

*Arithmetic is being able to count up to twenty without taking off your shoes. – Mickey Mouse*

# CHAPTER 1

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## Properties of Arithmetic

### 1.1 Why Start with Arithmetic?

You know how to add, subtract, multiply, and divide. In fact, you may already know how to solve many of the problems in this chapter. So why do we start this book with an entire chapter on arithmetic?

To answer this question, go back to the title of this book: *Prealgebra*. What is prealgebra? Not everybody agrees on what “prealgebra” means, but we (the writers of this book) like to think of prealgebra as the bridge between *arithmetic* and *algebra*.

*Arithmetic* refers to the basics of adding, subtracting, multiplying, dividing, and (maybe) more exotic things like squares and square roots. You probably learned most of these basics already. The hardest thing that you usually do in arithmetic is “word problems” like “If Jenny has 5 apples and Timmy has 7 apples, then how many apples do they have together?” As you get older, the numbers get bigger, but the problems don’t really get much harder. Arithmetic is great when trying to solve simple problems like counting apples. But when the problems get more complicated—like trying to compute a rocket’s trajectory, or trying to analyze a financial market, or trying to count the number of ways a text message can be routed through a cellular phone network—we need a more advanced toolbox.

That toolbox is *algebra*. Algebra is the language of all advanced mathematics. Algebra gives us tools to take our concepts from arithmetic and make them *general*, meaning that we can use the concepts not just for arithmetic problems, but for other sorts of problems, too.

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To take a simple example, you can use arithmetic to show that

$$2 \times (3 + 5) = (2 \times 3) + (2 \times 5),$$

because the left side equals  $2 \times 8$ , which is 16, and the right side equals  $6 + 10$ , which is also 16. But algebra gives us the much more general tool that

$$a \times (b + c) = (a \times b) + (a \times c),$$

no matter what numbers  $a$ ,  $b$ , and  $c$  are. And in higher mathematics,  $a$ ,  $b$ , and  $c$  might not even be numbers as you recognize them now, but might be more complicated mathematical objects. (Even more generally, “+” and “ $\times$ ” might not mean addition and multiplication as you think of them now, but instead might represent more complicated mathematical operations. But we’re getting ahead of ourselves a little bit!)

So our initial goal, in Chapter 1, is to carefully lay down the rules of arithmetic, and to give you some ideas as to *why* these rules are true. Once you know the rules, you’ll be ready to start thinking algebraically.

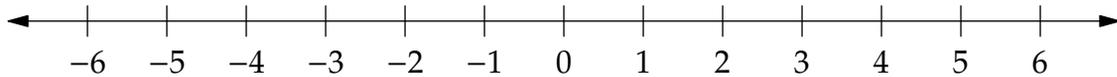
Also, by the time you start reading this book, you are mathematically mature enough to start thinking about not just *how* to perform various calculations, but *why* the techniques used in those calculations work. Understanding why mathematics works is the key to solving harder problems. If you only understand how techniques work but not why they work, you’ll have a lot more difficulty modifying those techniques to solve more complicated problems. So, throughout this book, we will rarely just tell you how something works—we’ll usually show you why it works.

By the end of this chapter, you should be able to explain *why* the following computations are true:

- $(-5) \times (-7) = 35$  (and not  $-35$ )
- $(1990 \times 1991) - (1989 \times 1990) = 3980$  (and be able to compute this in your head!)
- $8 \div \frac{1}{7} = 56$
- $(4 \times 10 \times 49) \div (2 \times 5 \times 7) = 28$  (again, in your head!)

You’ll know all these things not because you’ve blindly applied some calculation, or because you’ve memorized some formula—instead, you’ll understand the mathematics behind all these expressions.

Unfortunately, different mathematicians and different textbooks may use slightly different words for the same concept, in the same way that what an American calls a “truck” is called a “lorry” by people in Great Britain. So, before we go any farther, we want to make sure that we all agree on some of the words that we’re going to use.



The **number line** is shown above. It goes on forever in both directions. Every number that we will consider in this book is somewhere on the number line. The tick marks on the number line above indicate the **integers**. An integer is a number without a fractional part:

$$\dots, -7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, \dots$$

(The symbol  $\dots$  at either end of the above list means that the list goes on forever in that direction. The  $\dots$  symbol is called an **ellipsis**.) However, as you know, there are many numbers on the number line other than integers. For instance, most fractions such as  $\frac{1}{2}$  are not integers. (But some fractions are integers—we'll explore this further in Chapter 4.)

