a common side or angle.</li> <li>The SSS, SAS, and ASA Congruence Theorems can be applied to triangles on or off the coordinate plane.</li> </ul>
<b>Learning Individually with MATHia or Skills Practice</b>		G.CO.7 G.CO.8 G.CO.10	1	Students practice writing and identifying triangle congruence statements. They watch an animation introducing the triangle congruence theorems and match images of pairs of triangles to the theorem by which they are proven congruent. They then create flowchart proofs of the triangle congruence theorems.	

\*Pacing listed in 45-minute days

## Topic 2: Justifying Line and Angle Relationships

Students are introduced to formal geometric reasoning. They learn how to write formal proofs—flow chart, two-column and paragraph proofs, in addition to proof by construction and the algebraic proofs that they used previously—and then prove many of the conjectures that they made in the previous topic. Students begin by proving foundational theorems and then prove theorems related to angle pairs formed when parallel lines are intersected by a transversal. They prove conjectures about the angles on the interior and exterior of polygons and then focus on conjectures about the relationships between sides and angles in triangles. Finally, students prove theorems about angle relationships formed by chords and lines inside and outside of circles.

**Standards:** G.CO.9, G.CO.10, G.CO.11, G.C.2, G.C.3, G.C.4 **Pacing:** 16 Days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
1	<b>Proof Positive</b> Forms of Proof	G.CO.9	3	Students apply real number properties to angle measures, line segments, and distances. Next, they informally use constructions to reason about a conditional statement. Students are introduced to flowchart and two-column proofs and analyze worked examples of both forms to prove the same statement. They then complete partial proofs to prove the Right Angle Congruence Theorem and the Congruent Supplement Theorem. Students analyze a flowchart proof of the Vertical Angle Theorem before writing a two-column proof for the same theorem using the Congruent Supplement Theorem. They use these theorems to determine unknown angle measures. Finally, students are introduced to paragraph proofs and demonstrate what they have learned using complete sentences.	<ul style="list-style-type: none"> <li>• The Addition Property of Equality, the Subtraction Property of Equality, the Reflexive Property, the Substitution Property, and the Transitive Property can be applied to angle measures, segment measures, and distances.</li> <li>• A construction proof, two-column proof, flow chart proof, and paragraph proof are all acceptable forms of reasoning about geometric relationships.</li> <li>• The Right Angle Congruence Postulate states: "All right angles are congruent."</li> <li>• The Congruent Supplement Theorem states: "If two angles are supplements of the same angle or of congruent angles, then the angles are congruent."</li> <li>• The Vertical Angle Theorem states: "Vertical angles are congruent."</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
2	<b>A Parallel Universe</b> Proving Parallel Line Theorems	G.CO.9	2	Students explore theorems related to parallel lines cut by a transversal, proving both that special angle pairs are congruent given parallel lines and the converse statements—that two lines are parallel given the congruence of special angle pairs. Students begin by proving the Corresponding Angles Theorem using what they know about translations, and they prove the remaining theorems in flowcharts and two-column format using definitions, postulates, and already proven theorems. Students continue to investigate the process of creating proofs, building proof plans to help them connect if/then statements using deductive reasoning.	<ul style="list-style-type: none"> <li>• The Corresponding Angle Theorem states: “If two parallel lines are cut by a transversal, then corresponding angles are congruent.”</li> <li>• The Corresponding Angle Converse Theorem states: “If two lines cut by a transversal form congruent corresponding angles, then the lines are parallel.”</li> <li>• The Same-Side Interior Angle Theorem states: “If two parallel lines are cut by a transversal, then same-side interior angles are supplementary.”</li> <li>• The Same-Side Interior Angle Converse Theorem states: “If two lines cut by a transversal form supplementary same-side interior angles, then the lines are parallel.”</li> <li>• The Alternate Interior Angle Theorem states: “If two parallel lines are cut by a transversal, then alternate interior angles are congruent.”</li> <li>• The Alternate Interior Angle Converse Theorem states: “If two lines cut by a transversal form congruent alternate interior angles, then the lines are parallel.”</li> <li>• The Same-Side Exterior Angle Theorem states: “If two parallel lines are cut by a transversal, then same-side exterior angles are supplementary.”</li> <li>• The Same-Side Exterior Angle Converse Theorem states: “If two lines cut by a transversal form supplementary same-side exterior angles, then the lines are parallel.”</li> <li>• The Alternate Exterior Angle Theorem states: “If two parallel lines are cut by a transversal, then alternate exterior angles are congruent.”</li> <li>• The Alternate Exterior Angle Converse Theorem states: “If two lines cut by a transversal form congruent alternate exterior angles, then the lines are parallel.”</li> <li>• The Perpendicular/Parallel Line Theorem states: “If two lines are perpendicular to the same line, then the two lines are parallel to each other.”</li> </ul>
3	<b>Ins and Outs</b> Interior and Exterior Angles of Polygons	G.CO.10	2	Students investigate the Triangle Sum Theorem and then prove the theorem using what they know about congruent angle pairs formed from parallel lines and a transversal. Students then explain how the Exterior Angle Theorem can be demonstrated using the same diagram as the one used to prove the Triangle Sum Theorem. Students generalize this activity to derive a formula that can be used to determine the sum of the interior angle measures of any polygon and also determine the sum of the exterior angle measures of any polygon. Finally, students demonstrate what they have learned in the lesson by solving a variety of mathematical problems.	<ul style="list-style-type: none"> <li>• The Triangle Sum Theorem states: “The sum of the measures of the interior angles of a triangle is equal to <math>180^\circ</math>.”</li> <li>• The Exterior Angle Theorem states: “The measure of the exterior angle of a triangle is equal to the sum of the measures of the two remote interior angles.”</li> <li>• The sum of the measures of the interior angles of a quadrilateral is equal to <math>360^\circ</math>.</li> <li>• For a polygon with <math>n</math> sides, the sum of its interior angle measures is equal to <math>180(n - 2)^\circ</math>.</li> <li>• For a regular polygon with <math>n</math> sides, the measure of each interior angle is equal to <math>180(n - 2)^\circ/n</math>.</li> <li>• For a polygon with <math>n</math> sides, the sum of the measures of the exterior angles is equal to <math>360^\circ</math>.</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
4	<p><b>Identical Twins</b></p> <p>Perpendicular Bisector and Isosceles Triangle Theorems</p>	<p>N.RN.2 G.CO.9 G.CO.10</p>	2	<p>Students use their knowledge of Side-Angle- Side (SAS), Side-Side-Side (SSS), or Angle- Side-Angle (ASA) Theorems to explain why pairs of triangles are congruent. The term CPCTC (corresponding parts of congruent triangles are congruent) is defined as a reason that can be used after two triangles are proved congruent. Students then investigate and prove the Perpendicular Bisector Theorem using CPCTC and analyze a worked example of its converse. They then use CPCTC to prove the Isosceles Triangle Base Angles Theorem and its converse. Students use the converse of the Perpendicular Bisector Theorem to demonstrate the 30°-60°-90° Triangle Theorem using algebra. They then use the Isosceles Triangle Base Angles Theorem to algebraically demonstrate the 45°-45°-90° Triangle Theorem. Finally, students reason about the Hypotenuse-Angle Theorem and the Angle-Angle-Side Congruence Theorem and solve a variety of mathematical and real-world problems using what they learned in the lesson.</p>	<ul style="list-style-type: none"> <li>• The Perpendicular Bisector Theorem states: “The points on a perpendicular bisector of a line segment are equidistant from the segment’s endpoints.”</li> <li>• The Perpendicular Bisector Converse Theorem states: “If a point is equidistant from the endpoints of a line segment, then the point lies on the perpendicular bisector of the line segment.”</li> <li>• The Isosceles Triangle Base Angles Theorem states: “If two sides of a triangle are congruent, then the angles opposite these sides are congruent.”</li> <li>• The Isosceles Triangle Base Angles Converse Theorem states: “If two angles of a triangle are congruent, then the sides opposite these angles are congruent.”</li> <li>• The 30°-60°-90° Triangle Theorem states: “The length of the hypotenuse in a 30°-60°-90° triangle is 2 times the length of the shorter leg, and the length of the longer leg is <math>\sqrt{3}</math> times the length of the shorter leg.”</li> <li>• The 45°-45°-90° Triangle Theorem states: “The length of the hypotenuse in a 45°-45°-90° triangle is <math>\sqrt{2}</math> times the length of a leg.</li> <li>• The Hypotenuse-Angle (HA) Congruence Theorem states: “If the hypotenuse and an acute angle of one right triangle are congruent to the hypotenuse and an acute angle of another right triangle, then the two triangles are congruent.”</li> <li>• The Angle-Angle-Side (AAS) Congruence Theorem states: “If two angles and the non-included side of one triangle are congruent to two angles and the non-included side of another triangle, then the two triangles are congruent.”</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
5	<p><b>Corners in a Round Room</b></p> <p>Angle Relationships Inside and Outside Circles</p>	G.C.2 G.C.3 G.C.4(+)	2	<p>Students reason about arc measures associated with a clockface and conclude that the measures of two central angles of the same circle (or congruent circles) have corresponding congruent minor arcs. They use a two-column proof to prove one case of the Inscribed Angle Theorem and algebraic reasoning to prove the other two cases. Students then prove two theorems associated with inscribed polygons using the Inscribed Angle Theorem. Next, they explore and prove theorems for determining the measures of angles located on the inside and outside of a circle. They construct a tangent line to a circle from a point outside the circle. A proof by contradiction is provided to show a perpendicular relationship exists when the radius of a circle is drawn to a point of tangency. Finally, students use the theorems they have proven to determine the measures of arcs and angles of a circle.</p>	<ul style="list-style-type: none"> <li>• The Arc Addition Postulate states: “The measure of an arc formed by two adjacent arcs is the sum of the measures of the two arcs.”</li> <li>• The measure of a central angle is equal to the measure of its intercepted arc.</li> <li>• The Inscribed Angle Theorem states: “The measure of an inscribed angle is equal to half the measure of its intercepted arc.”</li> <li>• The Inscribed Right Triangle–Diameter Theorem states: “When a triangle is inscribed in a circle such that one side of the triangle is a diameter, the triangle is a right triangle.”</li> <li>• The Inscribed Quadrilateral–Opposite Angles Theorem states: “When a quadrilateral is inscribed in a circle, the opposite angles are supplementary.”</li> <li>• The Interior Angles of a Circle Theorem states: “If an angle is formed by two intersecting chords or secants of a circle such that the vertex of the angle is in the interior of the circle, then the measure of the angle is half of the sum of the measures of the arcs intercepted by the angle and its vertical angle.”</li> <li>• The Exterior Angles of a Circle Theorem states: “If an angle is formed by two intersecting chords or secants of a circle such that the vertex of the angle is in the exterior of the circle, then the measure of the angle is half of the difference of the measures of the arcs intercepted by the angle.”</li> <li>• The Tangent to a Circle Theorem states: “A line drawn tangent to a circle is perpendicular to a radius of the circle drawn to the point of tangency.”</li> </ul>
	<p><b>Learning Individually with MATHia or Skills Practice</b></p>	G.CO.1 G.CO.9	5	<p>Students identify given angle measures and justify their reasoning. They learn to write flowchart proofs and then convert them to two-column proofs. They calculate the measures of angles and sides in polygons and along parallel lines before writing formal proofs of the known relationships.</p>	

