for which our graph is below the x-axis with the points where the graph intersects the x-axis (because the inequality is nonstrict), we have the same answer as before, $x \in (-\infty, -5] \cup [-4, 4]$.

Our consideration of the graph of $y = f(x)$ is essentially the same approach to the problem as the table. In both, we determine the sign of $f(x)$ for different intervals of x. We use the roots of $f(x)$ to determine the intervals we must consider. □

Exercises

7.2.1 Find all roots of each of the following polynomials.

(a) $f(x) = x^3 - 4x^2 - 11x + 30$

(b) $g(t) = t^4 + 5t^3 - 19t^2 - 65t + 150$

(c) $f(x) = -x^5 - 12x^4 + 6x^3 + 64x^2 - 93x + 36$

(d) $h(y) = 6y^3 - 5y^2 - 22y + 24$

7.2.2 Find all solutions to the inequalities below.

(a) $t^3 + 10t^2 + 17t > 28$

(b) $r^2(6r + 12 - r^2) \leq 5(14r + 15)$

(c) $r^4 + r^3 + 5r^2 - 13r - 18 \leq 0$

7.2.3 There are four roots of $f(x) = x^4 - 8x^3 + 24x^2 - 32x + 16$. We can easily test to find that $f(2) = 0$. We can then check all the other divisors of 16, both positive and negative, and find that no divisors of 16 besides 2 are roots of the polynomial. Is it correct to deduce that the other three roots of $f(x)$ are not integers?

7.2.4★ Suppose that $f(x)$ is a polynomial with integer coefficients such that $f(2) = 3$ and $f(7) = -7$. Show that $f(x)$ has no integer roots. **Hints:** 1, 319

7.3 Rational Roots

A **rational number** is a number that can be expressed in the form p/q, where p and q are integers. A number that is not rational is called an **irrational number**. Just as we can use the coefficients of a polynomial with integer coefficients to narrow the search for integer roots, we can also use these coefficients to narrow the search for non-integer rational roots.

Problems

Problem 7.12: In this problem we find the roots of the polynomial $g(x) = 12x^3 + 16x^2 - 31x + 10$.

(a) Can you find any integers n such that $g(n) = 0$?

(b) Suppose $g(p/q) = 0$, where p and q are integers and p/q is in reduced form. Rewrite $g(p/q) = 0$ using the definition of g.

(c) Get rid of the fractions in your equation from (b) by multiplying by the appropriate power of q.

(d) What terms in your equation from (c) have p? Why must p divide 10?

(e) What terms in your equation from (c) have q? Why must q divide 12?

(f) Find all the roots of $g(x)$.

Problem 7.13: Let $f(x)$ be the polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$, where all the coefficients are integers and both a_n and a_0 are nonzero. Let p and q be integers such that p/q is a fraction in simplest terms, and $f(p/q) = 0$. In this problem, we show that p divides a_0 and q divides a_n.

(a) Suppose $f(p/q) = 0$. Why must we have

$$a_np^n + a_{n-1}p^{n-1}q + a_{n-2}p^{n-2}q^2 + a_{n-3}p^{n-3}q^3 + \cdots + a_1pq^{n-1} + a_0q^n = 0?$$

(b) Which terms in the equation in (a) must be divisible by p? Use this to show that a_0 must be divisible by p.

(c) Which terms in the equation in (a) must be divisible by q? Use this to show that a_n must be divisible by q.

Problem 7.14: Find all the roots of each of the following polynomials:

(a) $f(x) = 12x^3 - 107x^2 - 15x + 54$.

(b) $g(x) = 30x^4 - 133x^3 - 121x^2 + 189x - 45$.

Problem 7.15: Find all r such that $12r^4 - 16r^3 > 41r^2 - 69r + 18$.

We've found a way to narrow our search for integer roots. How about rational roots?

Problem 7.12: Find all x such that $12x^3 + 16x^2 - 31x + 10 = 0$.

Solution for Problem 7.12: We let $f(x) = 12x^3 + 16x^2 - 31x + 10$ and begin our search for roots of $f(x)$ by seeing if we can find any integer roots. An integer root of $f(x)$ must be a divisor of its constant term, which is $10 = 2^1 \cdot 5^1$. We try each divisor (positive and negative):

$$\begin{aligned}
f(1) &= 12 \cdot 1^3 + 16 \cdot 1^2 - 31 \cdot 1 + 10 = 7,\\
f(-1) &= 12 \cdot (-1)^3 + 16 \cdot (-1)^2 - 31 \cdot (-1) + 10 = 45,\\
f(2) &= 12 \cdot 2^3 + 16 \cdot 2^2 - 31 \cdot 2 + 10 = 108,\\
f(-2) &= 12 \cdot (-2)^3 + 16 \cdot (-2)^2 - 31 \cdot (-2) + 10 = 40,\\
f(5) &= 12 \cdot 5^3 + 16 \cdot 5^2 - 31 \cdot 5 + 10 = 1755,\\
f(-5) &= 12 \cdot (-5)^3 + 16 \cdot (-1)^2 - 31 \cdot (-5) + 10 = -935,\\
f(10) &= 12 \cdot 10^3 + 16 \cdot 10^2 - 31 \cdot 10 + 10 = 13300,\\
f(-10) &= 12 \cdot (-10)^3 + 16 \cdot (-10)^2 - 31 \cdot (-10) + 10 = -10080.
\end{aligned}$$

(We might also have checked some of these a little faster with synthetic division.) We know we don't have to check any other integers, since any integer root must divide the constant term of $f(x)$. Since all of these fail, we know there are no integer roots of $f(x)$. Back to the drawing board.

