

AN104 Smith Chart 101

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Abstract:

This paper gives a simplified explanation of the Smith chart. Prior exposure, at an introductory level, of complex numbers and vectors is assumed by this explanation.

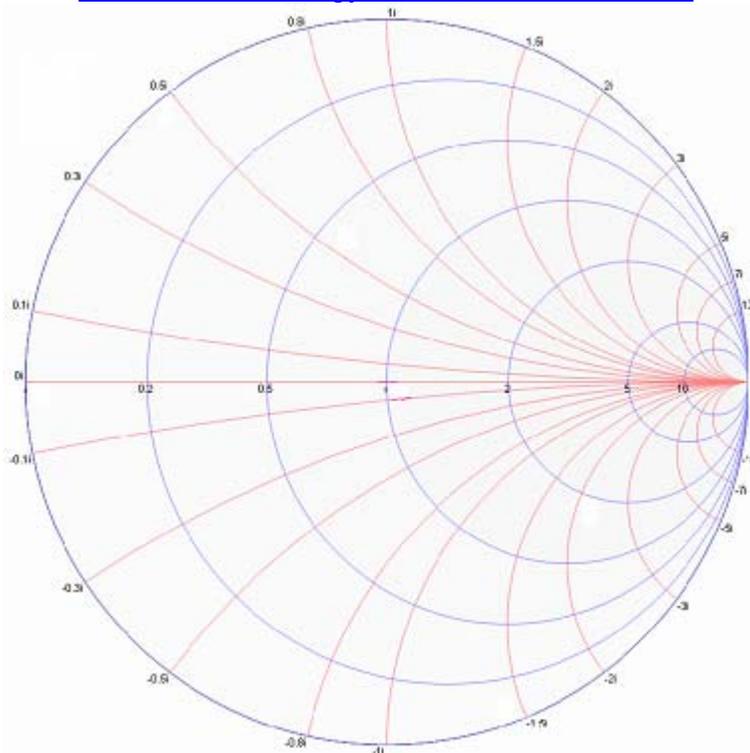
Introduction:

In the 1930's, Philip H. Smith developed a simple, yet truly *amazing* mathematical transformation that results in a graph known as the "Smith Chart". This chart maps all the possible impedances of any passive network onto a unit circle. As a tool, the Smith chart reduces complex algebraic calculations to simple graphical operations. A Smith chart tool set consists of a drafting compass, a fine pitched ruler, and a sharp pencil.

The Smith chart is available on the internet from numerous sources, Enter "Smith chart" and download' in any search engine to get results. Many of these web sites also present a deeper theoretical and less abbreviated historical explanation of the Smith chart.

The Smith charts in this paper were screen copied from the AEA Bravo PC Vision software downloaded from our web site. To use or see this chart, see Appendix A.

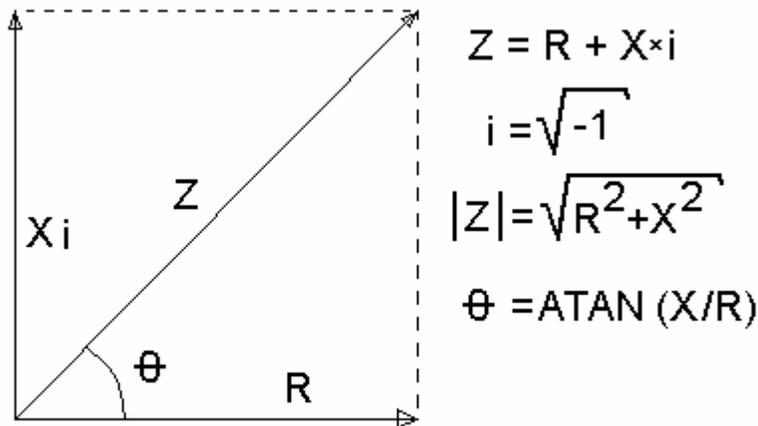
www.aeatechnology.com/html/software.htm.



The Smith Chart

Discussion:

Let's first take a look at impedance. Impedance (Z) has two components, the resistive (R) and reactive (X). Capacitance (C) or Inductance (L) create the reactive portion of Z , resistance creates the resistive portion. All passive networks, and many active ones as well, have non-negative resistances ($R \geq 0$). We will only consider these cases for the remainder of this paper. To illustrate a complex impedance we draw the R as a horizontal vector, pointing to the right (plotted on x axis), and draw the X as a vertical vector, up for inductive or down for capacitive (R always has an angle of 0 , X always has an angle of ± 90 degrees). The total impedance (Z) equals the vector sum of the R and X . The Z vector points in the direction described by the impedance angle (θ). Note that the diagram below depicts two components in series. Also note that in this paper, X = reactance, while x = RHP x axis.