\*Pacing listed in 45-minute days

### Topic 3: Using Congruence Theorems

Students use the theorems that they proved in Justifying Line and Angle Relationships to prove additional theorems. First, they prove congruence theorems specific to right triangles. Then, students prove properties of quadrilaterals. Finally, they prove theorems about relationships between chords of congruent circles.

**Standards:** G.CO.10, G.CO.11, G.C.2 **Pacing:** 13 Days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
1	<p><b>SSS, SAS, AAS, ... S.O.S!</b></p> <p>Using Triangle Congruence to Determine Relationships Between Segments</p>	G.CO.10	2	<p>Students construct a right triangle in a circle given a leg length and a hypotenuse length. They compare their constructions and notice that the triangles are congruent to each other. Students then prove this Hypotenuse-Leg Congruence Theorem using both a two-column proof and algebraic reasoning. Students use reasoning to demonstrate that the Leg-Leg Congruence Theorem and the Leg-Angle Congruence Theorem must be true, given the triangle congruence theorems they know. They then apply these congruence theorems to solve problems. Finally, students analyze a proof of the Tangent Segment Theorem and use it to solve additional problems.</p>	<ul style="list-style-type: none"> <li>The Hypotenuse-Leg Congruence Theorem states: "If the hypotenuse and leg of one right triangle are congruent to the hypotenuse and leg of another right triangle, then the triangles are congruent."</li> <li>The Leg-Leg Congruence Theorem states: "If the two corresponding shorter legs of two right triangles are congruent, then the triangles are congruent."</li> <li>The Leg-Angle Congruence Theorem states: "If the leg and an acute angle of one right triangle are congruent to the corresponding leg and acute angle of another triangle, then the triangles are congruent."</li> <li>The Leg-Leg and Leg-Angle Congruence Theorems can be justified using SSS, SAS, ASA, and/or AAS triangle congruence.</li> <li>The Tangent Segment Theorem states: "If two tangent segments are drawn from the same point on the exterior of a circle, then the tangent segments are congruent."</li> </ul>
2	<p><b>Props to You</b></p> <p>Properties of Quadrilaterals</p>	G.CO.11	2	<p>Students verify the properties of a parallelogram, a rhombus, a rectangle, a square, a trapezoid, and a kite using formal two-column or paragraph formats, as well as informal reasoning. Students apply the theorems to arrive at solutions to problem situations.</p>	<ul style="list-style-type: none"> <li>A parallelogram is defined as a quadrilateral with opposite sides parallel. The properties of a parallelogram include: <ul style="list-style-type: none"> <li>The opposite sides of a parallelogram are congruent.</li> <li>The opposite angles of a parallelogram are congruent.</li> <li>The diagonals of a parallelogram bisect each other.</li> <li>If one pair of opposite sides of a quadrilateral are both parallel and congruent, then the quadrilateral is a parallelogram.</li> </ul> </li> <li>A rhombus is defined as a parallelogram with all sides congruent. The properties of a rhombus include: <ul style="list-style-type: none"> <li>The diagonals of a rhombus bisect the vertex angles.</li> <li>The diagonals of a rhombus are perpendicular to each other.</li> </ul> </li> <li>A rectangle is defined as a parallelogram with all angles congruent. The diagonals of a rectangle are congruent.</li> <li>A square is defined as a parallelogram with all angles congruent and all sides congruent. The square has all of the properties of the rectangle and rhombus.</li> <li>An isosceles trapezoid is defined as a trapezoid with congruent non-parallel sides. The base angles of an isosceles trapezoid are congruent.</li> <li>The Trapezoid Midsegment Theorem states: "The midsegment of a trapezoid is parallel to each of the bases and its length is one half the sum of the lengths of the bases."</li> <li>A kite is defined as a quadrilateral with two pairs of consecutive congruent sides. The properties of a kite include: <ul style="list-style-type: none"> <li>One diagonal of a kite is a line of symmetry.</li> <li>One diagonal of a kite bisects a pair of opposite angles.</li> <li>One diagonal of a kite is the perpendicular bisector of the other diagonal.</li> </ul> </li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
3	<b>Three-Chord Song</b> Relationships Between Chords	G.C.2	1	Students explore a problem situation which asks them to determine the diameter of a circular plate given only a broken piece of the plate. Students conjecture about methods that may be used to determine the diameter. They then prove the Diameter-Chord Theorem, Equidistant Chord Theorem, and Equidistant Chord Converse Theorem. They also prove the Congruent Chord-Congruent Arc Theorem and its converse. Finally, students revisit and solve the broken plate problem from the Getting Started.	<ul style="list-style-type: none"> <li>• The Diameter-Chord Theorem states: "If a circle's diameter is perpendicular to a chord, then the diameter bisects the chord and</li> <li>• The Equidistant Chord Theorem states: "If two chords of the same circle or congruent circles are congruent, then they are equidistant from the center of the circle."</li> <li>• The Equidistant Chord Converse Theorem states: "If two chords of the same circle or congruent circles are equidistant from the center of the circle, then the chords are congruent."</li> <li>• The Congruent Chord-Congruent Arc Theorem states: "If two chords of the same circle or congruent circles are congruent, then their corresponding arcs are congruent."</li> <li>• The Congruent Chord-Congruent Arc Converse Theorem states: "If two arcs of the same circle or congruent circles are congruent, then their corresponding chords are congruent."</li> </ul>
<b>Learning Individually with MATHia or Skills Practice</b>		G.CO.10 G.CO.11	8	Students use the proof tool to prove triangles congruent using SSS, SAS, AAS, ASA, HL, and HA. They use CPCTC to prove other theorems related to angles and sides in triangles. They review the properties of parallelograms and use them to solve for unknown parts of quadrilaterals and parallelograms before proving that they are true in all cases.	

