Relativistic dissipative hydrodynamics and the quark-gluon plasma

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Meeting of Relativistic Viscous Hydro
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Overview

1. Formulation of relativistic hydrodynamics
   - Introduction
   - Thermodynamic variables
   - Relativistic dissipative hydrodynamics (1st and 2nd order theories)

2. Dissipative hydro in nuclear collisions
   - The quark-gluon plasma
   - Equation of state and transport coefficients
   - High-energy nucleus-nucleus collisions
   - Hydrodynamic model and elliptic flow
   - Latest developments: higher-order harmonics
   - Latest developments: longitudinal dynamics

3. Summary and outlook
2. Formulation of relativistic hydrodynamics

Previous: 1. Introduction
Next: 3. Dissipative hydro in nuclear collisions
Introduction

- Local thermalization; macroscopic variables are defined as fields

\[ \text{Flow } u^\mu(x) \quad u^\mu u_\mu = 1 \]
\[ \text{Temperature } T(x) \]
\[ \text{Chemical potentials } \mu_J(x) \]

**Gradient in the fields:** thermodynamic force

**Response to the gradients:** transport coefficients (= 0 if ideal hydro)

- Energy-momentum tensor & conserved current are

\[
T^{\mu\nu} = (e_0 + \delta e)u^\mu u^\nu - (P_0 + \Pi)\Delta^{\mu\nu} + 2W^{(\mu} u^{\nu)} + \pi^{\mu\nu}
\]
\[
N^\mu_j = (n_j + \delta n_j)u^\mu + V^\mu_j
\]

when decomposed with \( u^\mu \); \( \Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu \)
Thermodynamic quantities

- In local rest frame $u^\mu = (1, 0, 0, 0)$

\[
T^{\mu\nu} = T_0^{\mu\nu} + \delta T^{\mu\nu}
\]

\[
= \begin{pmatrix}
  e_0 & 0 & 0 & 0 \\
  0 & P_0 & 0 & 0 \\
  0 & 0 & P_0 & 0 \\
  0 & 0 & 0 & P_0
\end{pmatrix} + \begin{pmatrix}
  \delta e & W^x & W^y & W^z \\
  W^x & \Pi + \pi^{xx} & \pi^{xy} & \pi^{xz} \\
  W^y & \pi^{yx} & \Pi + \pi^{yy} & \pi^{yz} \\
  W^z & \pi^{zx} & \pi^{zy} & \Pi + \pi^{zz}
\end{pmatrix}
\]

\[
N^\mu_J = N_{J0}^\mu + \delta N^\mu_J \quad (J = 1, 2, \ldots, N)
\]

\[
= \begin{pmatrix}
  n_J^0 \\
  0 \\
  0 \\
  0
\end{pmatrix} + \begin{pmatrix}
  \delta n_J \\
  V^x_J \\
  V^y_J \\
  V^z_J
\end{pmatrix}
\]

- 2+N equilibrium quantities
  - Energy density: $e_0$
  - Hydrostatic pressure: $P_0$
  - $J$-th charge density: $n_J^0$

- 10+4N dissipative currents
  - Energy density deviation: $\delta e$
  - Bulk pressure: $\Pi$
  - Energy current: $W^\mu$
  - Shear stress tensor: $\pi^{\mu\nu}$
  - $J$-th charge density dev.: $\delta n_J$
  - $J$-th charge current: $V^\mu_J$
Hydrodynamic equations

- **Ideal** hydrodynamics

  \[ \text{Unknons (5+N)} \quad e_0, P_0, n_J, u\mu \quad \leftrightarrow \quad \text{Conservation laws (4+N) + EoS(1)} \quad \partial\mu T^{\mu
u} = 0, \partial\mu N_J = 0, P_0 = P_0(e_0, \{n_J\}) \]

- **Dissipative** hydrodynamics

  \[ \text{Additional unknowns (10+4N)} \quad \Pi, \delta e, W^{\mu}, \pi^{\mu
u}, \delta n_J, V_J^{\mu} \quad \leftrightarrow \quad \text{Constitutive equations} \]