R=Resistance

X=Reactance

Z=Total Impedance

θ =Z angle

Since R ranges from 0 to $+\infty$, and X ranges from $-\infty$ to $+\infty$, all possible impedances will plot somewhere on the Right Hand Plane (RHP) of a standard Cartesian coordinate system. The Smith chart will transform the RHP into a unit circle.

The Smith chart transformation uses the formula:

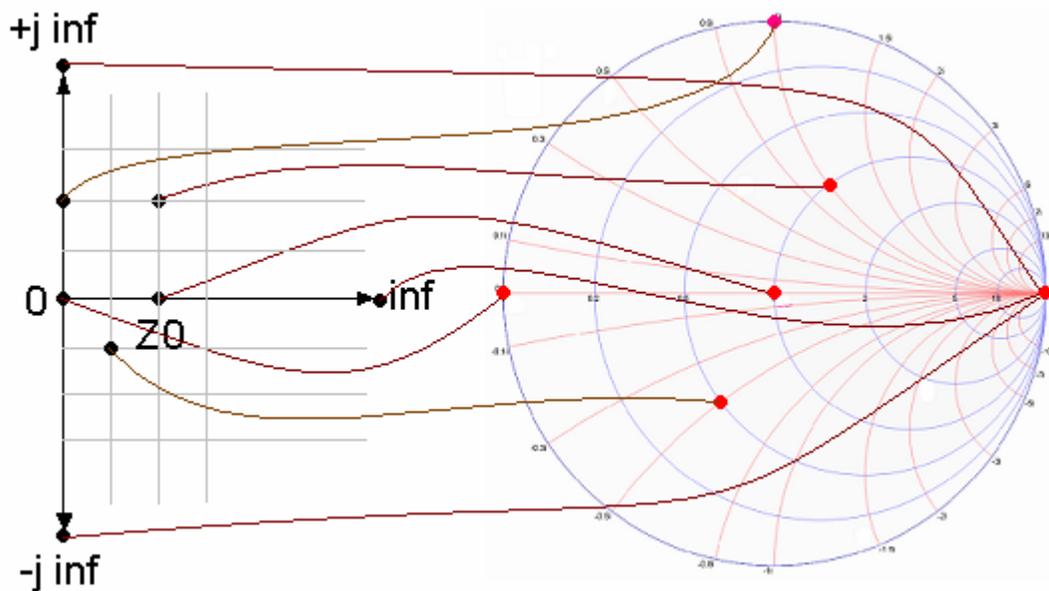
$$\Gamma = \frac{1 - \frac{Z_0}{Z}}{1 + \frac{Z_0}{Z}} \quad \text{or} \quad \Gamma = \frac{Z - Z_0}{Z + Z_0}$$

This transformation maps all the possible values on the RHP onto a unit circle or its interior (the unit circle has a radius of 1) by warping the entire RHP into a circular shape. The Z vector transforms into the gamma vector with a length ≤ 1 and an angle of 0 to 360 degrees (you may also use ± 180 degrees). Gamma may also be expressed in Cartesian format by separating it into its real and imaginary components:

$$\text{gamma} = \text{gamma}_{\text{re}} + j * \text{gamma}_{\text{im}} = |\text{gamma}| * (\cos(\lambda) + j * \sin(\lambda)).$$