\*Pacing listed in 45-minute days

# 3

## Investigating Proportionally

Pacing: 29 Days

### Topic 1: Similarity

Students build on what they know about rigid motion transformations to formally define similarity transformations. They perform dilations on figures and describe the similarity transformations necessary to map one figure onto another. Students establish triangle similarity criteria and use them to both determine the similarity of triangles and to prove theorems about proportionality. Geometric mean is defined, and students use it to solve problems and to prove the Pythagorean Theorem using similarity. Finally, they use the concepts of similarity and proportionality to partition directed line segments into given ratios.

**Standards:** G.SRT.1, G.SRT.1a, G.SRT.1b, G.SRT.2, G.SRT.3, G.SRT.4, G.SRT.5 **Pacing:** 15 Days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
1	<b>Big, Little, Big, Little</b> Dilating Figures to Create Similar Figures	G.SRT.1b G.SRT.2	1	Students perform dilations on triangles and other figures both on and off of the coordinate plane. They explore the ratios formed as a result of dilation and recall scale factor. Similar triangles are defined and students explore the relationships between the corresponding sides and between the corresponding angles. Students then use similarity statements to draw similar triangles, and describe the similarity transformations necessary to map one triangle onto another as an alternate approach to showing similarity.	<ul style="list-style-type: none"> <li>A dilation is a transformation that enlarges, reduces, or keeps congruent a pre-image to create an image.</li> <li>The center of dilation is a fixed point at which a figure is dilated.</li> <li>The scale factor of a dilation is the ratio of the distance from the center of dilation to a point on the image to the distance from the center of dilation to the corresponding point on the pre-image.</li> <li>When the scale factor is greater than 1, the dilation is an enlargement. When the scale factor is between 0 and 1, the dilation is a reduction. When the scale factor is exactly 1, the dilation produces a congruent figure.</li> </ul>
2	<b>Similar Triangles or Not?</b> Establishing Triangle Similarity Criteria	G.SRT.2 G.SRT.3	2	Students explore methods for proving triangles similar using construction tools and measuring tools. Then the Angle-Angle Similarity Theorem, Side-Side-Side Similarity Theorem, and Side-Angle-Side Similarity Theorem are stated, and students use these theorems to determine the similarity of triangles. The terms <i>included angle</i> and <i>included side</i> are defined. Some problem situations require the use of construction tools and others require measurement tools.	<ul style="list-style-type: none"> <li>The Angle-Angle Similarity Theorem states: "If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar."</li> <li>The Side-Side-Side Similarity Theorem states: "If all three corresponding sides of two triangles are proportional, then the triangles are similar."</li> <li>The Side-Angle-Side Similarity Theorem states: "If two of the corresponding sides of two triangles are proportional and the included angles are congruent, then the triangles are similar."</li> <li>An included angle is an angle formed by two consecutive sides of a figure.</li> <li>An included side is a line segment between two consecutive angles of a figure.</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
3	<b>Keep It in Proportion</b> Theorems About Proportionality	G.SRT.1a G.SRT.4	4	Students prove the Angle Bisector/Proportional Side Theorem, the Triangle Proportionality Theorem, the Proportional Segments Theorem, and the Triangle Midsegment Theorem. Students use these theorems to solve problems.	<ul style="list-style-type: none"> <li>The Angle Bisector/Proportional Side Theorem states: "A bisector of an angle in a triangle divides the opposite side into two segments whose lengths are in the same ratio as the lengths of the sides adjacent to the angle."</li> <li>The Triangle Proportionality Theorem states: "If a line parallel to one side of a triangle intersects the other two sides, then it divides the two sides proportionally."</li> <li>The Converse of the Triangle Proportionality Theorem states: "If a line divides the two sides of a triangle proportionally, then it is parallel to the third side."</li> <li>The Proportional Segments Theorem states: "If three parallel lines intersect two transversals, then they divide the transversals proportionally."</li> <li>The Triangle Midsegment Theorem states: "The midsegment of a triangle is parallel to the third side of the triangle and half the measure of the third side of the triangle."</li> <li>The medians of a triangle are concurrent.</li> </ul>
4	<b>This Isn't Your Average Mean</b> More Similar Triangles	G.SRT.3 G.SRT.4 G.SRT.5	1	The term <i>geometric mean</i> is defined and is used in triangle theorems to solve for unknown measurements. Students practice using the Right Triangle/Altitude Similarity Theorem, the Right Triangle Altitude/Hypotenuse Theorem, and the Right Triangle Altitude/Leg Theorem to solve problems. Next, they are guided through the steps necessary to prove the Pythagorean Theorem using similar triangles.	<ul style="list-style-type: none"> <li>The Right Triangle/Altitude Similarity Theorem states: "If an altitude is drawn to the hypotenuse of a right triangle, then the two triangles formed are similar to the original triangle and to each other."</li> <li>The geometric mean of two numbers <math>a</math> and <math>b</math> is the number <math>x</math> such that <math>a/x = x/b</math>.</li> <li>The Right Triangle Altitude/Hypotenuse Theorem states: "The measure of the altitude drawn from the vertex of the right angle of a right triangle to its hypotenuse is the geometric mean between the measures of the two segments of the hypotenuse."</li> <li>The Right Triangle Altitude/Leg Theorem states: "If the altitude is drawn to the hypotenuse of a right triangle, each leg of the right triangle is the geometric mean between the hypotenuse and the segment of the hypotenuse adjacent to the leg."</li> <li>The Pythagorean Theorem states: "If <math>a</math> and <math>b</math> are the lengths of the legs of a right triangle and <math>c</math> is the length of the hypotenuse, then <math>a^2 + b^2 = c^2</math>."</li> </ul>
5	<b>Run It Up the Flagpole</b> Application of Similar Triangles	G.SRT.5	2	Indirect measurement takes students out of their classroom and school building to measure the height of objects such as flagpoles, tops of trees, telephone poles, or buildings using similar triangles. Additionally, students are given several situations in which they create proportions related to similar triangles to solve for unknown measurements.	<ul style="list-style-type: none"> <li>Indirect measurement is the process of using proportions related to similar triangles to determine a measurement when direct measurement is inconvenient or difficult.</li> <li>The mirror method and the shadow method are used to diagram similar triangles.</li> <li>The Law of Reflection states that the angle of incidence is congruent to the angle of reflection.</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
6	<b>Jack's Spare Key</b> Partitioning Segments in Given Ratios	G.GPE.6	1	Students use the Midpoint Formula and Distance Formula to determine the midpoints of line segments on the coordinate plane. Students use the Distance Formula and other methods to partition directed line segments into given ratios. They then use geometric concepts of similarity and proportionality to partition directed line segments into given ratios.	<ul style="list-style-type: none"> <li>The midpoint of a line segment is the point on the segment that is equidistant from the endpoints of the line segment.</li> <li>The Midpoint Formula states that the midpoint between any two points on a coordinate plane, <math>(x_1, y_1)</math> and <math>(x_2, y_2)</math>, is <math>((x_1 + x_2)/2, (y_1 + y_2)/2)</math>.</li> <li>The Triangle Proportionality Theorem states: "If a line parallel to one side of a triangle intersects the other two sides, then it divides the two sides proportionally."</li> </ul>
<b>Learning Individually with MATHia or Skills Practice</b>		G.SRT.2 G.SRT.4 G.SRT.5	4	Students watch an animation that demonstrates that when figures are similar, a series of rigid motions and dilations can map one figure onto the other. They identify similar figures and determine corresponding side lengths and corresponding angle measures. Students then calculate corresponding parts of similar triangles, both in and out of context. They use the AA Similarity Theorem, SSS Similarity Theorem, and SAS Similarity Theorem to prove the Parallel Segment Proportionality Theorem and Triangle Midsegment Theorem.	

\*Pacing listed in 45-minute days

## Topic 2: Trigonometry

Students begin by exploring the relationships between the side lengths of similar right triangles. They use this investigation to define the trigonometric ratios: sine, cosine, and tangent ratios and the corresponding cosecant, secant, and cotangent ratios. They use these ratios and their inverses to solve real-world problems for unknown side lengths and angle measures. Students then explore the complementary relationships involved with trigonometric ratios and use them to solve problems. The final lesson addresses the (+) standards that apply trigonometry to general triangles. Students derive the formula  $A = (1/2)a \cdot b \cdot \sin(C)$ , the Law of Sines, and the Law of Cosines. They use these to solve real-world and mathematical problems.

