Viscous Hydrodynamics

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Outline

1. Introduction
   Relativistic hydrodynamics and heavy ion collisions

2. Relativistic Dissipative Hydrodynamics
   Extended Israel-Stewart theory from law of increasing entropy

3. Results and Discussion
   Constitutive equations in multi-component/conserved current systems

4. Summary
   Summary and Outlook
Introduction

- Quark-gluon plasma (QGP) at relativistic heavy ion collisions

**RHIC experiments (2000-)**

![Image of RHIC experiments]

- $\sqrt{s_{NN}} = 200$ (GeV)

- Discovery of QGP as nearly perfect fluid
- Thermodynamics is at work in QGP at RHIC energies
- Non-equilibrium effects need investigation for quantitative understandings
Introduction

- Hydrodynamic modeling of RHIC

- Intermediate stage (~1-10 fm) is described by hydrodynamics
- Results are dependent on the inputs:
  - Equation of state, Transport coefficients
  - Initial conditions
  - Hydrodynamic equations
  - Output

Hadronic cascade picture
- Hydro to particles
- Initial condition
- CGC/glasma picture?
Introduction

- Properties of QCD fluid

  Equation of state: relation among thermodynamic variables sensitive to degrees of freedom in the system

  Transport coefficients: responses to thermodynamic forces sensitive to interaction in the system

Naïve interpretation of dissipative processes

- Shear viscosity = response to deformation
- Bulk viscosity = response to expansion
- Energy dissipation = response to thermal gradient
- Charge dissipation = response to chemical gradients
Introduction

- Elliptic flow coefficients from RHIC data

Hirano et al. (‘09)

Viscosity

Initial cond.

Eq. of state

theoretical prediction ~ experimental data

Ideal hydro works well

Ideal hydro

Glauber

1\textsuperscript{st} order
Elliptic flow coefficients from RHIC data

Hirano et al. ('09)

Viscosity
Initial cond.
Eq. of state

Ideal hydro
Glauber
Lattice-based

theoretical prediction > experimental data

Ideal hydro shows slight overshooting
Elliptic flow coefficients from RHIC data

Viscosity

Initial cond.

Eq. of state

Viscosity in QGP phase plays important role in reducing $v_2$
## Introduction

- **Why viscous hydrodynamic models?**

<table>
<thead>
<tr>
<th>RHIC experiments (2000-)</th>
<th>( \sqrt{s_{NN}} = 200 \text{(GeV)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Success of ideal hydro</td>
<td>( \Rightarrow ) Necessity of viscous hydro</td>
</tr>
<tr>
<td>for improved inputs to the hydrodynamic models</td>
<td></td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>LHC experiments (2010-)</th>
<th>( \sqrt{s_{NN}} = 2.76, 5.5 \text{(TeV)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Asymptotic freedom in QCD</td>
<td>( \Rightarrow ) Viscous hydro?</td>
</tr>
<tr>
<td>QGP might become less-strongly coupled</td>
<td></td>
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</tbody>
</table>

- **CERN Press release, November 26, 2010:**
  “… confirms that the much hotter plasma produced at the LHC behaves as a very low viscosity liquid”

Viscous hydro is likely to work also at the LHC energies.
Introduction

- Viscous hydrodynamics needs improvement

1. **Form** of dissipative hydro equations
   - Fixing the equations is essential in fine-tuning viscosity from experimental data

2. Treatment of **conserved currents**
   - Low-energy ion collisions are planned at FAIR (GSI) & NICA (JINR)
   - Only 1 conserved current can be treated

3. Treatment of **multi-component systems**

   

   

   # of conserved currents ≠ # of particle species

   - baryon number, strangeness, etc.
   - pion, proton, quarks, gluons, etc.

   ➡️ We need to construct a firm framework of viscous hydro
Introduction

- Categorization of relativistic hydrodynamic formalisms

<table>
<thead>
<tr>
<th>Types of interactions</th>
<th>Number of components</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single component with binary collisions</td>
<td></td>
</tr>
<tr>
<td>Israel &amp; Stewart ('79), etc...</td>
<td>Multi-components with binary collisions</td>
</tr>
<tr>
<td>Prakash et al. ('91)</td>
<td></td>
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<tr>
<td>Single component with inelastic scatterings</td>
<td></td>
</tr>
<tr>
<td>(-)</td>
<td>Multi-components with inelastic scatterings</td>
</tr>
<tr>
<td>Monnai &amp; Hirano ('10)</td>
<td></td>
</tr>
</tbody>
</table>

