In Section 8.6 we'll explore geometric explanations for these possible outcomes.

Exercises

5.3.1 Solve each of the following systems of equations:

(a)
$$\begin{aligned} 3x - 7y &= 14,\\ 2x + 7y &= 6. \end{aligned}$$

(b)
$$\begin{aligned} 5u &= -7 - 2v,\\ 3u &= 4v - 25. \end{aligned}$$

(c)
$$\begin{aligned} \frac{2x}{13} + 2y &= -2(y+1),\\ -\frac{3x}{13} &= -5(6-y). \end{aligned}$$

(d)
$$\begin{aligned} -2.5a + 5b &= 25,\\ 42 + 10b &= 15 + 3.75a + 4b. \end{aligned}$$

5.3.2 Describe all solutions to each of the following systems of equations:

(a)
$$\begin{aligned} 2x + 3y &= 7,\\ 14x &= 49 - 21y. \end{aligned}$$

(b)
$$\begin{aligned} \frac{3x}{5} - \frac{4y}{5} &= 3,\\ 8y - 6x &= 5. \end{aligned}$$

5.3.3 For what value of the constant a does the system of equations below have infinitely many solutions?

$$\begin{aligned} 2x + 5y &= -8,\\ 6x &= 16 + a - 15y. \end{aligned}$$

5.3.4★ Let a, b, c, d, and e be constants in the system of equations

$$\begin{aligned} ax + by &= d,\\ ax + cy &= e. \end{aligned}$$

Suppose b and c are not equal and a is not 0. Must the system of equations have exactly one solution (x, y)?

5.4 Word Problems

Back in Section 3.3, we tackled word problems by converting them into one-variable linear equations. Sometimes, one variable isn't enough! Sometimes, we need to define a second variable, too. But the key step is still the same:

Concept: Convert the words into mathematics.

Extra! Simplify the expression $(x-a)(x-b)(x-c)(x-d)\cdots(x-z)$. *Solution on page 149.*

Problems

Problem 5.10: A football game was played between two teams, the Cougars and the Panthers. The two teams scored a total of 34 points, and the Cougars won by a margin of 14 points. How many points did the Panthers score? *(Source: AMC 12)*

(a) Let the Cougars' score be c and the Panthers' score be p. Write two equations using the information in the problem.

(b) Solve the equations you found in the first part.

(c) Check your answer; does your solution fit the information in the problem?

Problem 5.11: Marianna has only nickels and quarters in her piggy bank. Their combined value is \$9.15. Their combined weight is one pound. Ninety nickels weigh one pound. Eighty quarters weigh one pound. How many nickels does Marianna have in her piggy bank? *(Source: MATHCOUNTS)*

Problem 5.12: Tweedledum says, "The sum of your weight and twice mine is 361 pounds." Tweedledee says, "Contrariwise, the sum of your weight and twice mine is 362 pounds." If they are both correct, how much do Tweedledum and Tweedledee weigh together? *(Source: MATHCOUNTS)*

Problem 5.13: Two years ago, Gene was nine times as old as Carol. He is now seven times as old as she is.

(a) Find Gene's and Carol's ages now.

(b) In how many years from now will Gene be five times as old as Carol?

(Source: Mandelbrot)

Problem 5.10: A football game was played between two teams, the Cougars and the Panthers. The two teams scored a total of 34 points, and the Cougars won by a margin of 14 points. How many points did the Panthers score? *(Source: AMC 12)*

Solution for Problem 5.10: We could just keep guessing possibilities until we find the answer, but a little algebra finds the answer quickly. We convert the words to math by first defining two variables, one for each team:

Let c be the Cougars' score.
Let p be the Panthers' score.