**Standards:** G.SRT.6, G.SRT.7, G.SRT.8, G.SRT.9(+), G.SRT.10(+), G.SRT.11(+)

**Pacing:** 14 Days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
1	<b>Three Angle Measure</b> Introduction to Trigonometry	G.SRT.6	1	Students drop vertical lines from different points on the hypotenuses of $45^\circ\text{-}45^\circ\text{-}90^\circ$ and $30^\circ\text{-}60^\circ\text{-}90^\circ$ triangles to form similar right triangles and determine the lengths of the sides. They then convert the lengths into ratios and compare them. Students calculate the slope of the hypotenuse and realize it is the same as the opposite-to-adjacent ratio for both the $45^\circ\text{-}45^\circ\text{-}90^\circ$ triangle and the $30^\circ\text{-}60^\circ\text{-}90^\circ$ triangle. Students conclude that the ratios they studied are constant in similar right triangles, given the same reference angle. Students also discuss how these ratios change in general as the measure of the reference angle changes.	<ul style="list-style-type: none"> <li>Similar right triangles are formed by dropping vertical line segments from the hypotenuse perpendicular to the base of the right triangles.</li> <li>Given the same reference angle, the ratios (side opposite to reference angle)/hypotenuse, (side adjacent to reference angle)/hypotenuse, and (side opposite to reference angle)/(side adjacent to reference angle) are constant.</li> <li>The side length ratios(side opposite to reference angle)/hypotenuse, (side adjacent to reference angle)/hypotenuse, and (side opposite to reference angle)/(side adjacent to reference angle) are the same for all <math>45^\circ\text{-}45^\circ\text{-}90^\circ</math> given the same reference angle.</li> <li>The side length ratios(side opposite to reference angle)/hypotenuse, (side adjacent to reference angle)/hypotenuse, and (side opposite to reference angle)/(side adjacent to reference angle) are the same for all <math>30^\circ\text{-}60^\circ\text{-}90^\circ</math> given the same reference angle.</li> <li>The slope of the hypotenuse of a <math>45^\circ\text{-}45^\circ\text{-}90^\circ</math> triangle and the (side opposite to reference angle)/(side adjacent to reference angle) ratio are equal to 1.</li> <li>The slope of the hypotenuse of a <math>45^\circ\text{-}45^\circ\text{-}90^\circ</math> triangle and a <math>30^\circ\text{-}60^\circ\text{-}90^\circ</math> triangle is equal to the (side opposite reference angle)/(side adjacent to reference angle) ratio.</li> <li>The Pythagorean Theorem can be used to determine the exact ratios of side lengths in similar right triangles.</li> </ul>
2	<b>The Tangent Ratio</b> Tangent Ratio, Cotangent Ratio, and Inverse Tangent	G.SRT.6 G.SRT.8	2	The terms <i>tangent</i> , <i>cotangent</i> , and <i>inverse tangent</i> are introduced, and tangent is explicitly connected to the concept of slope. Applying the tangent ratio to similar triangles, students conclude that the value of the tangent of congruent angles of similar triangles is always equal and the measure of an acute angle increases as the value of the tangent increases. Students write expressions based on the complementary relationship between the two acute angles in right triangles. Students prove algebraically $\cot A = 1 / (\tan A)$ . When the inverse tangent is introduced, students use their calculators to solve for the measure of an acute angle in a right triangle.	<ul style="list-style-type: none"> <li>The tangent (tan) of an acute angle in a right triangle is the ratio of the length of the side that is opposite the angle to the length of the side that is adjacent to the angle.</li> <li>The cotangent (cot) of an acute angle in a right triangle is the ratio of the length of the side that is adjacent to the angle to the length of the side that is opposite the angle.</li> <li>The inverse tangent (or arc tangent) of <math>x</math> is the measure of an acute angle whose tangent is <math>x</math>.</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
3	<b>The Sine Ratio</b> Sine Ratio, Cosecant Ratio, and Inverse Sine	G.SRT.6 G.SRT.8	2	The terms <i>sine</i> , <i>cosecant</i> , and <i>inverse sine</i> are introduced. A real-world context is given for determining the sine ratio in right triangles. Students conclude that as the acute angle increases in measure, the sine ratio increases in value while the cosecant value decreases, and the value of sine will always be less than 1 because the hypotenuse—the denominator in the sine ratio—is the longest side of the right triangle. When the inverse sine is introduced, students use their calculators to solve for the measure of an acute angle in a right triangle.	<ul style="list-style-type: none"> <li>In a right triangle, the ratio (side opposite to reference angle)/hypotenuse increases as the reference angle increases.</li> <li>The sine (sin) of an acute angle in a right triangle is the ratio of the length of the side that is opposite the angle to the length of the hypotenuse.</li> <li>The cosecant (csc) of an acute angle in a right triangle is the ratio of the length of the hypotenuse to the length of the side that is opposite the angle.</li> <li>The inverse sine (or arcsine) of <math>x</math> is the measure of an acute angle whose sine is <math>x</math>.</li> </ul>
4	<b>The Cosine Ratio</b> Cosine Ratio, Secant Ratio, and Inverse Cosine	G.SRT.6 G.SRT.8	2	The terms <i>cosine</i> , <i>secant</i> , and <i>inverse cosine</i> are introduced. A real-world context is given for determining the cosine ratio in right triangles. Students conclude that as the acute angle increases in measure, the cosine ratio decreases in value while the secant value increases. Additionally, the value of cosine will always be less than 1 because the hypotenuse—the denominator in the sine ratio—is the longest side of the right triangle. When the inverse cosine is introduced, students use their calculators to solve for the measure of an acute angle in a right triangle.	<ul style="list-style-type: none"> <li>The cosine (cos) of an acute angle in a right triangle is the ratio of the length of the side that is adjacent to the angle to the length of the hypotenuse.</li> <li>The secant (sec) of an acute angle in a right triangle is the ratio of the length of the hypotenuse to the length of the side that is adjacent to the angle.</li> <li>The inverse cosine (or arccosine) of <math>x</math> is the measure of an acute angle whose cosine is <math>x</math>.</li> </ul>
5	<b>We Complement Each Other</b> Complement Angle Relationships	G.SRT.7	2	Students explore the complementary relationships involved with trigonometric ratios and use them to solve application problems. The Pythagorean Theorem in conjunction with complementary relationships is used to determine the value of the six trigonometric ratios of a $45^\circ$ angle, a $30^\circ$ angle, and a $60^\circ$ angle.	<ul style="list-style-type: none"> <li>When <math>\angle A</math> and <math>\angle B</math> are acute angles in a right triangle, <math>\sin \angle A = \cos \angle B</math> and <math>\cos \angle A = \sin \angle B</math>.</li> <li>When <math>\angle A</math> and <math>\angle B</math> are acute angles in a right triangle, <math>\csc \angle A = \sec \angle B</math> and <math>\sec \angle A = \csc \angle B</math>.</li> <li>When <math>\angle A</math> and <math>\angle B</math> are acute angles in a right triangle, <math>\tan \angle A = \cot \angle B</math> and <math>\cot \angle A = \tan \angle B</math>.</li> </ul>
6	<b>A Deriving Force</b> Deriving the Triangle Area Formula, the Law of Sines, and the Law of Cosines	G.SRT.9(+) G.SRT.10(+) G.SRT.11(+)	1	Students derive the area formula for a triangle when given the length of two sides and the measure of an included angle, the Law of Sines, and the Law of Cosines. They also apply these formulas to contextual problems that can be modeled using triangles to determine unknown measurements.	<ul style="list-style-type: none"> <li>The area formula for any triangle is <math>A = (1/2)ab \cdot (\sin C)</math>.</li> <li>The Law of Sines: <math>\sin A / a = \sin B / b = \sin C / c</math></li> <li>The Law of Cosines:  <math>a^2 = b^2 + c^2 - 2bc \cdot \cos A</math>  <math>b^2 = a^2 + c^2 - 2ac \cdot \cos B</math>  <math>c^2 = a^2 + b^2 - 2ab \cdot \cos C</math> </li> </ul>
<b>Learning Individually with MATHia or Skills Practice</b>		G.SRT.6 G.SRT.7 G.SRT.8	4	Students use similar triangles to define and understand the trigonometric ratios sine, cosine, and tangent. They then explore the sine, cosine, and tangent and estimate these ratios using an interactive Explore Tool with a unit circle, as well as describe the ratios as percentages of different lengths. Students solve problems in various contexts using the trigonometric ratios and the Explore Tool. They then explore the relationship between the sine and cosine of complementary angles. Students calculate the measures of sides and angles of right triangles using trigonometric ratios, the Pythagorean Theorem, and/or the Triangle Sum Theorem in both real-world and mathematical problems.	