Required for QGP/hadron gas at heavy ion collisions

In this work we formulate relativistic dissipative hydro for multi-component + multi-conserved current systems
Overview

- Formulation of relativistic dissipative hydrodynamics

\[ \partial_\mu T^{\mu \nu} = 0 \]
\[ \partial_\mu N^J_\nu = 0 \]
\[ \partial_\mu s^\mu \geq 0 \]

\[ \partial_\alpha I^{\mu \nu \alpha} = Y^{\mu \nu} , \quad \partial_\alpha I^{\mu \alpha}_J = Y^{\mu}_J \]

\[ \delta f^i = - f^i_0 (1 \pm f^i_0) \left( p_i^\mu \sum J q^J_i \epsilon^J_\mu + p_i^\mu p_i^\nu \epsilon_{\mu \nu} \right) \]

EoM for dissipative currents

Onsager reciprocal relations satisfied
Thermodynamic Quantities

Tensor decompositions by flow $u^\mu$

\[ T^{\mu\nu} = (e_0 + \delta e)u^\mu u^\nu - (P_0 + \Pi)\Delta^{\mu\nu} + 2W^{(\mu}u^{\nu)} + \pi^{\mu\nu} \]

\[ N^\mu_J = (n_J0 + \delta n_J)u^\mu + V^\mu_J \]

where $\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu$ is the projection operator

2+N equilibrium quantities

- Energy density: $e_0$
- Hydrostatic pressure: $P_0$
- $J$-th charge density: $n_J0$

10+4N dissipative currents

- Energy density deviation: $\delta e$
- Bulk pressure: $\Pi$
- Energy current: $W^{\mu}$
- Shear stress tensor: $\pi^{\mu\nu}$
- $J$-th charge density dev.: $\delta n_J$
- $J$-th charge current: $V^\mu_J$

*Stability conditions $\delta e = \delta n_J = 0$ should be considered afterward

Next slide: Thermodynamic Stability
Thermodynamic Stability

- Maximum entropy state condition

\[ s^\mu u_\mu = s_0 + \sum_i \int \frac{g_i d^3 p}{(2\pi)^3 E_i} p_i^\mu u_\mu y_0^i \delta f^i - \frac{1}{2} \sum_i \int \frac{g_i d^3 p}{(2\pi)^3 E_i} p_i^\mu u_\mu \frac{\delta f^i}{f_0^i (1 \pm f_0^i)} + \mathcal{O}(\delta f^3) \]

\[ = 0 \leq 0 \]

- Stability condition (1\textsuperscript{st} order)

\[ \sum_i \int \frac{g_i d^3 p}{(2\pi)^3 E_i} p_i^\mu u_\mu y_0^i \delta f^i = \frac{1}{T} u_\mu \delta T^{\mu \nu} u_\nu - \sum_J \frac{\mu_J}{T} u_\mu \delta N_J^\mu = 0 \]

\[ \Rightarrow \delta e = \delta n_J = 0 \]

- Stability condition (2\textsuperscript{nd} order) \( \Rightarrow \) automatically satisfied in kinetic theory

*Stability conditions are NOT the same as the law of increasing entropy
Relativistic Hydrodynamics

- **Ideal** hydrodynamics

  \[e_0, P_0, n, j, u^\mu \begin{array}{c} \leftrightarrow \end{array} \partial_\mu T^{\mu\nu} = 0, \partial_\mu N^\mu_j = 0, P_0 = P_0(e_0, \{n, j\})\]

- **Dissipative** hydrodynamics ("perturbation" from equilibrium)

  \[\Pi, \delta e, W^\mu, \pi^{\mu\nu}, \delta n, j, V^\mu_j \begin{array}{c} \leftrightarrow \end{array} \partial_\alpha I^{\mu\nu\alpha} = Y^{\mu\nu}, \partial_\alpha I^{\mu\alpha}_j = Y^\mu_j\]

  Defined in relativistic kinetic theory as

  \[
  \partial_\alpha I^{\mu\nu\alpha} = \sum_i \int \frac{q_i g^3}{(2\pi)^3 E_i} p_i^\mu p_i^\nu p_i^\alpha \partial_\alpha f^i = Y^{\mu\nu}
  \]

  \[
  \partial_\alpha I^{\mu\alpha}_j = \sum_i \int \frac{q_j g^3}{(2\pi)^3 E_i} p_i^\mu p_i^\alpha \partial_\alpha f^i = Y^\mu_j
  \]