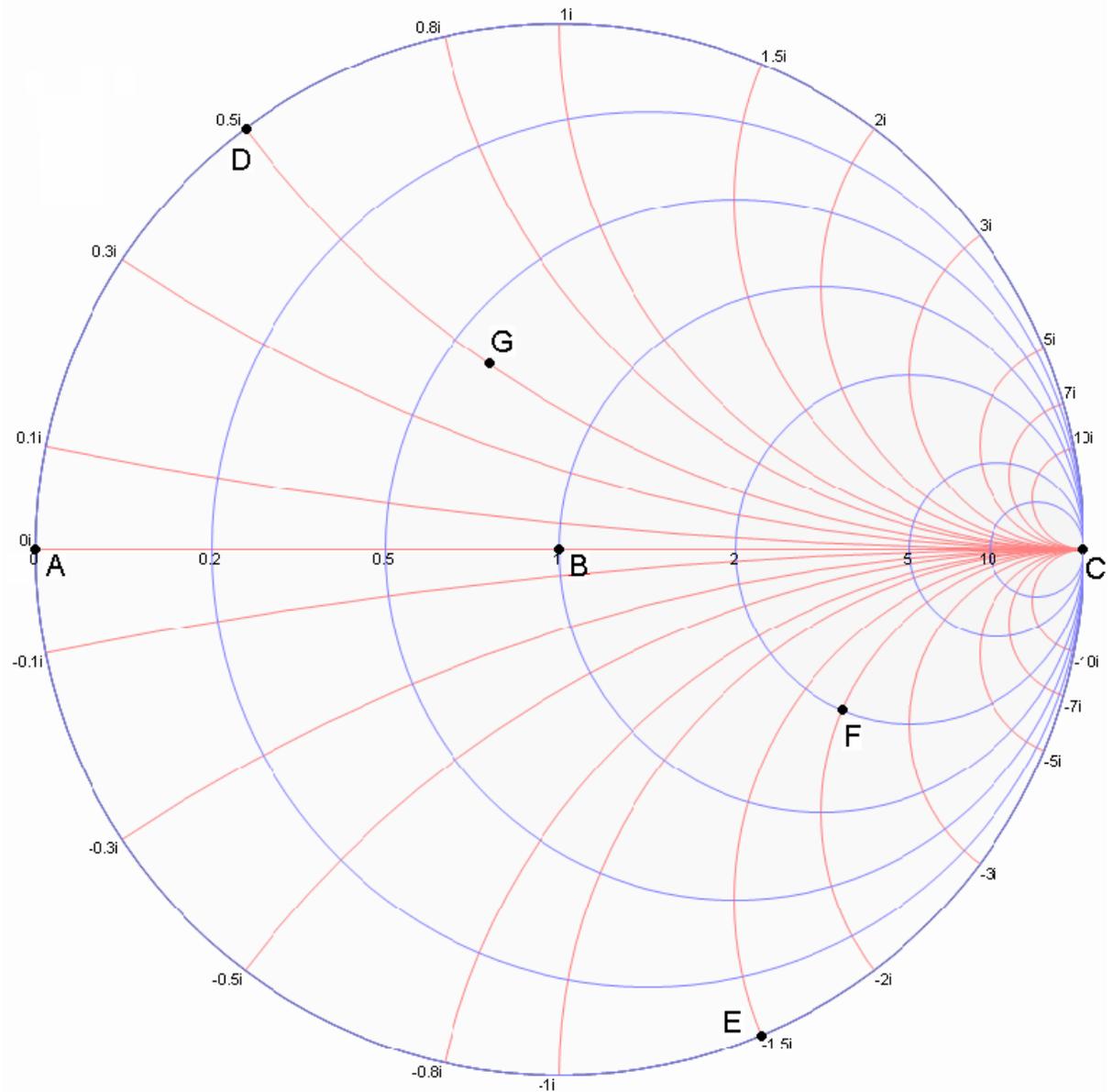
Some papers use rho, however, gamma and rho are one and the same, which term gets used depends solely on the author. Also, j and i are both terms that represent the square root of -1. There is only one point of discontinuity on the Smith chart, the points $Z = 0 \pm j * \text{inf}$ and $Z = \text{inf} \pm j * \text{anything}$ all map to the same point. Otherwise every point in the RHP maps to a unique point in the unit circle.

The Z_0 term in the formula normalizes the Z values by Z_0 . If you have a Z of $30 + j25$ ohms, this would normalize to $0.6 + j0.5$ ohms in a 50 ohm system ($Z_0 = 50$ ohms). Thus $50 + j0$ (a perfect 50 ohm load) normalizes to $1 + j0$.



Mapping of RHP to Unit Circle

Let's look at the Smith chart and point out a few of its features.



Example of several Z's Plotted on Chart

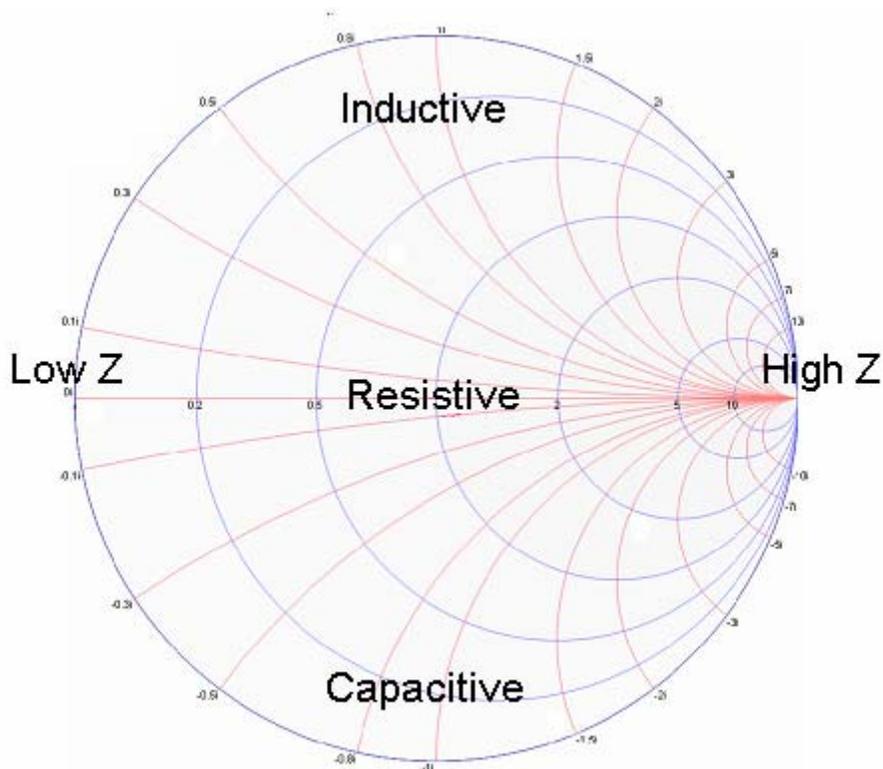
Point	Z	Z / Z0	Γ rect	Γ polar	Notes
A	0+0i	0+0j	-1+0i	1@180	Perfect short
B	50+0i	1+0i	0+0i	0@ undef	Perfect Z0 (50 ohms in this example)
C	Inf+0i	Inf+0i	1+0i	1@0	Perfect open (Z also = Inf +/- any i)
D	0+25i	0+0.5i	-.6+.8i	1@127	25i ohm inductor
E	0-75i	0-1.5i	.38-.92i	1@-67	-75i ohm capacitor
F	100-100i	2-2i	.54-.31i	.62@-30	100 ohm R - 100i ohm C (series)
G	30+25i	.6+.5i	-.14+.36i	.38@111	30 ohm R + 25i ohm L (series)