\*Pacing listed in 45-minute days

4

# Connecting Geometric and Algebraic Descriptions

Pacing: 25 Days

## Topic 1: Circles and Volume

Students use similarity transformations to prove that all circles are similar. The term arc length is defined as a proportion of the circumference. Students establish that the length of the arc intercepted by an angle is proportional to the radius and define the radian measure of the angle as the constant of proportionality. They define the area of a sector in terms of a proportion of the area of the circle that contains it. They use these formulas to solve real world problems. Then students analyze the three-dimensional solid formed when translating a figure along a line segment and use Cavalieri's Principles to derive the volume formulas for prisms and cylinders. Limit arguments and Cavalieri's Principles are used to derive the volume formulas for pyramids and cones. Students use the formulas for the volume and surface area of geometric solids to solve real-world problems.

**Standards:** G.C.1, G.C.5, G.GMD.1, G.GMD.3, G.GMD.4, G.MG.1, G.MG.2 **Pacing:** 11 Days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
1	<b>All Circles Great and Small</b> Similarity Relationships in Circles	G.C.1 G.C.5 G.GMD.1	1	Students informally explore a different way of measuring lengths on a circle using radius measures. They then use similarity transformations to demonstrate that all circles are similar. Students use the fact that all circles are similar to demonstrate that $C = 2\pi r$ . <i>Arc length</i> is defined as a portion of the circumference of a circle and it is distinguished from arc measure. Students convert between arc measure and arc length and solve problems involving arc lengths. Finally, they explore radian measures, formalizing the ideas they explored in the opening of the lesson.	<ul style="list-style-type: none"> <li>The radius of a circle, <math>r</math>, maps onto the circumference of the circle <math>2\pi r</math> times.</li> <li>All circles are similar figures.</li> <li>There is a proportional relationship between the measure of an arc length of a circle, <math>s</math>, and the circumference of the circle.</li> <li>The formula for arc length can be written as <math>s = m/(360^\circ)(2\pi r)</math> where <math>s</math> is the arc length and <math>m</math> is the central angle measure.</li> <li>One radian is the measure of a central angle whose arc length is the same as the radius of the circle.</li> <li>The formula for arc length can be written <math>s = \theta r</math>, where <math>s</math> is the arc length and <math>\theta</math> is the central angle measure in radians.</li> <li>When converting degrees to radians, multiply a degree measure by <math>\pi/180^\circ</math> and when converting radians to degrees, multiply a radian measure by <math>180^\circ/\pi</math>.</li> </ul>
2	<b>A Slice of Pi</b> Sectors and Segments of a Circle	G.C.5 G.GMD.1	1	The terms <i>sector of a circle</i> and <i>segment of a circle</i> are defined. Students explore and describe methods for determining the area of a sector and the area of a segment of a circle. The formula for each is stated and students apply them to solve problem situations. The formulas for linear and angular velocity are presented and are used to solve problems.	<ul style="list-style-type: none"> <li>A sector of a circle is a region of the circle bounded by two radii and the included arc.</li> <li>The area for the sector of a circle can be determined by multiplying the area of the circle, <math>A = \pi r^2</math>, by the fraction <math>m/360^\circ</math>, where <math>m</math> represents the central angle measure of the sector.</li> <li>The area for a sector of a circle can be determined by the formula <math>A_{\text{sector}} = m/360^\circ (\pi r^2)</math>.</li> <li>The segment of a circle is a region of the circle bounded by a chord and the included arc. Each segment of a circle can be associated with a sector of the circle.</li> <li>The strategy for calculating the area of a segment of a circle is to calculate the area of the sector associated with the segment and from that, subtract the area of the triangle within the sector formed by the two radii and the chord or <math>A_{\text{segment}} = A_{\text{sector}} - A_{\text{triangle}}</math>.</li> <li>The area for a segment of a circle can be determined by the formula <math>A_{\text{segment}} = m/360^\circ (\pi r^2) - 1/2(b)(h)</math>.</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
3	<b>Do Me a Solid</b> Building Three-Dimensional Figures	G.GMD.1 G.GMD.4	2	Students first rotate and then stack congruent two-dimensional figures to create three-dimensional figures. Students analyze the three-dimensional solid figures associated with the set of all the translations along a line segment. Stacking congruent figures creates prisms and cylinders and develops an informal argument for the corresponding volume formulas. Finally, students use Cavalieri's Principles to show that areas and volumes remain the same when bases and heights remain the same.	<ul style="list-style-type: none"> <li>• Rigid motion is used in the process of redrawing two-dimensional plane figures as three-dimensional solids.</li> <li>• Models of three-dimensional solids are formed using translations of plane figures through space.</li> <li>• Models of two-dimensional plane figures are stacked to create models of three-dimensional solids.</li> <li>• The volume formula, <math>V = Bh</math>, where <math>B</math> is the area of the base and <math>h</math> is the height, applies to all right and oblique cylinders and right and oblique prisms.</li> <li>• Cavalieri's Principle for area states that if the lengths of one-dimensional slices—just line segments—of two figures are the same, then the figures have the same area.</li> <li>• Cavalieri's Principle for volume states that, given two solids included between parallel planes, if every plane cross section parallel to the given planes has the same area in both solids, then the volumes of the solids are equal.</li> </ul>
4	<b>Get to the Point</b> Building Volume and Surface Area Formulas for Pyramids, Cones, and Spheres	G.GMD.1 G.GMD.3 G.MG.1 G.MG.2 G.MG.3	3	Students consider what happens when you stack similar figures instead of congruent figures. In a hands-on activity, students relate the volume of a pyramid to the volume of a cube with the same base. They use a spreadsheet to explore the limit of the ratio as the prisms and pyramids grow and observe that the volume of a pyramid is one-third the volume of a cube with the same base. They then use Cavalieri's Principle to derive the formula for the volume of a cone. Students solve a problem about the volume of two cylinders with the same lateral area and then determine the lateral and total surface areas of various three-dimensional solid figures. Students are introduced to the formulas for the volume and surface area of a sphere. Finally, students solve a variety of real-world problems using reasoning and the volume and surface area formulas.	<ul style="list-style-type: none"> <li>• The volume formula for a pyramid is <math>V = (1/3)Bh</math>, where <math>V</math> is the volume, <math>B</math> is the area of the base, and <math>h</math> is the height of the pyramid.</li> <li>• The volume of a pyramid is one third the volume of a prism with the same base area and height</li> <li>• The volume formula for a cone is <math>V = (1/3)\pi r^2 h</math>, where <math>V</math> is the volume, <math>r</math> is the radius of the base, and <math>h</math> is the height of the cone.</li> <li>• The volume of a cone is one third the volume of a cylinder with the same base area and height.</li> <li>• The lateral surface area of a three-dimensional figure is the sum of the areas of its lateral faces. The total surface area of a three-dimensional figure is the sum of its bases and lateral faces.</li> <li>• The volume formula for a sphere is <math>V = (4/3)\pi r^3</math>, where <math>V</math> is the volume, and <math>r</math> is the radius of the sphere.</li> <li>• The total surface area of a sphere is <math>4\pi r^2</math>.</li> </ul>
	<b>Learning Individually with MATHia or Skills Practice</b>	G.C.2 G.C.5 G.GMD.3 G.GMD.4	4	Students explore the difference between the degree measure of an arc and the length of an arc. They then practice calculating the fraction of a circle's circumference that an arc occupies and writing an expression that can be used to calculate an arc length. Students then calculate the arc length given the radius or diameter of the circle. Next, they relate the arc length to the circle's radius and are introduced to radians and the theta symbol. Finally, students practice determining different measurements of a circle using the formula $\theta = s/r$ . Students calculate the measure of an arc or an angle using the Interior Angles of a Circle Theorem and Exterior Angles of a Circle Theorem. Students are given the definition of a sector of a circle and practice identifying sectors. They then work through an example that develops the formula for determining the area of a sector of a circle before using the formula to find areas of different sectors of circles. Students rotate two-dimensional figures about an axis to create three-dimensional shapes and relation the dimension of the plane figure to the solid. They differentiate between right and oblique solids and then create solids by stacking content of similar shapes. They then use mathematical and real-world objects to determine the volume of cylinders, pyramids, cones, and spheres.	

\*Pacing listed in 45-minute days

## Topic 2: Conic Sections

Students begin by recalling the Pythagorean Theorem and using it to write equations of circles on a coordinate plane. They complete the square to convert between general and standard forms of equations when necessary to determine the radius and center of a given circle. Students then use the Pythagorean Theorem, the Distance Formula, and symmetry to verify whether a point lies on a given circle. They then incorporate the trigonometric ratios—considering right triangles formed by the radius of the unit circle—to develop the Pythagorean identity  $\sin^2 \theta + \cos^2 \theta = 1$ . This is used to calculate the value of trigonometric ratios for angles in given quadrants of the coordinate plane. Finally, students use concentric circles around a focus and distance from a directrix to draw a parabola and develop the general and standard forms of a parabola.