  **Constitutive equations from the law of increasing entropy** \( \partial\mu s^{\mu} \geq 0 \)

  "perturbation" from equilibrium \( s^{\mu} = s_0 u^{\mu} + \delta s^{\mu}_{(1)} + \delta s^{\mu}_{(2)} + O(\delta^3) \)

  - **0\textsuperscript{th} order theory**
    - ideal hydro; no entropy production
  - **1\textsuperscript{st} order theory**
    - linear response theory
  - **2\textsuperscript{nd} order theory**
    - relaxation equations
Dissipative hydrodynamics

- 1st order theory; relativistic Navier-Stokes equations

Linear response theory

\[ J_i = C_{ij} X_j \]

- Dissipative current \( J_i \)
- Transport coefficient matrix \( C_{ij} \)
- Thermodynamic force \( X_j \)

- Onsager reciprocal relation \( C_{ij} = C_{ji} \) is satisfied
- Thermodynamic forces are derivatives of \( u^\mu, T \) and \( \mu J \)

ex. 1) scalar process

\[ \Pi = -\zeta \nabla_\mu u^\mu \]

- Bulk pressure \( \Pi \)
- (Effective) bulk viscosity \( \zeta \)
- Expansion scalar \( \nabla_\mu u^\mu \)

\[
\begin{align*}
D &= u^\mu \partial_\mu \quad : \text{time-like derivative} \\
\nabla_\mu &= \Delta^{\mu\nu} \partial_\nu \quad : \text{space-like derivative} \\
\n\Pi &= \text{bulk pressure} \\
\zeta &= \text{(effective) bulk viscosity} \\
\nabla_\mu u^\mu &= \text{expansion scalar} \\
\end{align*}
\]

\[ \nabla_\mu u^\mu > 0 \]

Expansion

\[ \text{Pressure is reduced as } P = P_0 + \Pi \]
Dissipative hydrodynamics

- **1st order** theory; relativistic *Navier-Stokes* equations

ex. 2) tensor process

\[
\pi^{\mu\nu} = 2\eta \nabla \langle \mu u^\nu \rangle = \eta (\nabla^\mu u^\nu + \nabla^\nu u^\mu - \frac{2}{3} \Delta^{\mu\nu} \nabla^\rho u^\rho)
\]

- \(\pi^{\mu\nu}\): shear stress tensor
- \(\eta\): shear viscosity
- \(\nabla \langle \mu u^\nu \rangle\): traceless symmetric tensor

The constitutive equations can be obtained from the law of increasing entropy

\[
\partial_\mu s^\mu = \delta e D \frac{1}{T} - \Pi \frac{1}{T} \nabla_\mu u^\mu + \bar{W}^\mu \left( \nabla_\mu \frac{1}{T} + \frac{1}{T} D u_\mu \right) + \pi^{\mu\nu} \frac{1}{T} \nabla \langle \mu u_\nu \rangle - \sum_J \delta n_J D \frac{\mu J}{T} - \sum_J V_J^\mu \nabla_\mu \frac{\mu J}{T} > 0
\]
Dissipative hydrodynamics

1\textsuperscript{st} order theory; relativistic \textit{Navier-Stokes} equations

The entropy production be quadratic forms

\[ \Pi = -\zeta_{\Pi\Pi} \frac{1}{T} \nabla_\mu u^\mu - \zeta_{\Pi \epsilon} D \frac{1}{T} + \sum_J \zeta_{\Pi \eta_{JJ}} D \frac{\mu_J}{T} \]

\[ \delta e = \zeta_{e\Pi} \frac{1}{T} \nabla_\mu u^\mu + \zeta_{e\epsilon} D \frac{1}{T} - \sum_J \zeta_{e \eta_{JJ}} D \frac{\mu_J}{T} \]

\[ \delta n_{JJ} = \zeta_{n\Pi J} \frac{1}{T} \nabla_\mu u^\mu + \zeta_{n \epsilon J} D \frac{1}{T} - \sum_K \zeta_{n \eta_{JJ K}} D \frac{\mu_K}{T} \]

\[ W^\mu = -\kappa_{WW} \left( \nabla_\mu \frac{1}{T} + \frac{1}{T} Du^\mu \right) + \sum_J \kappa_{WV,J} \nabla_\mu \frac{\mu_J}{T} \]

\[ V_{J\mu} = -\kappa_{VJ,W} \left( \nabla_\mu \frac{1}{T} + \frac{1}{T} Du^\mu \right) + \sum_K \kappa_{VJ,VK} \nabla_\mu \frac{\mu_K}{T} \]

\[ \pi^{\mu\nu} = 2\eta_{\pi\pi} \frac{1}{T} \nabla \langle \mu u^\nu \rangle \]