Note how the grid lines are all “warped”. These grid lines actually are circles (for a constant R) or circular arcs (for constant X). Note that the labels on the grid lines are normalized Z (third column in table). When you plot the Z, you locate the point by plotting the normalized Z using the circular grid lines. A step by step example of this is shown later.

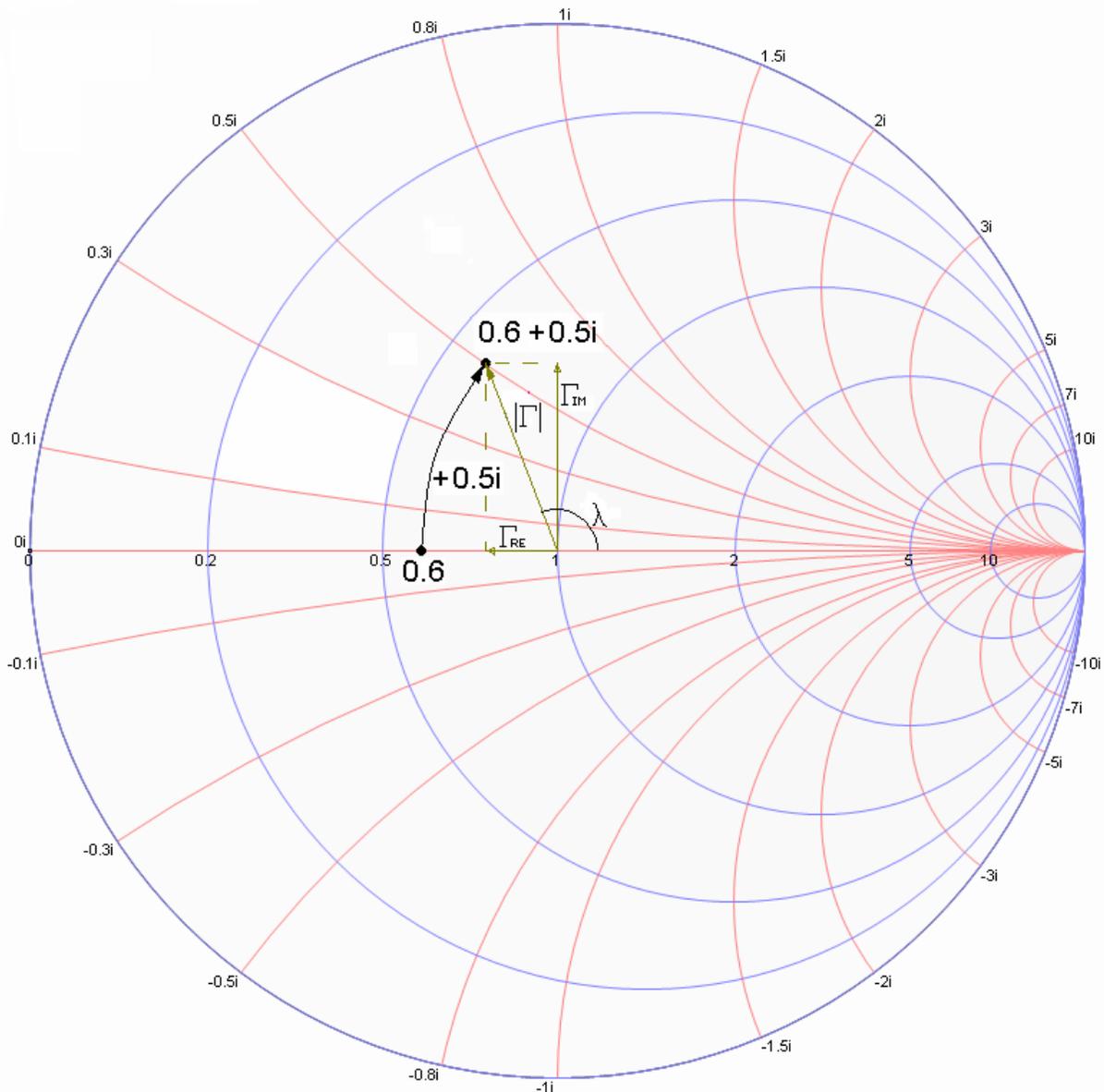
If the circular grids were replaced with a set of orthogonal grid lines (0, 0 in center), the gamma values listed in the fourth column could be plotted. Start in the center of the chart, where gamma is $0 + j0$. Plot the real portion of gamma by moving right or left by the real amount (right for positive, left for negative). From that point, move straight up or straight down by the imaginary portion of gamma. This will locate Z using the rectangular gamma format.