  Estimated from the law of increasing entropy \(\partial_\mu S^\mu \geq 0\)

  New equations in our work
Moment Equations

- Introduce distortion of distribution

\[
\delta f^i = -f^i_0(1 \pm f^i_0)(p_i^\mu \sum_J q_i^J \varepsilon_{\mu}^J + p_i^\mu p_i^\nu \varepsilon_{\mu\nu})
\]

*Grad’s moment method extended to multi-conserved current systems

- 10+4N unknowns \( \varepsilon_{\mu\nu}^i, \varepsilon_{\nu}^J \) are determined in self-consistency conditions

\[
\delta T_{\mu\nu} = \sum_i \int \frac{g_i d^3p}{(2\pi)^3 E_i} p_i^\mu p_i^\nu \delta f^i \\
\delta N_{\mu}^J = \sum_i \int \frac{q_i^J g_i d^3p}{(2\pi)^3 E_i} p_i^\mu \delta f^i
\]

- The entropy production is expressed in terms of \( Y_{\mu\nu} \) and \( Y_J^\mu \)

\[
\partial_\mu s^\mu = \sum_i \int \frac{g_i d^3p}{(2\pi)^3 E_i} y_i p_i^\mu \partial_\mu f^i = \sum_J \varepsilon_{\nu}^J Y_{\nu}^J + \varepsilon_{\nu\rho} Y_{\nu\rho} \geq 0
\]
Constitutive Equations

- **Bulk Pressure**

\[
\Pi = -\zeta \nabla \mu u^\mu - \zeta \Pi \delta e \frac{1}{T} + \sum_j \zeta \Pi \delta n_j \frac{D \mu_j}{T} \\
- \tau \Pi D \Pi + \sum_J \chi^{aJ} \Pi D \frac{\mu_J}{T} + \chi^b \Pi \Pi D \frac{1}{T} + \chi^c \Pi \nabla \mu u^\mu \\
+ \sum_J \chi^{aJ} W^\mu D \frac{\mu_J}{T} + \chi^b W^\mu \nabla \frac{1}{T} + \chi^c W^\mu D u^\mu + \chi^d W^\mu W^\mu \\
+ \sum_{J,K} \chi^{aJ} V^J \nabla \frac{\mu_K}{T} + \sum_J \chi^b V^J \nabla \frac{1}{T} + \sum_J \chi^c V^J D u^\mu + \sum_J \chi^d V^J \nabla \mu V^J \\
+ \chi^\Pi \Pi \nabla \langle \mu, u^\nu \rangle
\]

- **Cross terms** appear (reciprocal relations)
- **2\text{nd order terms}** in full form (multi-conserved currents)
- **Relaxation term** appears (causality is preserved)
Constitutive Equations

- **Energy current**

\[
W^\mu = -\kappa_W \left( \frac{1}{T} D u^\mu + \nabla^\mu \frac{1}{T} \right) + \sum_j \kappa_W V_j \nabla^\mu \frac{\mu_j}{T}
\]

- **response to temperature gradient**

\[
- \tau_W \Delta^{\mu\nu} D W_{\nu} + \sum_j \chi_W^{a,j} W^\mu D \frac{\mu_j}{T} + \chi_W W^\mu D \frac{1}{T}
\]

- **cross terms (linear)**

\[
+ \chi_W^c W^\mu \nabla_\nu u^\nu + \chi_W^d W^\nu \nabla_\nu u^\mu + \chi_W^e W^\nu \nabla^\mu u_\nu
\]

- **relaxation term**

\[
- \sum_j \tau_W V_j \Delta^{\mu\nu} D V^j_{\nu} + \sum_{j,K} \chi_W^{a,k} V^\mu D \frac{\mu_k}{T} + \sum_j \chi_W^{b,j} V^\mu D \frac{1}{T}
\]

- **2nd order corrections**

\[
+ \sum_j \chi_W^c V^\mu_j \nabla^\nu u_\nu + \sum_j \chi_W^d V^\nu_j \nabla^\mu u^\mu + \sum_j \chi_W^e V^\nu_j \nabla^\mu u_\nu
\]

\[
+ \sum_j \chi_W^{a,j} \pi^{\mu\nu} \nabla_\nu \frac{\mu_j}{T} + \chi_W^{b,j} \pi^{\mu\nu} \nabla_\nu \frac{1}{T} + \chi_W^{c,j} \pi^{\mu\nu} u_\nu + \chi_W^{d,j} \Delta^{\mu\nu} \nabla^\rho \pi_{\nu\rho}
\]

\[
+ \sum_j \chi_W^{a,j} \Pi^{\mu\nu} \nabla_\nu \frac{\mu_j}{T} + \chi_W^{b,j} \Pi^{\mu\nu} \nabla_\nu \frac{1}{T} + \chi_W^{c,j} \Pi^{\mu\nu} D u^\mu + \chi_W^{d,j} \nabla^\mu \Pi
\]
Constitutive Equations