Scalar

Vector

Tensor

Diagonal terms

Onsager cross terms

However,

- It breaks down \textit{causality}; superluminal propagation is allowed
- It is unstable against perturbation

\textit{Hiscock & Lindblom} (‘85)
Dissipative hydrodynamics

- **2nd order** theory; relaxation equations

Introducing the relaxation term

\[ J_i = C_{ij} X_j - \tau J_i \frac{D}{\tau J_i} \]

- Relaxation term

- Finite propagation velocity
  - Causal for sufficiently large \( \tau J_i \)

- Dissipative currents relax to the Navier-Stokes values in about \( \tau \sim \tau J \)

- There can/should be other 2nd order terms

ex.) Bulk pressure

\[ D\Pi = \frac{1}{\tau_{\Pi}} (-\Pi - \zeta \nabla \mu u^{\mu}) \]

- Navier-Stokes
- 2nd order
2nd order theory

The Israel-Stewart theory

\[ T^{\mu\nu} = \sum_i \int \frac{g_i d^3p}{(2\pi)^3 E_i} p_i^\mu p_i^\nu f^i \]
\[ \mathcal{N}_J^\mu = \sum_i \int \frac{q_i^J g_i d^3p}{(2\pi)^3 E_i} p_i^\mu f^i \]

\[ g_i : \text{degeneracy} \quad q_i^J : \text{conserved charge number} \]

Conventional formalism

\[ \partial_\alpha I^{\mu\nu\alpha} = \int \frac{gd^3p}{(2\pi)^3 E} p^\mu p^\nu p^\alpha \partial_\alpha f = Y^{\mu\nu} \]

Dissipative currents (9)
\[ \delta_\epsilon, \Pi, W^\mu, \pi^{\mu\nu}, \delta_\omega, V^\mu \]

Moment equations (9)
\[ \partial_\alpha I^{\mu\nu\alpha} = Y^{\mu\nu} \]

Frame fixing, stability conditions

Not extendable for multi-component/conserved current systems

Single-component, elastic scattering

Meeting of relativistic viscous hydro, 25 May 2012, RIKEN, Japan
### 2nd order theory

**Extended I-S theory**

\[
\partial_\alpha I_{\mu\nu\alpha} = \sum_i \int \frac{g_i d^3 p}{(2\pi)^3 E_i} p_i^\mu p_i^\nu p_i^\alpha \partial_\alpha f^i = Y_{\mu\nu}
\]

\[
\partial_\alpha I_{J}^{\mu\alpha} = \sum_i \int \frac{q_i^J g_i d^3 p}{(2\pi)^3 E_i} p_i^\mu p_i^\alpha \partial_\alpha f^i = Y_J^{\mu}
\]

- **Unknowns (10+4N)**
  - \(\delta T^{\mu\nu}, \delta N_J^{\mu}\)

- **Moment eqs. (10+4N)**
  - \(\partial_\alpha I_{\mu\nu\alpha} = Y_{\mu\nu}, \partial_\alpha I_{J}^{\mu\alpha} = Y_J^{\mu}\)

- **All viscous quantities determined in arbitrary frame**

**Expressions of \(Y_{\mu\nu}\) and \(Y_J^{\mu}\)**

Determined through the 2nd law of thermodynamics

\[
\partial_\mu S^\mu = \sum_i \int \frac{g_i d^3 p_i}{(2\pi)^3 E_i} p_i^\mu y^i \partial_\mu f^i
\]

where \(f^i = \frac{1}{\exp(y^i) + 1}\)