The polar representation of gamma, in the fifth column, uses point C as the zero angle reference, rotate gamma CCW for positive angles, CW for negative angles. Since this is a unit circle (radius = 1) a gamma magnitude of 1 will lie somewhere on the circle (proper) and gamma magnitudes <1 lie inside the circle. Start at the center again, and draw a line straight to the right with a length equal to the gamma magnitude. From this point, rotate the vector by the angle’s value (λ). Using the drafting compass makes this step easy.

Looking at the previous chart, the line defined by points A-B-C is the RHP x axis, where the R values are located. This line happens to be the only straight grid line in the Smith chart. The RHP y axis, where we find the jX values, wraps about the circumference of the unit circle.



Let us try to plot point G, which has a Z of $30 + 25i$ ohms, on the Smith chart. This normalizes to the value in the second column, $0.6 + 0.5i$ ($Z_0 = 50$ ohms). This normalized value is what gets plotted. First find the resistance, 0.6 , on the central horizontal line. Note the non linearity of the scales; you must estimate your interpolations with this in mind. From the 0.6 point, we need to move $0.5i$ in the positive direction. Try to imagine a circle that is drawn through 0.6 and stays between the two circles that go through 0.5 and 1.0 . (you may use your compass and ruler for this, the circle's center point is halfway between 0.6 and infinity). Since the X we are plotting is positive we move upward on this imaginary circle until we reach the $0.5i$ gridline. That is the point which represents $0.6 + 0.5i$. For negative X values, we would move downward instead of up. If the value of X does not correspond to a grid line, we would interpolate the X position also. Note how the rectangular and polar representations of gamma correspond to point G.



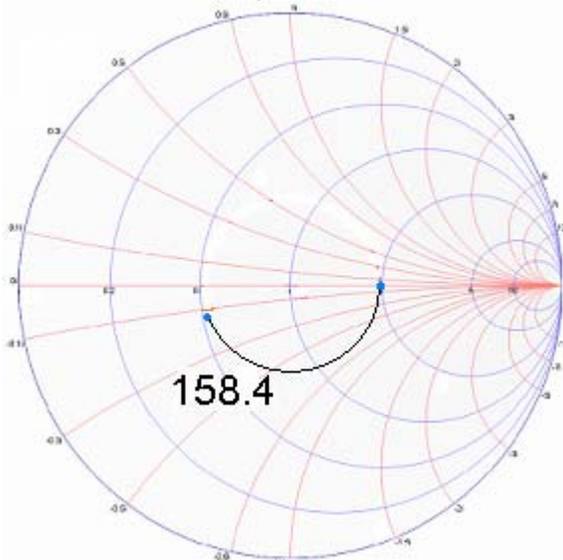
Plot a Point and Compare Normalized Z with Gamma Formats

Wow! Why would I want to use such a complicated chart? Actually, the Smith chart simplifies transmission line and complex math operations, so if you work with reactive loads, RF circuits, or transmission lines, the Smith chart becomes an invaluable tool.

How can the Smith chart simplify anything?