**Standards:** G.GPE.1, G.GPE.2, F.TF.8, A.REI.7, A.CED.2 **Pacing:** 14 Days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
1	<b>Any Way You Slice It</b> Cross-Sections	G.GMD.4	1	Students reason about when the intersection of a plane and a geometric solid creates a cross-section that is a single point or a line segment. A variety of cross-sections are created as geometric solids are sliced with planes drawn parallel to the base, perpendicular to the base, and on an angle to the base. Students practice identifying solids given their cross-sections, and cross-sections given their solids.	<ul style="list-style-type: none"> <li>A cross-section of a three-dimensional solid can be a point, a line segment, or a two-dimensional figure that is formed by the intersection of the solid and a plane.</li> <li>The maximum number of sides of a cross-section equals the number of sides of faces of the solid, if it is a polyhedron.</li> <li>Circles, ellipses, hyperbolas, and parabolas are conic sections because each is the intersection of a plane and a double-napped cone.</li> </ul>
2	<b><math>X^2</math> Plus <math>Y^2</math> Equals Radius<sup>2</sup></b> Deriving the Equation for a Circle	G.GPE.1	2	Students derive the standard form for the equation of a circle, with the center point and the radius, using the Pythagorean Theorem. Next, the general form for the equation of a circle is introduced. Students are shown how to rewrite the equation of a circle from general form to standard form by completing the square. In standard form, the radius and center point can be readily identified.	<ul style="list-style-type: none"> <li>The standard form of the equation of a circle centered at <math>(h, k)</math> with radius <math>r</math> can be expressed as <math>(x - h)^2 + (y - k)^2 = r^2</math>.</li> <li>The equation for a circle in general form is <math>Ax^2 + Cy^2 + Dx + Ey + F = 0</math>, where <math>A, C, D, E,</math> and <math>F</math> are constants, <math>A = C</math>, and <math>x \neq y</math>.</li> <li>An equation written in general form can be written in standard form using the algebraic procedure called completing the square.</li> </ul>
3	<b>A Blip on the Radar</b> Determining Points On a Circle	A.REI.7 A.CED.2 G.GPE.1	1	Students use the Pythagorean Theorem and the Distance Formula to determine whether a point lies on a circle, given the location of the center point (which may or may not be at the origin) and either the coordinates of a point on the circle or the circle's radius or diameter.	<ul style="list-style-type: none"> <li>The Pythagorean Theorem can be used to determine whether a point lies on the circumference of a circle when the center point is located at the origin and the length of the radius is given.</li> <li>The Pythagorean Theorem, the Distance Formula, and symmetry can be used to determine whether a point lies on the circumference of a circle when the center point is not located at the origin and the coordinates of a point on the circumference of a circle are given.</li> <li>The coordinates of the points at which a circle and line intersect can be determined algebraically by writing equations for the line and for the circle, substituting the expression representing the <math>y</math>-value of the line into the equation of the circle, and then solving the quadratic equation.</li> <li>Segments drawn tangent to the same circle from the same exterior point are congruent.</li> <li>The equation of a line drawn tangent to a circle can be determined given the center point of the circle, a radius drawn to the point of tangency, and the coordinates of the point of tangency.</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
4	<b>Sin<sup>2</sup> θ Plus Cos<sup>2</sup> θ Equals 1<sup>2</sup></b> The Pythagorean Identity	F.TF.8	1	Students investigate the sine, cosine, and tangent of angle measures that form right triangles in Quadrants II, III, and IV. They use a unit circle to determine the signs of each trigonometric ratio in each quadrant of the coordinate plane. Students then prove the Pythagorean Identity, $\sin^2 \theta + \cos^2 \theta = 1^2$ , and write this identity in different forms. Students use the Pythagorean Identity to determine the sine, cosine, and tangent of angle measures in given quadrants. Students summarize by labeling different angle measures on the unit circle with degrees and radians, identifying the sign of the sine, cosine, and tangent of each angle measure, and then calculating each trigonometric ratio.	<ul style="list-style-type: none"> <li>The Pythagorean identity states that <math>\sin^2 \theta + \cos^2 \theta = 1</math>, where <math>\theta</math> represents an angle measure.</li> <li>The ratios sine, cosine, and tangent may be positive or negative, depending on which quadrant of the coordinate plane the reference angle and reference triangle are drawn.</li> <li>The sines and cosines of angle measures in different quadrants of the coordinate plane are related by symmetry.</li> </ul>
5	<b>Going the Equidistance</b> Equation of a Parabola	G.GPE.2	2	The focus and directrix of a parabola are introduced through an exploratory activity. Next, students use the focus and directrix to write the equation of a parabola in both general and standard form. They derive the standard form of a parabola algebraically to make sense of the constant $p$ in the equation and use this constant to graph parabolas. Students then use the Distance Formula to determine the equation for the set of points equidistant from the focus and the directrix when the vertex is not at the origin. Finally, students apply characteristics of parabolas—including the axis of symmetry, vertex, and concavity—to solve problems.	<ul style="list-style-type: none"> <li>A parabola is the locus of points in a plane that are equidistant from a fixed point (the focus) and a fixed line (the directrix).</li> <li>The focus and directrix of a parabola can be used to derive the equation of the parabola.</li> <li>Parabolas can be described by their concavity.</li> <li>The standard form for the equation of a parabola with vertex at the origin can be written in the form <math>x^2 = 4py</math> (symmetric with respect to the <math>y</math>-axis) or <math>y^2 = 4px</math> (symmetric with respect to the <math>x</math>-axis), where <math>p</math> is the distance from the vertex to the focus.</li> <li>The standard form for the equation of a parabola with vertex at the origin, <math>y^2 = 4px</math> or <math>x^2 = 4py</math> can be derived using the Distance Formula and the definitions of focus, directrix, and parabola.</li> <li>In the standard form for the equation of a parabola centered at the origin, <math>y^2 = 4px</math> or <math>x^2 = 4py</math> the value of <math>p</math> is positive when the parabola is concave up or concave right and the value of <math>p</math> is negative when the parabola is concave down or concave left.</li> <li>The standard forms of parabolas with vertex <math>(h, k)</math> are <math>(x - h)^2 = 4p(y - k)</math> and <math>(y - k)^2 = 4p(x - h)</math>.</li> <li>The characteristics of parabolas can be used to solve real-world problems.</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
6	<b>It's a Stretch</b> Ellipses	G.GPE.3(+)	3	Students use patty paper to explore the structure of an ellipse, which is defined as the locus of all points in a plane for which the sum of the distances from two given points is constant. Students discover that the sum of the distances from any point on an ellipse to the foci equals the distance from one vertex to the other. They then graph an ellipse on a coordinate plane and investigate the relationship between the $a$ -, $b$ -, and $c$ -values. Students solve a real-world problem to make sense of the foci points. Next, they graph ellipses whose centers are not at the origin, making connections among the various representations and the various forms of the equations. Finally, Kepler's Laws of Orbits are introduced, and students explore the equations of space travel.	<ul style="list-style-type: none"> <li>An ellipse is the locus of points in a plane for which the sum of the distances from two given points is a constant.</li> <li>The equation of an ellipse with the center at the origin is <math>(x^2/a^2) + (y^2/b^2) = 1</math> when the longer or major axis is on the <math>x</math>-axis, and <math>(x^2/b^2) + (y^2/a^2) = 1</math> when the major axis is on the <math>y</math>-axis.</li> <li>The endpoints of the major axis are called the vertices of the ellipse and are <math>a</math> units from the center.</li> <li>The endpoints of the minor axis of the ellipse are <math>b</math> units from the center. The given points are called the foci and are on the major axis <math>c</math> units from the center.</li> <li>The relationship between <math>a</math>, <math>b</math>, and <math>c</math> is <math>a^2 - b^2 = c^2</math>.</li> <li>The semi-major axis has length <math>a</math>, and the semi-minor axis has length <math>b</math>.</li> <li>The eccentricity, <math>e</math>, is a number that describes the shape of a conic section. It is defined as the ratio <math>c/a</math> where <math>c</math> is the distance from the center of the conic to a focus and <math>a</math> is the length of the semi-major axis. For an ellipse, <math>0 &lt; e &lt; 1</math>. If <math>e</math> is close to 0 then the ellipse will appear more circular. If <math>e</math> is close to 1 then the ellipse will appear more elongated.</li> <li>The equation of an ellipse with the center at the point <math>(h, k)</math> is <math>((x - h)^2/a^2) + ((y - k)^2/b^2) = 1</math> when the longer or major axis is on the <math>x</math>-axis and is <math>((x - h)^2/b^2) + ((y - k)^2/a^2) = 1</math> when the major axis is on the <math>y</math>-axis.</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
7	<p><b>More Asymptotes</b> Hyperbolas</p>	G.GPE.3(+)	2	<p>Students use patty paper to explore the structure of a hyperbola. They write an equation for a hyperbola and then use a table to discover its end behavior. Students are provided the equations of a hyperbola with respect to the position of the transverse axis and the relationship between <math>a</math>, <math>b</math>, and <math>c</math> and the orientation of the hyperbola. They then explore a hyperbola not centered at the origin; they derive the equation, graph it on the coordinate plane, and label key characteristics. Students rewrite equations of hyperbolas written in general form to reveal the center, the semi-major and semi-minor axes, and the asymptotes. Both hyperbolic reflectors and parabolic reflectors are given as contexts for applying equations for conic sections.</p>	<ul style="list-style-type: none"> <li>• A hyperbola is the locus of all points in a plane, the difference of whose distances from two given points is a constant.</li> <li>• The equation of a hyperbola with the center at the origin is either <math>(x^2/a^2) - (y^2/b^2) = 1</math> when the transverse axis is on the <math>x</math>-axis or <math>(y^2/a^2) - (x^2/b^2) = 1</math> when the transverse axis is on the <math>y</math>-axis.</li> <li>• The vertices, or endpoints, of the major axis of the hyperbola are <math>a</math> units from the center and on the transverse axis.</li> <li>• The endpoints of the minor axis of the hyperbola are <math>b</math> units from the center and on the conjugate axis.</li> <li>• The given points are called the foci and are on the major axis <math>c</math> units from the center.</li> <li>• The relationship between <math>a</math>, <math>b</math>, and <math>c</math> is <math>a^2 + b^2 = c^2</math>.</li> <li>• The semi-major axis has length <math>a</math>, and the semi-minor axis has length <math>b</math>.</li> <li>• The eccentricity, <math>e</math>, is a number that describes the shape of a conic section. It is defined as the ratio <math>c/a</math>, where <math>c</math> is the distance from the center of the conic to a focus and <math>a</math> is the length of the semimajor axis. For a hyperbola, <math>e &gt; 1</math>. A hyperbola with a larger eccentricity will appear wider than a hyperbola with a smaller eccentricity.</li> <li>• The equations of the asymptotes are <math>y = \pm(a/b)x</math> for a horizontal hyperbola and <math>y = \pm(b/a)x</math> for a vertical hyperbola.</li> <li>• To sketch the graph of a hyperbola, determine and plot the vertices and the endpoints of the minor axes, draw a rectangle with these points as midpoints of its sides. Draw lines through the opposite vertices of this rectangle to form your asymptotes, and sketch the graph.</li> </ul>
	<p><b>Learning Individually with MATHia or Skills Practice</b></p>	G.GPE.1 G.GMD.4	2	<p>Students watch an animation showing two different intersections of a plane and a solid. They then describe cross-sections of different solids given the intersection of a plane and identify the solid from a given cross-section. Students are given a circle on the coordinate plane with a defined center. They use the Pythagorean Theorem to derive the standard form for the equation of a circle. Students are given an equation for a circle. They rewrite the equation in standard form if necessary to identify the radius and center of the circle.</p>	