Now we convert the language in the problem into the language of mathematics. The two teams scored a total of 34 points:

$$c + p = 34.$$

The Cougars won by 14 points:

$$c - p = 14.$$

Adding the equations gives $2c = 48$, so $c = 24$. Substituting this into either equation gives $p = 10$. Our solution is $(c, p) = (24, 10)$, so the Panthers scored 10 points. □

Notice that we didn't stop at noting $(c, p) = (24, 10)$. We answered the question asked by stating that the Panthers scored 10 points.

Important: Make sure you answer the question that is asked.

Notice also that we choose c for Cougars and p for Panthers in solving the last problem, rather than using x and y.

Concept: Choose variables that are related to their meanings so you can remember what they stand for.

This isn't such a big deal for simple problems, but as the problems get more complex it will help prevent errors and save time.

Problem 5.11: Marianna has only nickels and quarters in her piggy bank. Their combined value is \$9.15. Their combined weight is one pound. Ninety nickels weigh one pound. Eighty quarters weigh one pound. How many nickels does Marianna have in her piggy bank? *(Source: MATHCOUNTS)*

Solution for Problem 5.11: We first define our variables:

Let n be the number of nickels in the piggy bank.
Let q be the number of quarters in the piggy bank.

Each nickel is worth \$0.05 and each quarter is worth \$0.25, so

$$0.05n + 0.25q = 9.15.$$

We can get rid of decimals by multiplying by 100:

$$5n + 25q = 915.$$

Now we can divide by 5 to simplify the equation:

$$n + 5q = 183.$$

That's much nicer than $0.05n + 0.25q = 9.15$.

Concept: Don't work with ugly equations if you don't have to; manipulate them into nicer-looking equations.

We need another equation, so we turn to the weight information. Our change together weighs a pound. We are given that each nickel is 1/90 of a pound and each quarter is 1/80 of a pound, so

$$\frac{n}{90} + \frac{q}{80} = 1.$$

We can make this equation nicer to work with by multiplying by 720 to get rid of the fractions:

$$8n + 9q = 720.$$

Now we're ready to solve. From our first equation, we have $n = 183 - 5q$. Substituting this into the second equation gives

$$8(183 - 5q) + 9q = 720.$$

Expanding and simplifying the left side gives $1464 - 31q = 720$, and solving this equation gives us $q = 24$. We then substitute this into our expression for n to find $n = 183 - 5q = 183 - 5(24) = 63$, so the piggy bank has 63 nickels.

As a quick check, we note that $63/90 + 24/80 = 7/10 + 3/10 = 1$, so our solution does give us the correct weight of coins. □

Problem 5.12: Tweedledum says, "The sum of your weight and twice mine is 361 pounds." Tweedledee says, "Contrariwise, the sum of your weight and twice mine is 362 pounds." If they are both correct, how much do Tweedledum and Tweedledee weigh together? *(Source: MATHCOUNTS)*

Solution for Problem 5.12: As usual, we define the variables first:

Let e be Tweedledee's weight.
Let m be Tweedledum's weight.

From Tweedledum's statement, we have

$$e + 2m = 361.$$

From Tweedledee's statement, we have

$$2e + m = 362.$$

We could use either substitution or multiply one equation by −2 and use elimination, but the similar forms of the two equations gives us an idea. Let's try adding the two equations as is, which will let us find $e + m$:

$$\begin{array}{rcrcl} e & + & 2m & = & 361 \\ 2e & + & m & = & 362 \\ \hline 3e & + & 3m & = & 723 \end{array}$$

Dividing this equation by 3 gives us $e + m = 241$. We could go on and find Tweedledee's and Tweedledum's weights, but looking back at the problem, all we're asked for is the sum of their weights. We already have that, since we found $e + m = 241$. Therefore, the two together weigh 241 pounds. □

Concept: Keep your eye on the ball. Make sure you know what you're looking for in a problem, so you know when you've found what you need. Sometimes you don't even need to find all the variables you define in order to answer a question.

Notice that adding the equations in the previous problem not only answers the question, but also makes the original system easier to solve. We can easily use our new equation to eliminate e or m from one of our original equations. We'll explore combining equations in clever ways to make systems of equations easy to solve in Chapter 22.