1. Rotating around the Smith chart (use your compass), represents moving some distance through a transmission line. Try to solve this using algebraic techniques... I'd rather eat a bug!
2. Adding a series component to a load is as simple as moving a normalized distance along a grid line. If the added component is complex, it takes just 2 moves along the grid lines.
3. Adding a parallel component is as easy as adding the susceptance and conductance. Finding the susceptance and conductance only requires a ruler and pencil.
4. An SWR value maps to some gamma magnitude, so a circle drawn around the center of the Smith chart represents all the values of Z that produce a specific SWR.
5. Easily design minimum loss matching networks between arbitrary source and load impedances.
6. Not covered in this paper: Represent amplifier stability regions, and Noise Figure plots

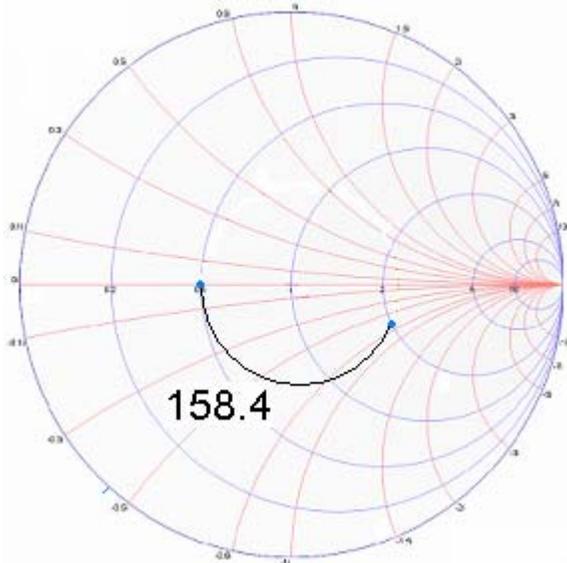
Let's examine the rotation. One lap around the Smith chart represents $\frac{1}{2}$ wavelength of transmission line. Rotating by a smaller angle represents a shorter length of transmission line. What happens if my transmission line is longer than $\frac{1}{2}$ wavelength? You take more than one lap around the chart. This implies that the transmission line effects repeat every $\frac{1}{2}$ wavelength. So all you have to do is rotate by the remainder once you divide the total length by $\frac{1}{2}$ wavelength. For example, a piece of coax is 1.72 wavelengths long; the remainder portion would be .22 wavelengths. $(.22/.5)*360 = 158.4$ degrees. If this coax is 50 ohms ($Z_0 = 50$) and terminated with a 100 ohm load, the other end looks like $0.52 - j0.13$ (or $26 - j6.5$ ohms denormalized).



Effect of Transmission Line

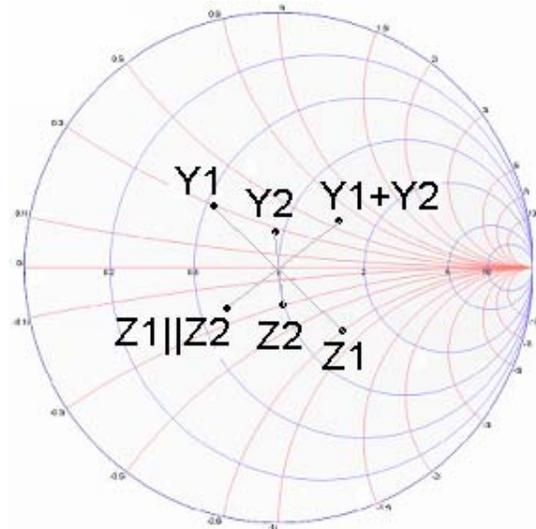
To solve this algebraically would take a lot of time and paper. Note that we moved 158.4 degrees CCW. This is the direction to rotate when you move down the coax away from the load (called "towards generator" on most Smith charts). If this coax were connected to a network analyzer, the reading would be $26 - j6.5$ ohms. If we start there ($.52 - j.13$) on the Smith chart move along the coax towards the load, we now rotate CW for 158.4 degrees, and we end up at $2 - j0$ ($100 - j0$ denormalized) which equals the load attached to the end of the 1.72 wavelengths of coax.

Let's add a series component to the 100 ohm load, say a capacitor with reactance of $-j35$ ohms. The result is found by moving along the 2.0 resistance grid circle by $-j.7$ ohms. Connect this to a network analyzer and what should it read? Start at 2.0, shift down the circle by $-j.7$, now rotate CCW by 158.4 degrees, we end up at $0.42-j0$. The network analyzer would read $21+j0$ ohms.



Series Impedance Plus Coax Line

To add a complex load, you shift along the red grid lines by the resistance that you've added, and then shift along the blue lines by the added reactance.