A number is called **positive** if it is to the right of 0 on the number line. In other words, a number is positive if it is greater than 0. A number is called **negative** if it is to the left of 0 on the number line. That is, a number is negative if it is less than 0. For example, 2 is positive, while  $-2$  is negative. Note that 0 itself is neither positive nor negative, and that every number is either positive, negative, or 0.

A number is called **nonnegative** if it is not negative. In other words, a nonnegative number is positive or 0. Similarly, a number is called **nonpositive** if it is not positive. Finally, a number is called **nonzero** if it is not equal to 0. Note that 0 is nonnegative and nonpositive.

**Sidenote:** A lot of people use the term **whole number** to mean a nonnegative integer. In other words, a whole number is one of the numbers 0, 1, 2,  $\dots$ . These same people use the term **natural number** to mean a positive integer. In other words, a natural number is one of the numbers 1, 2, 3,  $\dots$ .

However—and this is the really irritating part—a lot of people use the term **natural number** to mean a nonnegative integer. In other words, a natural number is one of the numbers 0, 1, 2,  $\dots$ . These same people use the term **whole number** to mean a positive integer. In other words, a whole number is one of the numbers 1, 2, 3,  $\dots$ .

And some of both of these groups of people might also use the term **counting number** to mean either whole number or natural number.

These people have been arguing for centuries, and they will likely never agree. Since we don't like to argue, we will stick with the very clear terms **positive integers** for the numbers 1, 2, 3,  $\dots$ , and **nonnegative integers** for the numbers 0, 1, 2,  $\dots$ .

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### 1.2 Addition

We'll start by exploring the simplest arithmetic operation: **addition**. There's not a whole lot to explore, but as we'll see, a solid knowledge of the basic properties of addition makes complicated-looking calculations easy.

**Important:** **How to use this book:** Most sections will begin with problems, like those shown below. You should first try to solve the problems. Then, continue reading the section, and compare your solutions to the solutions presented in the book.



#### Problems

**Problem 1.1:** Using the two pictures below, explain why  $2 + 3 = 3 + 2$ .



**Problem 1.2:** Using the two pictures below, explain why  $(2 + 3) + 4 = 2 + (3 + 4)$ .



Reminder: The parentheses tell you what to compute first. For example,  $(3 + 4) \times 5$  equals  $7 \times 5$ , whereas  $3 + (4 \times 5)$  equals  $3 + 20$ .

**Problem 1.3:**

(a) Using the properties from Problems 1.1 and 1.2, explain why

$$472 + (219 + 28) = (472 + 28) + 219.$$

(b) Compute the sum  $472 + (219 + 28)$ . (Source: MATHCOUNTS)

**Problem 1.4:** Compute  $(2 + 12 + 22 + 32) + (8 + 18 + 28 + 38)$ . (Source: MATHCOUNTS)

**Problem 1.5:** Find the sum  $1 + 2 + 3 + \cdots + 19 + 20$ . Reminder: The ellipsis  $\cdots$  means that we should include all the numbers in the pattern. So we are adding the positive integers from 1 to 20.

**Problem 1.6:** Using the picture below, explain why  $2 + 0 = 2$ .



**Problem 1.1:** Using the two pictures below, explain why  $2 + 3 = 3 + 2$ .



*Solution for Problem 1.1:* First let's look at the picture on the left. The first row has 2 squares; the second row has 3 squares. So in total, there are  $2 + 3$  squares.

Now let's look at the picture on the right. The first row has 3 squares; the second row has 2 squares. In total, there are  $3 + 2$  squares.

The picture on the right, however, is just an upside-down version of the picture on the left. Flipping a picture upside down doesn't change the number of squares. So we conclude that  $2 + 3 = 3 + 2$ .  $\square$

Whenever we add two numbers, the order of the numbers does not matter. For example,  $5 + 17 = 17 + 5$  and  $32 + 999 = 999 + 32$ . There are an infinite number of such examples. Of course, we can't write down an infinite number of examples. Instead, we can write the equation

$$\text{first number} + \text{second number} = \text{second number} + \text{first number}.$$

This is a long equation to write. We can shorten it by letting  $a$  represent the first number and  $b$  represent the second number. Then our equation becomes

$$a + b = b + a.$$

Here  $a$  and  $b$  stand for any numbers (and possibly the same number). For instance, if we let  $a = 2$  and  $b = 3$ , then we have our original example:  $2 + 3 = 3 + 2$ . If we let  $a = 100$  and  $b = 200$ , then we get  $100 + 200 = 200 + 100$ . Note that  $a$  has the same value throughout the equation, as does  $b$ ; both, however, may change from one problem to the next. Letters such as  $a$  and  $b$  that represent numbers are called **variables**.