- **It is essential to know the off-equilibrium distribution**

A. Monnai, T. Hirano, NPA 847, 283 (2010)
2nd order theory

- Introduce distortion of distribution

\[ \delta f^i = -f^i_0 (1 \pm f^i_0) (p^\mu_i \sum J q^J_i \varepsilon^J_{\mu} + p^\mu_i p^\nu_i \varepsilon_{\mu\nu}) \]

*Grad’s moment method extended to multi-conserved current systems

\[ \begin{align*}
\delta T^{\mu\nu} &= \sum_i \int \frac{g_i d^3p}{(2\pi)^3} p^\mu_i p^\nu_i \delta f^i \\
\delta N^\mu_J &= \sum_i \int \frac{q^J_i g_i d^3p}{(2\pi)^3} p^\mu_i \delta f^i
\end{align*} \]

The entropy production is expressed in terms of \( Y^{\mu\nu} \) and \( Y^\mu_J \)

\[ \partial_\mu s^\mu = \sum_i \int \frac{g_i d^3p}{(2\pi)^3} y^i p^\mu_i \partial_\mu f^i = \sum_J \varepsilon^J_\nu Y^\nu_J + \varepsilon_\nu_\rho Y^{\nu\rho} \geq 0 \]

Dissipative currents
\( \Pi, \delta e, \delta n_J, W^\mu, V^\mu_J, \pi^{\mu\nu} \)

Matching matrices for \( \delta f^i \)

Viscous distortion tensor & vector
\( \varepsilon^{\mu\nu}, \varepsilon^\mu_J \)

Moment equations
\( \partial_\alpha I^{\mu\nu\alpha} = Y^{\mu\nu} \)
\( \partial_\alpha I^{\mu\nu_J}_J = Y^\mu_J \)

Semi-positive definite condition
2nd order theory

- **Bulk Pressure**

\[ \Pi = -\zeta \nabla_\mu u^\mu + \zeta T \delta \delta_{J} D \frac{1}{T} + \sum_{J} \zeta_{I} \delta_{I} D \frac{\mu}{T} \]

**Cross terms (linear)**

\[ -\tau \Pi D II + \sum_{J} \chi_{II}^{J} D \frac{1}{T} + \chi_{III}^{I} D \frac{1}{T} + \chi_{III}^{I} \nabla_\mu u^\mu \]

**Relaxation term**

\[ + \sum_{J} \chi_{II}^{J} W_\mu \nabla_\mu \frac{1}{T} + \chi_{II}^{B} W_\mu \nabla_\mu \frac{1}{T} + \chi_{II}^{C} W_\mu D u^\mu + \chi_{II}^{D} \nabla_\mu W_\mu \]

**2nd order corrections**

\[ + \sum_{J,K} \chi_{II}^{K} V_\mu \nabla_\mu \frac{1}{T} + \sum_{J} \chi_{II}^{B} V_\mu \nabla_\mu \frac{1}{T} + \sum_{J} \chi_{II}^{C} V_\mu D u^\mu + \sum_{J} \chi_{II}^{D} \nabla_\mu V_\mu \]

- **Cross terms** appear (reciprocal relations)
- **2nd order terms** in full form (multi-conserved currents)
- **Relaxation term** appears (causality is preserved)
2\textsuperscript{nd} order theory

- Energy dissipation current

\[ W^\mu = -\kappa_W \left( \frac{1}{T} D u^\mu + \nabla^\mu \frac{1}{T} \right) + \sum_j \kappa_{Wj} \nabla^\mu \frac{\mu_j}{T} \]

- Relaxation term

\[ -\tau_W \Delta^{\mu\nu} D W^\nu + \sum_j \chi^{a,j}_{Wj} W^\mu D^{\mu,j} + \chi^{b}_{W} W^\mu D \frac{1}{T} \]

- Cross terms (linear)

\[ + \chi^{c}_{WW} W^\mu \nabla_\nu u^\nu + \chi^{d}_{WW} W^\nu \nabla_\nu u^\mu + \chi^{e}_{WW} W^\nu \nabla^\mu u_\nu \]