In our next problem, we see that sometimes setting up equations and solving them is only a first step in solving a word problem.

Problem 5.13: Two years ago, Gene was nine times as old as Carol. He is now seven times as old as she is. How many years from now will Gene be five times as old as Carol? *(Source: Mandelbrot)*

Solution for Problem 5.13: Word problems are sometimes called "story problems." We start solving such a story by giving mathematical names to our characters:

Let c be Carol's age.
Let g be Gene's age.

What's wrong with this solution:

Bogus Solution: Gene was nine times older than Carol, so $g = 9c$. Two years from then, he will be seven times older than she is, so $g + 2 = 7(c + 2)$. We solve these equations by substituting $g = 9c$ into our second equation:

$$9c + 2 = 7(c + 2).$$

Solving this equation gives $c = 6$, so $g = 9c = 54$.

We wish to know in how many years Gene will be five times as old as Carol. We let t be the number of years from now until Gene is five times as old as Carol. Since Gene is now 54 and Carol is now 6, we must have

$$54 + t = 5(6 + t).$$

Solving this equation gives $t = 6$, so Gene will be five times as old as Carol 6 years from now.

All the algebraic manipulation in this bogus solution is correct, but our answer is incorrect. Our mistake is that we use c and g to mean Carol's and Gene's ages *two years ago* (when Gene's age is 9 times Carol's) to solve for c and g, but then use c and g to mean their ages *now* when we answer the question.

WARNING!! Define your variables clearly and stick to your definitions throughout the problem.

And now, back to our story. We name our mathematical characters by defining our variables:

Let c be Carol's age *now*.
Let g be Gene's age *now*.

Since Gene is now seven times as old as Carol, we have

$$g = 7c.$$

Two years ago, Gene was $g - 2$ years old and Carol was $c - 2$ years old. Gene was also then nine times as old as Carol, so

$$g - 2 = 9(c - 2).$$

Substituting $g = 7c$ into this equation gives $7c - 2 = 9(c - 2)$, from which we find $c = 8$. Since Carol is 8 years old now, Gene is $7c = 56$ years old now. We use this information to answer the question.

We let t be the number of years until Gene is five times as old as Carol. In t years, Gene will be $56 + t$ years old and Carol will be $8 + t$. Therefore, we have

$$56 + t = 5(8 + t).$$

Solving this equation gives $t = 4$, so Gene will be five times as old as Carol in 4 years.

Why did our Bogus Solution give us an answer that is exactly 2 years greater then our correct solution? □

Problem 5.13 is really two word problems in one. After solving for Gene's and Carol's ages, we then had a second problem to solve: finding the number of years until Gene is five times as old as Carol. To tackle this second problem, we had to define a new variable.

Concept: You won't always realize all the variables you need at the beginning of a problem. Define new variables as you need them.

Exercises

5.4.1 Find Tweedledum's and Tweedledee's weights in Problem 5.12.

5.4.2 My parents started a small farm after they retired. On their farm, they have chickens and pigs. In total, there are 40 animal legs among the chickens and the pigs, and there are 16 animal heads. How many chickens do my parents have?

5.4.3 The sum of Eric's and Bob's weights is 9 times the difference of their weights. The positive difference of their weights is also 240 pounds less than the sum. If Eric weighs less than Bob, find Bob's weight.

5.4.4 5 green balls and 2 red balls together weigh 10 pounds, and 1 green ball and 4 red balls together weigh 7 pounds. If all red balls weigh the same amount and all green balls weigh the same, then what is the weight of 8 red and 8 green balls together?

5.4.5 At a certain time, Janice notices that her digital watch reads a minutes after two o'clock. Fifteen minutes later, it reads b minutes after three o'clock. She is amused to note that a is six times b. What time was it when she looked at her watch for the second time? *(Source: Mandelbrot)*