The rule that  $a + b = b + a$  for all numbers  $a$  and  $b$  is called the **commutative property** of addition.

**Important: Addition is commutative:** Let  $a$  and  $b$  be numbers. Then



$$a + b = b + a.$$

In Problem 1.1, we explained why  $a + b = b + a$  is true for one particular example, when  $a = 2$  and  $b = 3$ . But one example doesn't prove that  $a + b = b + a$  for all  $a$  and  $b$ . We don't have the

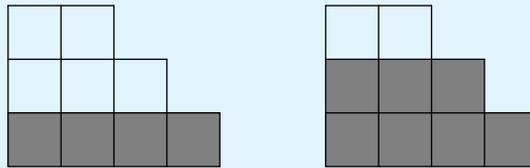
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tools in this book to explain why this must hold for any two numbers, but Problem 1.1 should give you good intuition for why it is true for positive integers.

Throughout the rest of this chapter, we will explore many more arithmetic rules. We will use examples and pictures to give intuition for why these rules work. These examples and pictures are not proofs, but they should give you a feel for why these rules must be true.

The commutative property is concerned with adding two numbers. What if we add three numbers?

**Problem 1.2:** Using the two pictures below, explain why  $(2 + 3) + 4 = 2 + (3 + 4)$ .



Reminder: The parentheses tell you what to compute first. For example,  $(3 + 4) \times 5$  equals  $7 \times 5$ , whereas  $3 + (4 \times 5)$  equals  $3 + 20$ .

*Solution for Problem 1.2:* For each picture, we will first count the number of light squares, then count the number of dark squares, and finally add the two counts.

Let's start with the picture on the left. It has  $(2 + 3)$  light squares and 4 dark squares. So altogether it has  $(2 + 3) + 4$  squares.

Next, let's look at the picture on the right. It has 2 light squares and  $(3 + 4)$  dark squares. So altogether it has  $2 + (3 + 4)$  squares.

What's the difference between the two pictures? The only difference is the color of the middle row. Changing the color doesn't change the number of squares. So we conclude that  $(2 + 3) + 4 = 2 + (3 + 4)$ .  $\square$

We get a similar equation for any three numbers:  $(a + b) + c = a + (b + c)$ . In other words, first adding  $a$  and  $b$  and then adding  $c$  is the same as adding  $a$  to  $b + c$ . This property is called the **associative property** of addition.

**Important:** Addition is associative: Let  $a$ ,  $b$ , and  $c$  be numbers. Then



$$(a + b) + c = a + (b + c).$$

**WARNING!!** Students sometimes mix up the names "commutative" and "associative." In the commutative property, the numbers are moved around ("commuted") on the two sides of the equation. In the associative property, the numbers stay in the same place, but are grouped ("associated") differently.

Together, the commutative and associative properties are sneakily powerful, as they let us add a list of numbers in any order. The next problem is an illustration of this **any-order principle**.

**Problem 1.3:**

- (a) Using the properties from Problems 1.1 and 1.2, explain why

$$472 + (219 + 28) = (472 + 28) + 219.$$

- (b) Compute the sum  $472 + (219 + 28)$ . (Source: MATHCOUNTS)

*Solution for Problem 1.3:*

- (a) Let's start with the left side of the equation:

$$472 + (219 + 28).$$

We will try to make it look like the right side. To do that, we need to switch the order of the 219 and the 28. We can do so by the commutative property. In other words, we replace  $472 + (219 + 28)$  with the equal quantity

$$472 + (28 + 219).$$

This expression is close to what we want:  $(472 + 28) + 219$ . In fact, these two expressions are equal by the associative property.

Let's combine the pieces of our explanation into a nice chain of equations:

$$\begin{aligned} 472 + (219 + 28) &= 472 + (28 + 219) && \text{commutative property} \\ &= (472 + 28) + 219. && \text{associative property} \end{aligned}$$

- (b) By part (a), the quantities  $472 + (219 + 28)$  and  $(472 + 28) + 219$  are equal, so we can compute  $(472 + 28) + 219$  instead. But this is easy to compute: the sum  $472 + 28$  is 500, so we are left with  $500 + 219$ . The answer is 719.