- 2\textsuperscript{nd} order corrections

\[ - \sum_j \tau_{Wj} \Delta^{\mu\nu} D V^J_j + \sum_{j,k} \chi^{a,K}_{Wj} V^J_k D^{\mu,k} + \sum_j \chi^{b}_{Wj} V^J_j D \frac{1}{T} \]

\[ + \sum_j \chi^{c}_{Wj} V^J_j \nabla^\nu u_\nu + \sum_j \chi^{d}_{Wj} V^J_j \nabla_\nu u^\mu + \sum_j \chi^{e}_{Wj} V^J_j \nabla^\mu u_\nu \]

\[ + \sum_{j} \chi^{a,j}_{W} \pi^{\mu\nu} \nabla_\nu \frac{\mu_j}{T} + \chi^{b}_{W} \pi^{\mu\nu} \nabla_\nu \frac{1}{T} + \chi^{c}_{W} \pi^{\mu\nu} D u_\nu + \chi^{d}_{W} \Delta^{\mu\nu} \nabla^\rho \pi_{\nu\rho} \]

\[ + \sum_j \chi^{a,j}_{W} \Pi^{\mu\nu} \nabla_\nu \frac{\mu_j}{T} + \chi^{b}_{W} \Pi^{\mu\nu} \nabla_\nu \frac{1}{T} + \chi^{c}_{W} \Pi^{\mu\nu} D u_\mu + \chi^{d}_{W} \nabla^\mu \Pi \]
2nd order theory

- Charge dissipation current(s)

\[ V^\mu_J = \sum_K \kappa_{V_J V_K} \nabla^\mu \frac{\mu K}{T} - \kappa_{V_J W} \left( \frac{1}{T} D u^\mu + \nabla^\mu \frac{1}{T} \right) \]

response to chemical gradient (+ cross terms)  cross terms (linear)

- \( - \sum_K \tau_{V_J V_K} \Delta^{\mu \nu} D V^K_\nu \) relaxation terms

+ \( \sum_K \chi^{a L}_{V_J V_K} V^K_\mu D^{\mu L}_\nu \frac{1}{T} \)  + \( \sum_K \chi^{b}_{V_J V_K} V^K_\mu D^{\mu}_\nu \frac{1}{T} \)

- \( \tau_{V_J W} \Delta^{\mu \nu} D W_\nu \)  + \( \sum_K \chi^{a}_{V_J W} W^{\mu} D^{\mu K}_\nu \frac{1}{T} \)  + \( \chi^{b}_{V_J W} W^{\mu} D^{\mu}_\nu \frac{1}{T} \)

+ \( \chi^{c}_{V_J W} W^{\mu} \nabla^\nu u_\nu \)  + \( \chi^{d}_{V_J W} W^{\nu} \nabla^\mu u^\mu \)  + \( \chi^{e}_{V_J W} W^{\nu} \nabla^\mu u_\nu \)

+ \( \sum_K \chi^{a K}_{V_J \pi} \pi^{\mu \nu} \nabla^\nu \frac{\mu K}{T} \)  + \( \chi^{b}_{V_J \pi} \pi^{\mu \nu} \nabla^\nu \frac{1}{T} \)  + \( \chi^{c}_{V_J \pi} \pi^{\mu \nu} D u_\nu \)  + \( \chi^{d}_{V_J \pi} \Delta^{\mu \nu} \nabla^\rho \pi_{\nu \rho} \)

+ \( \sum_K \chi^{a K}_{V_J \Pi} \Pi \nabla^\mu \frac{\mu K}{T} \)  + \( \chi^{b}_{V_J \Pi} \Pi \nabla^\mu \frac{1}{T} \)  + \( \chi^{c}_{V_J \Pi} \Pi D u^\mu \)  + \( \chi^{d}_{V_J \Pi} \nabla^\mu \Pi \)

\[ D = u^\mu \partial_\mu \]

\[ \nabla^\mu = \Delta^{\mu \nu} \partial_\nu \]
2nd order theory

- **Shear stress tensor**

\[
\pi^{\mu\nu} = 2\eta \nabla^{\langle \mu} u^{\nu \rangle}
\]