\*Pacing listed in 45-minute days

# 5

## Making Informed Decisions

Pacing: 21 Days

### Topic 1: Independence and Conditional Probability

Students are introduced to compound probability. They explore various probability models and calculate compound probabilities with independent and dependent events in a variety of problem situations. The emphasis is on modeling and analyzing sample spaces to determine rules for calculating probabilities in different situations.

**Standards:** S.CP.1, S.CP.2, S.CP.7, S.CP.8(+) **Pacing:** 10 Days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
1	<p><b>What Are the Chances?</b></p> <p>Compound Sample Spaces</p>	S.CP.1	2	<p>Students learn strategies for determining the sample space of compound events. They begin with a review of terms associated with probability such as sample space, event, and probability model. For each given situation, students list the sample space, construct a probability model, and differentiate between uniform and nonuniform probability models. They examine examples of tree diagrams and then, for different situations, create their own tree diagrams and organized lists of the corresponding sample spaces. They analyze the sample space in each situation, distinguishing between situations that involve independent events from disjoint sets and those that involve dependent events from intersecting sets. Students identify a situation comprising dependent events that does not allow repetitions. The Counting Principle is stated and discussed as a shortcut for determining the size of a sample space.</p>	<ul style="list-style-type: none"> <li>The probability of an event is the ratio of the number of desired outcomes to the total number of possible outcomes.</li> <li>An outcome is a result of an experiment. A sample space is all of the possible outcomes in a probability situation. An event is an outcome or set of outcomes in a sample space.</li> <li>A probability model lists the possible outcomes and the probability of each outcome. The sum of the probabilities in the model equals one.</li> <li>The complement of an event is an event that contains all the outcomes in the sample space that are not outcomes in the event.</li> <li>A non-uniform probability model is a model in which all of the outcomes are not equal.</li> <li>Disjoint sets do not have common elements. Intersecting sets have at least one common element.</li> <li>Independent events are events for which the occurrence of one event has no impact on the occurrence of the other event. Dependent events are events for which the occurrence of one event has an impact on the occurrence of the following events.</li> <li>The Counting Principle states: "If an action <math>A</math> can occur in <math>m</math> ways and for each of these <math>m</math> ways, an action <math>B</math> can occur in <math>n</math> ways, then actions <math>A</math> and <math>B</math> can occur in <math>m \cdot n</math> ways."</li> </ul>
2	<p><b>And?</b></p> <p>Compound Probability with "And"</p>	S.CP.2 S.CP.8(+)	2	<p>Students determine the probability of two or more independent events and two or more dependent events. The Rule of Compound Probability involving and is stated and is used to compute compound probabilities. Various situations present students with opportunities to construct tree diagrams, create organized lists, and compute the probabilities of compound events.</p>	<ul style="list-style-type: none"> <li>A compound event is an event that consists of two or more events.</li> <li>The Rule of Compound Probability Involving and states: "If Event <math>A</math> and Event <math>B</math> are independent, then the probability that Event <math>A</math> happens and Event <math>B</math> happens is the product of the probability that Event <math>A</math> happens and the probability that Event <math>B</math> happens, given that Event <math>A</math> has happened." However <math>A</math> happening has no influence on <math>B</math> happening. Using probability notation, the Rule of Compound Probability involving and is <math>P(A \text{ and } B) = P(A) \cdot P(B)</math>.</li> <li>The Rule of Compound Probability Involving and states: "If Event <math>A</math> and Event <math>B</math> are dependent, then the probability that Event <math>A</math> happens and Event <math>B</math> happens is the product of the probability that Event <math>A</math> happens and the probability that Event <math>B</math> happens, given that Event <math>A</math> has happened." Using probability notation, the Rule of Compound Probability involving and is <math>P(A \text{ and } B) = P(A) \cdot P(B A)</math>.</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
3	<b>Or?</b> Compound Probability with "Or"	S.CP.7	2	Students determine the probability of one or another independent events and the probability of one or another dependent events. The Addition Rule for Probability is stated and used to compute probabilities. Several situations present students with the opportunity to construct tree diagrams, create organized lists, complete tables, and compute $P(A)$ , $P(B)$ , $P(A \text{ and } B)$ , and $P(A \text{ or } B)$ with respect to the problem situation. Students create a graphic organizer to record the different types of compound events they have studied; independent events $P(A \text{ and } B)$ , independent events $P(A \text{ or } B)$ , dependent events $P(A \text{ and } B)$ , dependent events $P(A \text{ or } B)$ .	<ul style="list-style-type: none"> <li>A compound event is an event that consists of two or more events.</li> <li>The Addition Rule for Probability states: "The probability that Event <math>A</math> occurs or Event <math>B</math> occurs is the probability that Event <math>A</math> occurs plus the probability that Event <math>B</math> minus the probability that both <math>A</math> and <math>B</math> occur." Using probability notation, the Addition Rule for Probability is <math>P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)</math>.</li> </ul>
4	<b>And, Or, and More!</b> Calculating Compound Probability	S.CP.2 S.CP.7 S.CP.8(+)	2	Students analyze scenarios involving a standard deck of playing cards, choosing committee members, and selecting items from a menu and determine compound probabilities. Students determine the probability of independent events $P(A \text{ and } B)$ with replacement, independent events $P(A \text{ or } B)$ with replacement, dependent events $P(A \text{ and } B)$ without replacement, and dependent events $P(A \text{ or } B)$ without replacement.	<ul style="list-style-type: none"> <li>Situations "with replacement" generally involve independent events. Whether or not the first event happens has no effect on the second event.</li> <li>Situations "without replacement" generally involve dependent events. If the first event occurs, it has an impact on the probability of subsequent events.</li> </ul>
<b>Learning Individually with MATHia or Skills Practice</b>		S.CP.2 S.CP.3 S.CP.6	2	Students define independent events. They investigate different scenarios to determine whether the events given are independent or not independent. Students then investigate compound probability with "and" and use the equation $P(A \text{ and } B) = P(A)P(B)$ to verify whether two events are independent or not. They use an interactive Explore Tool to explore probability using area and random points. Students then explore the idea of conditional probability, using the interactive tool to visualize the conditional probability formula $P(A   B) = \frac{P(A \text{ and } B)}{P(B)}$ . Finally, they apply what they know about conditional probability to make predictions and check for the independence of events using the Explore Tool.	