□

The point of Problem 1.3 is that we can rearrange the numbers in our addition to make the addition easier to compute. It's easier to first compute  $472 + 28$ , and then compute  $500 + 219$ , than it would have been to start with  $219 + 28$  and then add that sum to 472.

In a similar way, any addition problem can be rearranged without changing the sum. Usually we won't bother to write all the individual steps of the rearrangement, like we did in Problem 1.3. Instead, we'll use our knowledge of the commutative and associative properties to just go ahead and rearrange a sum in whatever way is best. Let's apply that principle to solve the next problem.

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**Problem 1.4:** Compute  $(2 + 12 + 22 + 32) + (8 + 18 + 28 + 38)$ . (Source: MATHCOUNTS)

*Solution for Problem 1.4:* We could start with 2, then add 12, then add 22, and so on, but that's too much work. Instead, let's try to rearrange the sum in a useful way. Let's pair up the numbers so that each pair has the same sum. Specifically, let's pair each number in the first group with a number in the second group:

$$(2 + 38) + (12 + 28) + (22 + 18) + (32 + 8).$$

The first pair of numbers adds up to 40; so does the second pair, the third pair, and the fourth pair. So our sum becomes

$$40 + 40 + 40 + 40.$$

The answer is 160.  $\square$

Let's use the any-order principle to compute a longer sum.

**Problem 1.5:** Find the sum  $1 + 2 + 3 + \cdots + 19 + 20$ . Reminder: The ellipsis  $\cdots$  means that we should include all the numbers in the pattern. So we are adding the positive integers from 1 to 20.

*Solution for Problem 1.5:* We definitely don't want to add the 20 numbers one at a time. Instead, let's try again to rearrange the numbers into pairs, so that each pair has the same sum. We pair the smallest number with the largest, the second-smallest with the second-largest, and so on:

$$(1 + 20) + (2 + 19) + (3 + 18) + \cdots + (10 + 11).$$

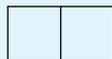
We have grouped the 20 numbers into 10 pairs. Each pair adds up to 21. So our sum becomes

$$21 + 21 + 21 + 21 + 21 + 21 + 21 + 21 + 21 + 21.$$

Adding 10 copies of 21 is the same as multiplying 10 and 21. So the answer is 210.  $\square$

Finally, let's look at one more property of addition. What happens when we add zero to a number?

**Problem 1.6:** Using the picture below, explain why  $2 + 0 = 2$ .



*Solution for Problem 1.6:* On one hand, there are 2 squares. On the other hand, we can say there are 2 light squares and 0 dark squares. So we get the equation  $2 + 0 = 2$ .  $\square$

Adding zero to any number doesn't change the number.

**Important: Adding zero:** Let  $a$  be a number. Then



$$a + 0 = a.$$

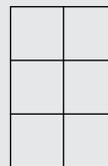
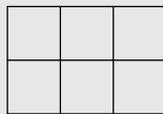
**Exercises**

- 1.2.1 Compute  $99 + 99 + 99 + 101 + 101 + 101$ .
- 1.2.2 Compute  $1999 + 2001 + 1999 + 2001 + 1999 + 2001 + 1999 + 2001$ .
- 1.2.3 Compute  $(3 + 13 + 23 + 33 + 43) + (7 + 17 + 27 + 37 + 47)$ .
- 1.2.4 Compute  $(1 + 2 + 3 + \cdots + 49 + 50) + (99 + 98 + 97 + \cdots + 51 + 50)$ .

## 1.3 Multiplication

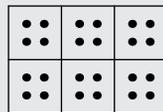
**Problems**

**Problem 1.7:** Using the two pictures below, explain why  $2 \times 3 = 3 \times 2$ .



**Problem 1.8:** By counting the dots in the picture below in two different ways, explain why

$$(2 \times 3) \times 4 = 2 \times (3 \times 4).$$



**Problem 1.9:** Compute  $25 \times 125 \times 4 \times 6 \times 8$ .

**Problem 1.10:** Using the picture below, explain why  $2 \times 1 = 2$ .



**Problem 1.11:**

- (a) Compute  $(5 + 6) \times 7$ .
- (b) Compute  $5 + (6 \times 7)$ .