\[
\tau_\pi D\pi^{\langle \mu\nu \rangle} + \sum_J \chi^{a,J}_\pi \pi^{\mu\nu} \frac{D^{\mu J}}{T} + \chi^b_\pi \pi^{\mu\nu} \frac{1}{T} + \chi^c_\pi \nabla \rho u^\rho + \chi^d_\pi \pi^{\rho} \nabla^{\rho} u^{\nu} + \sum_J \chi^{a,J}_\pi W^{\langle \mu \nabla u^{\nu \rangle} \frac{\mu J}{T} + \chi^b_\pi W^{\langle \mu \nabla u^{\nu \rangle} \frac{1}{T} + \chi^c_\pi W^{\langle \mu D u^{\nu \rangle} + \chi^d_\pi W^{\langle \mu \nabla W^{\nu \rangle} + \sum_J \chi^{a,J}_\pi V_J^{\langle \mu \nabla u^{\nu \rangle} \frac{\mu K}{T} + \sum_J \chi^b_\pi V_J^{\langle \mu \nabla u^{\nu \rangle} \frac{1}{T} + \sum_J \chi^c_\pi V_J^{\langle \mu D u^{\nu \rangle} + \sum_J \chi^d_\pi V_J^{\langle \mu \nabla V_J^{\nu \rangle}}
\]

- **Thermodynamic stability**

\[
s^{\mu} u_\mu = s_0 + \sum_i \int \frac{g_i d^3p}{(2\pi)^3 E_i} p_i^{\mu} u_\mu y_0 \delta f_i - \frac{1}{2} \sum_i \int \frac{g_i d^3p}{(2\pi)^3 E_i} p_i^{\mu} u_\mu \frac{\delta f_i^2}{f_0^2(1 \pm f_i^0)} \leq 0 \quad \text{automatically satisfied}
\]

where \( \delta e = \delta n_J = 0 \)
2nd order theories

■ The AdS/CFT + phenomenological approach

\[
\pi^{\mu\nu} = 2\eta \nabla \langle \mu u^\nu \rangle - \tau_\pi D\pi \langle \mu u^\nu \rangle - \frac{d}{d-1} \tau_\pi \pi^{\mu\nu} \nabla_\rho u^\rho \\
+ \frac{\lambda_1}{\eta} \pi^\rho \langle \mu u^\nu \rangle - \frac{\lambda_2}{\eta} \rho^\rho \langle \mu \omega^\nu \rangle + \lambda_3 \omega^\rho \langle \mu u^\nu \rangle
\]

R. Baier et al., JHEP 0804, 100 (2008)

- It is subject to the arbitrariness in the choice of holographic theories
- Vorticity-vorticity terms do not appear in kinetic theory

■ The renormalization group approach

\[
\pi^{\mu\nu} = 2\eta \nabla \langle \mu u^\nu \rangle - \tau_\pi D\pi \langle \mu u^\nu \rangle \\
+ \tau_\pi \left[ -\frac{1}{2}\frac{T\eta}{\tau_\pi} \partial_\rho \left( \pi^\mu u^\nu \right) \right] + \frac{1}{2} \left( -D^\mu \frac{\pi}{T} + T\delta_{\pi}^{(0)} D \frac{1}{T} + \frac{7}{3} \delta_{\pi}^{(1)} \nabla_\rho u^\rho \right) \pi^{\mu\nu} \\
+ \tau_\pi \delta_{\pi}^{(1)} 4\pi^\rho \langle \mu \sigma^\nu \rangle + l_\pi V \left[ -\nabla \langle \mu \frac{\pi}{T} + T\delta_{\pi} \nabla \left( \nabla \frac{1}{T} + \frac{1}{T} D u^\langle \mu \rangle \right) \right] V^\nu \\
- l_\pi V \nabla \langle \mu V^\nu \rangle + l_{\pi \Pi} \nabla \langle \mu u^\nu \rangle \Pi
\]

K. Tsumura et al., PLB 646, 134 (2007)

- Boltzmann equation + renormalization group equations
- Equations are dependent on the definitions of the flow frame
3rd order theory

- Yes, there is a 3rd order theory

El et al., PRC 81, 041901 (2010)