- Charge currents

\[ V_J^\mu = \sum_K \kappa_{VJ} V_K \frac{\mu K}{T} - \kappa_{VJ} W \left( \frac{1}{T} Du^\mu + \nabla^\mu \frac{1}{T} \right) \]

\[ + \sum_K \chi_{VJ} V_K \Delta^\mu \nabla u^\nu \]

\[ = \sum_K \chi_{VJ} V_K \nabla u^\nu \]

\[ + \sum_K \chi_{VJ} V_K \nabla u^\mu \]

\[ + \sum_K \chi_{VJ} V_K \nabla u^\nu \]

\[ - \tau_{VJ} W \Delta^\mu \nabla u^\nu \]

\[ + \sum_K \chi_{VJ} W \nabla^\mu \nabla u^\nu \]

\[ + \sum_K \chi_{VJ} W \nabla^\mu \nabla u^\nu \]

\[ + \sum_K \chi_{VJ} W \nabla^\mu \nabla u^\nu \]

\[ + \sum_K \chi_{VJ} \pi^\mu \nabla^\nu \frac{\mu K}{T} + \sum_K \chi_{VJ} \pi^\mu \nabla^\nu \frac{1}{T} \]

\[ + \sum_K \chi_{VJ} \pi \nabla^\mu \frac{1}{T} \]

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\[ + \sum_K \chi_{VJ} \pi \nabla^\mu \frac{1}{T} \]
Constitutive Equations

- **Shear stress tensor**

\[
\pi^{\mu\nu} = 2\eta \nabla^{\langle \mu} u^{\nu \rangle}
\]

**Relaxation term**

\[
\tau \pi D\pi^{\langle \mu\nu \rangle} + \sum_J \chi_{\pi\pi}^{aJ} \pi^{\mu\nu} D^{\mu\nu} + \chi_{\pi\pi}^{b} \pi^{\mu\nu} D^{\mu\nu} 1 \frac{1}{T} + \chi_{\pi\pi}^{c} \pi^{\mu\nu} \nabla u^\rho + \chi_{\pi\pi}^{d} \pi^{\rho\mu} \nabla \rho^{\nu}
\]

\[
+ \sum_J \chi_{\pi\pi}^{aJ} W^{\langle \mu\nu \rangle} \frac{\mu J}{T} + \chi_{\pi\pi}^{b} W^{\langle \mu\nu \rangle} \frac{1}{T} + \chi_{\pi\pi}^{c} W^{\langle \mu D u^\nu \rangle} + \chi_{\pi\pi}^{d} W^{\langle \mu W^\nu \rangle}
\]

\[
+ \sum_{J,K} \chi_{\pi\pi}^{aJ} V_J^{\langle \mu\nu \rangle} \frac{\mu K}{T} + \sum_J \chi_{\pi\pi}^{b} V_J^{\langle \mu\nu \rangle} \frac{1}{T} + \sum_J \chi_{\pi\pi}^{c} V_J^{\langle \mu D u^\nu \rangle} + \sum_J \chi_{\pi\pi}^{d} V_J^{\langle \mu V_J^\nu \rangle}
\]

\[
+ \chi_{\pi\pi} V^{\langle \mu u^\nu \rangle}
\]

- **Discussion - Relaxation term**

- Linear response ⟷ Acausal and unstable in relativistic systems
- Relaxation effect to limit the propagation faster than the light speed

*Hiscock & Lindblom (’85)*
Discussion - Cross terms

- Coupling of thermodynamic forces in the dissipative currents

\[
W^{\mu} = -\kappa W W \left( \nabla^{\mu} \frac{1}{T} + \frac{1}{T} D u^{\mu} \right) + \sum_{J} \kappa_{WJ} \nabla^{\mu} \frac{\mu_{J}}{T} \\
V_{J}^{\mu} = -\kappa_{VJ} W \left( \nabla^{\mu} \frac{1}{T} + \frac{1}{T} D u^{\mu} \right) + \sum_{K} \kappa_{VJ} V_{K} \nabla^{\mu} \frac{\mu_{K}}{T}
\]