\*Pacing listed in 45-minute days

## Topic 2: Computing Probabilities

Students encounter more compound probability concepts and counting strategies. Compound probability concepts are presented using two-way frequency tables, conditional probability, and independent trials. The counting strategies include permutations, permutations with repetition, circular permutations, and combinations. The last lesson focuses on geometric probability and expected value.

**Standards:** S.CP.3, S.CP.4, S.CP.5, S.CP.6, S.CP.9(+), S.MD.6, S.MD.7 **Pacing:** 11 Days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
1	<p><b>Table Talk</b></p> <p>Compound Probability for Data Displayed in Two-Way Tables</p>	S.CP.4	1	Two number cubes and the results of surveys are the contexts for creating sample spaces, organized lists, and tables. The converse of the multiplication rule is stated and used to determine when events are independent. The terms <i>frequency</i> , <i>frequency table</i> , <i>two-way frequency table</i> , <i>relative frequency</i> , and <i>two-way relative frequency table</i> are introduced. Students complete these types of tables and use the tables to answer questions related to the situations. Students convert ratios to percents.	<ul style="list-style-type: none"> <li>A two-way table is a table that shows the relationship between two data sets, one organized in rows and one organized in columns.</li> <li>A frequency table is a table that shows the frequency of an item, number, or event appearing in a sample space.</li> <li>A two-way frequency table or contingency table shows the number of data points and their frequencies for two variables.</li> <li>A relative frequency is the ratio of occurrences within a category to the total number of occurrences.</li> <li>A two-way relative frequency table displays the relative frequencies for two categories of data.</li> <li>Two-way tables can be used to determine the probabilities of compound events.</li> <li>The converse of the multiplication rule for probability states: "If the probability of two events <math>A</math> and <math>B</math> occurring together is <math>P(A) \cdot P(B)</math>, then the two events are independent."</li> </ul>
2	<p><b>It All Depends</b></p> <p>Conditional Probability</p>	S.CP.3 S.CP.5 S.CP.6	2	Rolling two number cubes and calculating the sum is once again used to generate a two-way data table listing the possible outcomes. Different events are described and students calculate $P(A)$ , $P(B)$ , and $P(A \text{ and } B)$ . The term <i>conditional probability</i> , $P(B A)$ is defined. Students derive a formula for computing conditional probability, $P(B A) = (P(A \text{ and } B))/P(A)$ . The conditional probability formula is applied to several different situations.	<ul style="list-style-type: none"> <li>Conditional probability is the probability of Event <math>B</math>, given that Event <math>A</math> has already occurred. The notation for conditional probability is <math>P(B A)</math>, which reads, "the probability of Event <math>B</math>, given Event <math>A</math>."</li> <li>When <math>P(B A) = P(B)</math>, the two events, <math>A</math> and <math>B</math>, are independent.</li> <li>When <math>P(B A) \neq P(B)</math>, the two events, <math>A</math> and <math>B</math>, are dependent.</li> <li>The conditional probability formula is stated as <math>P(B A) = P(A \text{ and } B)/P(A)</math>.</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
3	<b>Give Me 5!</b> Permutations and Combinations	S.CP.9(+)	2	The terms <i>factorial</i> , <i>permutation</i> , and <i>combination</i> are defined. Students derive the formulas to calculate permutations and combinations, then apply them in different situations. Situations involve permutations with and without repeated elements. Students answer questions, complete tables, and make the connections necessary to develop additional formulas related to combinations and permutations. Circular permutations are introduced. Students conclude that the formula for the circular permutation of $n$ objects is $(n - 1)!$ .	<ul style="list-style-type: none"> <li>The factorial of <math>n</math>, which is written with an exclamation mark as <math>n!</math>, is the product of all non-negative integers less than or equal to <math>n</math>: <math>n(n - 1)(n - 2) \dots</math></li> <li>A permutation is an ordered arrangement of items without repetition.</li> <li>The notation denoting a permutation of <math>r</math> elements taken from a collection of <math>n</math> items is: <math>{}_n P_r = P(n, r) = P_r^n</math>.</li> <li>The formula used to compute the number of permutations, <math>P</math>, of <math>r</math> elements chosen from <math>n</math> elements is: <math>{}_n P_r = n!/(n - r)!</math>.</li> <li>A combination is an unordered collection of items.</li> <li>The notation denoting a combination of <math>r</math> elements taken from a collection of <math>n</math> elements is: <math>{}_n C_r = C(n, r) = C_r^n</math>.</li> <li>The formula used to compute the number of combinations, <math>C</math>, of <math>r</math> elements chosen from <math>n</math> elements is: <math>{}_n C_r = n!/(n - r)!</math>.</li> <li>The formula for the number of permutations of <math>n</math> elements with <math>k</math> copies of an element is <math>n!/k!</math>.</li> <li>The formula for the number of permutations of <math>n</math> elements with <math>k</math> copies of one element and <math>h</math> copies of another element is <math>n!/(k!h!)</math>.</li> <li>The circular permutation of <math>n</math> objects is <math>(n - 1)!</math>.</li> </ul>
4	<b>A Different Kind of Court Trial</b> Independent Trials	S.CP.9(+)	2	Situations in this lesson focus on multiple trials for two independent events. Making free throw shots, rolling number cubes, and rolling a tetrahedron are used to generate the probabilities of two independent events. Outcomes are organized in a table and the table is connected to Pascal's Triangle. Students use Pascal's Triangle to compute the probability of an occurrence. A formula using combinations is applied to different situations to calculate probabilities for two independent events over multiple trials.	<ul style="list-style-type: none"> <li>If the probability of Event <math>A</math> is <math>p</math> and the probability of Event <math>B</math> is <math>1 - p</math>, then the probability of Event <math>A</math> occurring <math>r</math> times and Event <math>B</math> occurring <math>n - r</math> times in <math>n</math> trials is: <math>P(A \text{ occurring } r \text{ times and } B \text{ occurring } n - r \text{ times})</math> or <math>{}_n C_r (p)^r (1 - p)^{n - r}</math>.</li> </ul>

\*Pacing listed in 45-minute days

Lesson	Title / Subtitle	Standards	Pacing*	Lesson Summary	Essential Ideas
5	<b>What Do You Expect?</b> Expected Value	S.MD.6 S.MD.7	2	The terms <i>geometric probability</i> and <i>expected value</i> are introduced in this lesson. Dartboards containing geometric shapes are used to determine geometric probabilities. Money wheels divided into eight equal regions are used to determine expected values.	<ul style="list-style-type: none"> <li>Geometric probability is the likelihood of an event occurring based on geometric relationships such as area, surface area, volume, and so on.</li> <li>Expected value is the sum of the values of a random variable with each value multiplied by its probability of occurrence.</li> </ul>
<b>Learning Individually with MATHia or Skills Practice</b>		S.CP.4 S.CP.5 S.CP.7	2	Students review how to read a two-way frequency table and construct a relative frequency table. They then use two-way frequency tables to determine probabilities, including conditional and other compound probabilities; they use information from frequency tables to check for the independence of events. Students apply the concept of conditional probability in a variety of different situations involving a change in the sample space as a result of an event. Students determine probabilities of compound events from two-way frequency tables via the Addition Rule.	

**Total Days: 153**

Learning Together: 114  
Learning Individually: 39

\*Pacing listed in 45-minute days