Shear stress tensor

\[
\dot{\pi}^{\alpha\beta} = -\frac{\pi^{\alpha\beta}}{\tau \pi} + \frac{\sigma^{\alpha\beta}}{\beta_2} - \pi^{\alpha\beta} \frac{T}{\beta_2} \partial_\mu \left( \frac{\beta_2}{2T} u^\mu \right) - \frac{8}{9} \frac{T}{\beta_2} \partial_\mu \left( \frac{\beta_2^2}{T} u^\mu \right) \pi_\sigma^{(\alpha \pi \sigma \beta)} - \frac{8}{3} \frac{T}{\pi} \beta_2 \pi_\sigma^{(\alpha \pi \sigma \beta)}
\]

Comparison to transport calculations exhibits better agreements

- Thermodynamic stability at the third order

\[
s^\mu \approx - \int \frac{d^3 p}{E} f_0 p^\mu \left( \ln f_0 - 1 + \phi + \phi \ln f_0 + \frac{\phi^2}{2} - \frac{\phi^3}{6} \right)
\]

\[
= s_0 u^\mu - \frac{\beta_2}{2T} \pi_\alpha^{\alpha \beta} \pi^{\alpha \beta} u^\mu - \frac{8}{9} \frac{\beta_2^2}{T} \pi_\sigma^{\alpha \pi \sigma \beta} u^\mu
\]

This term may be violating the stability; further checking is needed
3. Dissipative hydro in nuclear collisions

Previous: 2. Formulation of relativistic hydrodynamics
Next: 4. Summary and outlook
Motivation

- Quark-gluon plasma (QGP): many-body system of deconfined quarks and gluons at high energies

Where to find: early universe, high-energy heavy ion collisions

Goals: Examine the human-made QGP to...

- Quantify the observed particle distribution
- Extract the static (equation of state) and dynamical (transport coefficients, etc.) properties of the QGP
Macroscopic aspects

- Observables of the hot QCD matter

- Electromagnetic probes:
  - Jet quenching, heavy quarks:
  - Hydrodynamic medium:

- EM transparent
- Color opaque
- Strongly-coupled
Microscopic aspects

- Properties of the hot QCD matter

**Equation of state:** static relation among thermodynamic variables; sensitive to degrees of freedom in the system

\[ P_0 = P_0(e_0, \{n_{J0}\}) \]

where

- \( e_0 \): energy density
- \( P_0 \): hydrostatic pressure
- \( n_{J0} \): type-\( J \) charge density

- Lattice QCD calculations with zero net-baryon density suggest
  - Type of transition: crossover
  - \( T_C \): \( \sim 170 \) MeV

- The temperature suggested by photon measurements > 300 MeV
Microscopic aspects

Properties of the QCD fluid

Transport coefficients: dynamical responses to thermodynamic forces; sensitive to interaction in the system

Naïve interpretation of dissipative processes

- Shear viscosity $\eta$ = response to deformation
- Bulk viscosity $\zeta$ = response to expansion
- Energy dissipation $\kappa_w$ = response to thermal gradient
- Charge dissipation $\kappa_v$ = response to chemical gradients

- Difficult to calculate in lattice QCD (yet)
- Minimum boundary for shear viscosity $\eta = s/4\pi$ is proposed in AdS/CFT

P. Kovtun et al., PRL 94, 111601 (2005)
High-energy nuclear collisions

- **Relativistic Heavy Ion Collider** (2000-)
  \[ \sqrt{s_{NN}} = 200 \text{ GeV} \]
  - First evidence of the nearly perfect fluid QGP
  - Viscosity is important for “improved” inputs based on first-principle calculations (initial conditions, equation of state, etc.)
  - Au+Au collisions

- **Large Hadron Collider (LHC)** (2010-)
  - New!
  \[ \sqrt{s_{NN}} = 2.76 \text{ TeV} \]
  - Heavy ion collisions at the highest energy
  - Seems consistent with the fluid picture (*i.e.* not too perfect nor viscous)
  - Pb+Pb collisions
High-energy nuclear collisions

- A “standard” view of high-energy heavy ion collisions

- Color glass condensate (CGC): saturated gluons in the colliding nuclei (τ < 0 fm/c)
  - Local thermalization (instabilities in glasma?)