Parallel Impedances

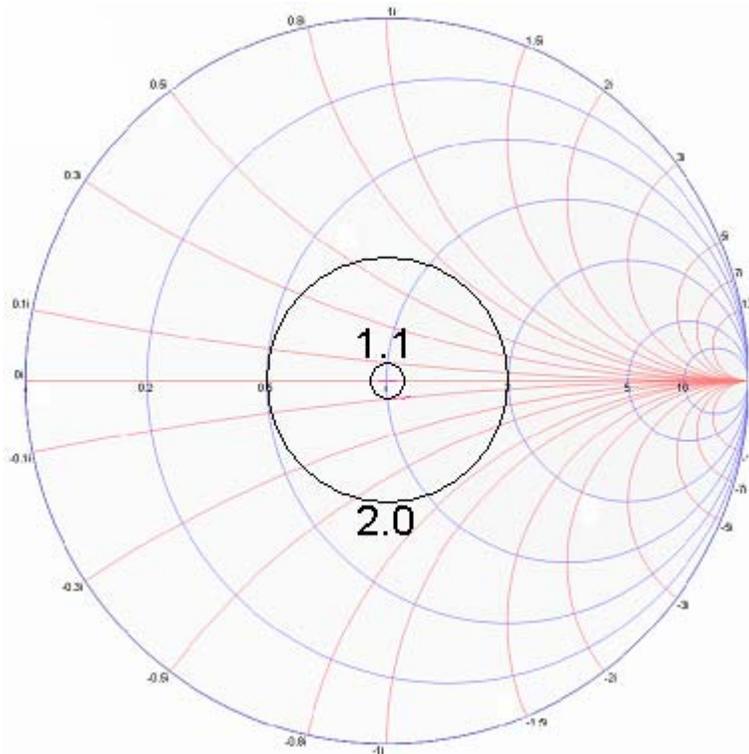
To add a parallel component, we first must learn about conductance and susceptance. Conductance (G) is the inverse of resistance; i.e. $1/R$. Susceptance (B) is the inverse of the reactance; $1/X$. The inverse of Z is the admittance (Y). The Smith chart lets us convert Z to Y in one easy step. All you have to do is to rotate Z by 180 degrees. An alternate method is to draw a line from Z through the center and beyond. Find the point, Y, on the line that places the chart center exactly in the middle of Z and Y. Lets find the admittance of $Z=1.4-j0.8$. This ends up on $0.55 + j0.26$. what happens if we add a Z of $1-j0.3$ in parallel? First we find that its Y is $0.83+j0.38$. Now add the two admittances together, the total Y is $1.38+j0.64$. To convert back to Z we just rotate by 180 again and we get $0.59 -j.27$ (denorm: $29.5 - j13.5$ ohms). Since 180 is exactly $\frac{1}{2}$ a lap, it doesn't matter if you go CW or CCW; you end up at the same spot. That was easy!

Next we look at SWR. The gamma magnitude can be calculated into an SWR value:

$$\text{SWR} = (1 + |\Gamma|) / (1 - |\Gamma|) \text{ If } \text{SWR} < 1.0 \text{ then invert it: } \text{SWR} = (1 - |\Gamma|) / (1 + |\Gamma|)$$

Since all the Z's with the same gamma magnitude lie on a circle (radius = gamma), the distance from the center of the Smith chart Calculate the return loss using SWR:

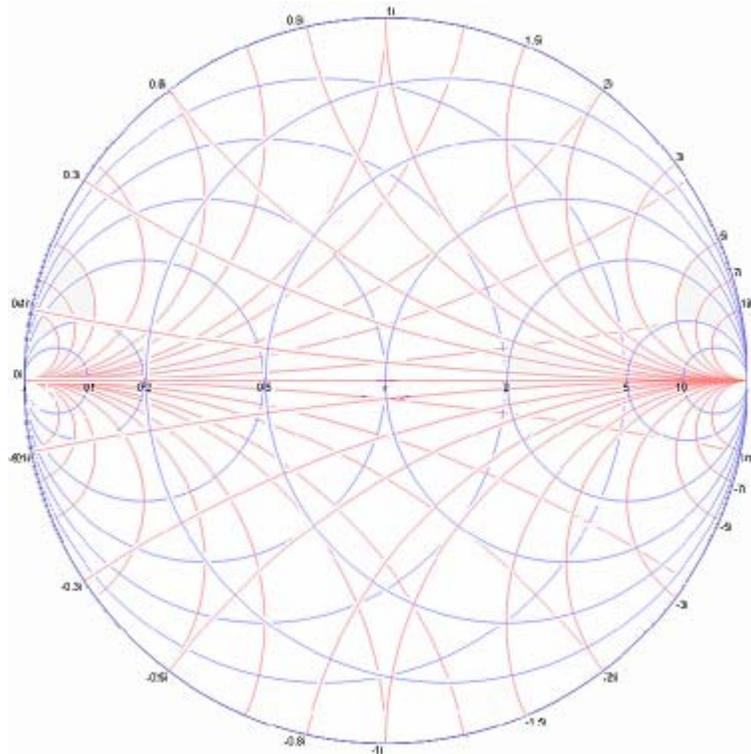
$$R_L = -20 * \log(\text{SWR}) \text{ and } \text{SWR} = 10^{R_L/20}$$



Constant SWR Circles

SWR and return loss are figures of merit that are commonly used in the industry. However, these quantities lose some detail when compared to the Z or gamma, so try to use Z or gamma when practical.