- Onsager reciprocal relations \((\kappa_{WJ} = \kappa_{VJ})\) is satisfied

Cf: “Cooling” process for cooking tasty *oden* (Japanese soup)

Chemical diffusion via thermal gradient \(\Rightarrow\) Soret effect

It should play an important role in our “quark soup”
Discussion – 2\(^{nd}\) order terms

- Comparison with AdS/CFT+phenomenological approach

\[\pi^{\mu\nu} = 2\eta \nabla \langle \mu u^\nu \rangle - \tau_\pi D\pi^{\mu\nu} - \frac{d}{d-1} \tau_\pi \pi^{\mu\nu} \nabla_\rho u^\rho \]

\[+ \frac{\lambda_1}{\eta^2} \pi^{\rho\langle \mu \pi^{\nu} \rangle} - \frac{\lambda_2}{\eta} \pi^{\rho\langle \mu \omega^{\nu} \rangle} + \lambda_3 \omega^{\rho\langle \mu \omega^{\nu} \rangle} \]

-\textit{Our approach goes beyond the limit of conformal theory}

-\textit{Vorticity-vorticity terms do not appear in kinetic theory}

- Comparison with Renormalization group approach

\[\pi^{\mu\nu} = 2\eta \nabla \langle \mu u^\nu \rangle - \tau_\pi D\pi^{\mu\nu} \]

\[+ \tau_\pi \left[ - \frac{1}{2} \frac{T\eta}{\tau_\pi} \partial_\rho \left( \frac{\tau_\pi u^\rho}{T\eta} \right) + \frac{1}{2} \left( - \frac{D\mu}{T} + T\delta_\pi^{(0)} D \frac{1}{T} + \frac{7}{3} \delta_\pi^{(1)} \nabla_\rho u^\rho \right) \right] \pi^{\mu\nu} \]

\[+ \tau_\pi \delta_\pi^{(1)} 4\pi^{\rho\langle \mu \sigma^{\nu} \rangle} + l_\pi V \left[ - \nabla \langle \mu \frac{1}{T} + T \delta_\pi V \left( \nabla \langle \mu \frac{1}{T} + \frac{1}{T} D u^{\langle \mu} \right) \right] V^{\nu} \]

\[- l_\pi V \nabla \langle \mu V^{\nu} \rangle + l_\Pi \nabla \langle \mu u^{\nu} \rangle \Pi \]

-\textit{Consistent, as vorticity terms are added in their recent revision}

-\textit{Frame-dependent equations}
Discussion – 2\textsuperscript{nd} order terms

- Comparison with Grad’s 14-moment approach

\[
\pi^{\mu\nu} = 2\eta \nabla \langle \mu u^\nu \rangle - \tau_\pi D_\pi \langle \mu \nu \rangle \\
- 2\eta \delta_{2} \pi^{\mu\nu} \nabla \lambda u^\lambda - 2\tau_\pi \pi_\lambda \langle \mu \sigma^\nu \rangle^\lambda + 2\tau_\pi \pi_\lambda \langle \mu \omega^\nu \rangle^\lambda \\
- 2\lambda_{\pi q} q \langle \mu \nabla^\nu \rangle \frac{\mu}{T} + 2\tau_\pi q \langle \mu D u^\nu \rangle + 2l_{\pi q} \partial \langle \mu q^\nu \rangle \\
+ 2\lambda_{\pi \Pi \Pi} \Pi \nabla \langle \mu u^\nu \rangle
\]

- The form of their equations are consistent with that of ours
- Multiple conserved currents are not supported in 14-moment method

Consistencies suggest we have successfully extended 2\textsuperscript{nd} order theory to multi-component + conserved current systems
Summary and Outlook

- We formulated generalized 2nd order dissipative hydro from the entropy production w/o violating causality
  
  1. **Multi-component systems with multiple conserved currents**
     
     Inelastic scattering (e.g. pair creation/annihilation) included
  
  2. **1st order cross terms are present**
     
     Onsager reciprocal relations are satisfied
  
  3. **Frame independent**
     
     Independent equations for energy and charge currents

- Future prospects include applications to...
  
  - Numerical estimation of viscous hydrodynamic models for relativistic heavy ion collisions
  
  - Cosmological fluid, and more

AM & T. Hirano, in preparation
The End

Thank you for listening!