- Relativistic hydrodynamics: collective motion of the QGP fluid (τ ~ 1-10 fm/c)
  - Freezeout (flow to particle)

- Hadronic cascade, UrQMD: transport approach to hadrons (τ > 10 fm/c)
High-energy nuclear collisions

- Geometrical setup of colliding nuclei

\[ \tau = \sqrt{t^2 - z^2} \]
\[ \eta_s = \frac{1}{2} \ln \frac{t+z}{t-z} \]
\[ t = \tau \cosh \eta_s \]
\[ z = \tau \sinh \eta_s \]

- Centrality
  Groups of events per number of participants (20-30 %, etc.)

central events
peripheral events
Elliptic flow

- Fourier analyses of particle spectra in momentum space

\[
\frac{dN}{d\phi p_T dp_T} = \frac{dN}{(2\pi)p_T dp_T} \left[ 1 + 2v_2(p_T) \cos(\phi) + 2v_4(p_T) \cos(\phi) + \ldots \right]
\]

elliptic flow coefficient

Azimuthal anisotropy in coordinate space: \( \varepsilon_2 \)

Azimuthal anisotropy in momentum space: \( v_2 \)

- If the medium is strongly interacting (= hydro-like), \( v_2 \) is large
- If the medium is weakly interacting (= gas-like), \( v_2 \) is small
Hydrodynamics at the RHIC

- Experimental data from the RHIC

- Hydro limit is achieved in 200 GeV collisions at the RHIC

- Reasonable agreement with ideal hydro calculation; small viscous corrections would improve the agreement

Constraints on the transport coefficients
Higher-order harmonics $v_n$

Event-by-event fluctuations

- Fourier analysis on momentum anisotropy
  
  \[
  \frac{dN}{d\phi p_T dp_T} = \frac{dN}{(2\pi)p_T dp_T} \left[ 1 + 2v_1(p_T) \cos(\phi) + 2v_2(p_T) \cos(\phi) \\
  + 2v_3(p_T) \cos(\phi) + 2v_4(p_T) \cos(\phi) + \ldots \right]
  \]

  Odd-order harmonics can appear

- If the higher-order harmonics is hydrodynamic, they can work as additional constraints on viscosity, etc.

Participant trianlarity

\[
\varepsilon_3 = \frac{\sqrt{\langle r^2 \cos(3\phi) \rangle^2 + \langle r^2 \sin(3\phi) \rangle^2}}{\langle r^2 \rangle}
\]

Q: “Does $v_3$ disappear for accumulated events?”

A: “No, because $\psi_3$ is used instead of $\psi_{RP}$.”

J. Takahashi et al., PRL 103, 242301 (2009),
B. Alver & G. Roland, PRC 81, 054905 (2010)
Higher-order harmonics $v_n$

- Experimental evidence at the LHC

Centrality dependence is *small* for $v_3$

Fluctuation is the origin of $v_3$ because otherwise it is sensitive to centrality like $v_2$

$v_3$ supports different $\eta/s$ than $v_2$

- Temperature dependence of $\eta/s$?
- Different initial conditions?
- Bulk viscosity $\zeta/s$ at freezeout?
Longitudinal dynamics

- Recent developments: relativistic expansion of viscous fluids

  ▶ Elliptic and triangular flow of charged particles
    - Viscous correction for better agreement; needs improvement on initial condition

  ▶ Finite net baryon distribution
    - Hydrodynamic effects reduce rapidity loss; could be sensitive to baryon diffusion

A. Monnai, arXiv:1204.4713

B. Schenke et al, PRL 106, 042301
4. Summary and outlook

Previous: 3. Viscous hydro for heavy ion collisions
Summary and outlook

- Relativistic dissipative hydrodynamics involves non-trivialities due to causality and stability
  - The 1\textsuperscript{st} order theory is not sufficient; the 2\textsuperscript{nd} order theory with relaxation effects is necessary
  - Multiple conserved currents can be introduced to Israel-Stewart theory

- The QGP in high-energy nucleus-nucleus collisions are quantified as low viscous fluid
  - Hydrodynamic analyses on the higher-order harmonics of azimuthal anisotropy provide rich information on the properties of the QGP
  - Longitudinal viscous hydrodynamic expansion is important
The end

- Thank you for listening!
- Website: http://tkynt2.phys.s.u-tokyo.ac.jp/~monnai/