What happened to this chart? This form of Smith chart has two sets of grid lines. The first set (called Z grid) we are familiar with. The second set (called Y grid) eases admittance operations. We form the Y grid lines by performing a horizontal flip on the Z grid. Now when we need to operate with admittance, we just move along the Y grid. A little study will reveal that this accomplishes the same thing as converting Z to Y and then moving along the Z grid as we had done in the parallel impedances example above.



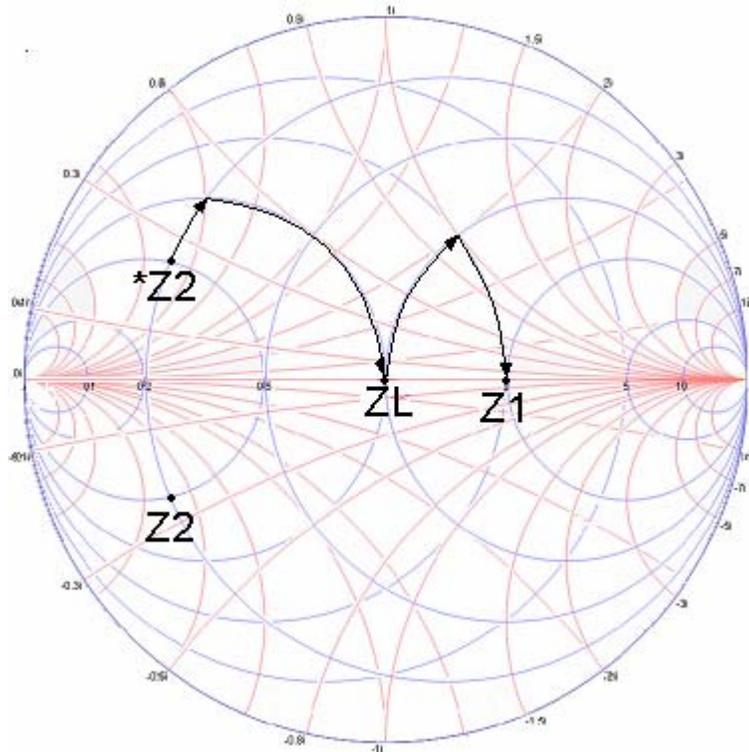
Smith Chart with Z and Y Grids

Summary of chart shifts:

Shift direction	Z grid	Y grid
“Left” (along red lines)	R decrease (G increase)	R increase (G decrease)
“Right” (along red lines)	R increase (G decrease)	R decrease (G increase)
CW on Blue lines	X increase (series inductor)	X decrease (parallel Cap)
CCW on Blue lines	X decrease (series Cap)	X increase (parallel inductor)

Let's design a matching network. Wideband matching networks are generally resistive and therefore lossy. We show two narrow band examples, outlined in the table below (*Z denotes the complex conjugate of Z):

Example	Load	Source	*Source	1 st Shift (Z)	2 nd Shift (Y)
Z1	$1 - j0$	$2 - j0$	$2 + j0$	$+j1$	$-j0.5$
Z2	$1 - j0$	$0.2 - j0.25$	$0.2 + j0.25$	$+j0.15$	$-j1.5$



The first example matches 50 ohms to 100, we can work this problem from either source to load, or as shown here from load to source, the results are the same. Also note that you could work this problem using $Z_0=100$ ohms, the load and source would be $0.5-j0$ and $1.0-j0$ respectively. Back to $Z_0=50$ ohms start from Z_L , we make a $+j1.0$ shift (series inductor). See how the Y grid makes it easier to decide how much Z (jX) shift to apply. Now we shift down the Y grid by $-j0.5$. Shifts on the Y grid represents susceptance, in this case a parallel capacitor. The second example shows a reactive source. The matching network must present the complex conjugate of the source to the source (or load to load) for proper matching. The two reactances cancel out so the net result is a resistive node. The second example shows that the matching network starts at the complex conjugate of Z_2 (i.e. $*Z_2$). From there the operations are similar to the first example.



Conclusion:

Learning to use a Smith chart will save you lots of time if you need to work with transmission lines or complex impedances. Understanding what a Smith chart represents makes visualizing transmission line effects easy.

Appendices:

A: Instructions for Using AEA Smith Chart.

Download and install the Bravo Vision Software. Run the program, select ignore (if you have no Bravo unit connected), select Get Data tab, press the Archive Data button, select File|Open|Archived Files, and click on the demo.aea file. Now select Chart Plots|Smith Chart.

To get a blank chart, replace the data section of the demo file with all zeroes using a word processor (like Notebook). It is a good idea to save the original demo somewhere safe in case the edited version doesn't operate. There are 81 lines of data, each of these lines will now contain eleven comma separated zeros: 0,0,0,0,0,0,0,0,0,0,0. Save the file, it is ready to be read by the PC Vision software. There will be a point plotted at $0+0i$, but it is barely noticeable.