A little number theory helped us find integer roots, so we hope that a similar method might help us find rational roots. Letting p and q be relatively prime integers (meaning they have no common positive divisor besides 1), we examine the equation $f(p/q) = 0$:

$$f\left(\frac{p}{q}\right) = 12\left(\frac{p}{q}\right)^3 + 16\left(\frac{p}{q}\right)^2 - 31\left(\frac{p}{q}\right) + 10 = 0.$$

Multiplying by q^3 to get rid of all the denominators, we have the equation

$$12p^3 + 16p^2q - 31pq^2 + 10q^3 = 0. \tag{7.1}$$

Previously, we isolated the last term of such an expansion in order to find information about integer roots. Doing so here gives us

$$10q^3 = p(-12p^2 - 16pq + 31q^2).$$

Dividing this equation by p gives us

$$\frac{10q^3}{p} = -12p^2 - 16pq + 31q^2.$$

The right side of this equation must be an integer, so $10q^3/p$ must also be an integer. Therefore, p divides $10q^3$. But, since p and q are relatively prime, p must divide 10.

If we isolate the first term of the left side of Equation (7.1) instead of the last term, we have

$$12p^3 = q(-16p^2 + 31pq - 10q^2).$$

Dividing this equation by q gives us

$$\frac{12p^3}{q} = -16p^2 + 31pq - 10q^2.$$

As before, the right side is an integer, so the left side must be as well. Thus q divides $12p^3$. Because p and q are relatively prime, we know that q must divide 12.

Combining what we have learned about p and q, we see that if p/q is a root of $f(x)$, then p is a divisor of 10 and q is a divisor of 12.

Now, we make a list of possible rational roots p/q:

$$\pm 1, \pm 2, \pm 5, \pm 10, \pm\frac{1}{2}, \pm\frac{5}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}, \pm\frac{10}{3}, \pm\frac{1}{4}, \pm\frac{5}{4}, \pm\frac{1}{6}, \pm\frac{5}{6}, \pm\frac{1}{12}, \pm\frac{5}{12}.$$

While this is not a terribly small list, it is a *complete list of possible rational roots of a polynomial with leading coefficient 12 and constant term 10*. We already know that $f(x)$ has no integer roots, so we continue our search for roots with the fractions in the list above. We start with the fractions with small denominators, since these will be easiest to check.

After a little experimentation, we find that $f(1/2) = 0$ via the synthetic division at right. (You may have found one of the other roots first.) We therefore have

$$\begin{array}{c|cccc} \frac{1}{2} & 12 & 16 & -31 & 10 \\ & & 6 & 11 & -10 \\ \hline & 12 & 22 & -20 & \multicolumn{1}{|c}{0} \end{array}$$

$$f(x) = \left(x - \frac{1}{2}\right)(12x^2 + 22x - 20).$$

We continue our search by finding roots of $12x^2 + 22x - 20$. We find $12x^2 + 22x - 20 = 2(6x^2 + 11x - 10) = 2(3x - 2)(2x + 5)$, so we have

$$f(x) = 2\left(x - \frac{1}{2}\right)(3x - 2)(2x + 5).$$

When factoring polynomials, we often factor constants out of each linear factor so that the coefficient of x in each factor is 1. This makes identifying the roots particularly easy:

$$f(x) = 2\left(x - \frac{1}{2}\right)(3)\left(x - \frac{2}{3}\right)(2)\left(x + \frac{5}{2}\right) = 12\left(x - \frac{1}{2}\right)\left(x - \frac{2}{3}\right)\left(x + \frac{5}{2}\right).$$

Now we can easily see that the roots of $f(x)$ are $\frac{1}{2}$, $\frac{2}{3}$, and $-\frac{5}{2}$. □

In our solution, we found a way to narrow our search for rational roots of $f(x)$ in much the same way we can narrow our search for integer roots. Let's see if we can apply this method to any polynomial with integer coefficients.

Problem 7.13: Let $f(x)$ be the polynomial $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$, where all the coefficients are integers and both a_n and a_0 are nonzero. Let p and q be integers such that p/q is a fraction in simplest terms, and $f(p/q) = 0$. Show that p divides a_0 and q divides a_n.

Solution for Problem 7.13: We use our solution to Problem 7.12 as a guide. Since $f(p/q) = 0$, we have

$$a_n\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + a_{n-2}\left(\frac{p}{q}\right)^{n-2} + \cdots + a_1\left(\frac{p}{q}\right) + a_0 = 0.$$

We get rid of the fractions by multiplying both sides by q^n, which gives us

$$a_np^n + a_{n-1}p^{n-1}q + a_{n-2}p^{n-2}q^2 + \cdots + a_1pq^{n-1} + a_0q^n = 0. \tag{7.2}$$

To see why q must divide a_n, we divide Equation (7.2) by q to produce a_n/q in the first term:

$$\frac{a_np^n}{q} + a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q + \cdots + a_1pq^{n-2} + a_0q^{n-1} = 0.$$

Isolating this first term gives us

$$\frac{a_np^n}{q} = -a_{n-1}p^{n-1} - a_{n-2}p^{n-2}q - \cdots - a_1pq^{n-2} - a_0q^{n-1}.$$

Each term on the right side of this equation is an integer, so the entire right side is an integer. Therefore, a_np^n/q must be an integer. Since p/q is in lowest terms and a_np^n/q is an integer, we know that q divides a_n.

We take essentially the same approach to show that $p \mid a_0$. We divide Equation (7.2) by p and isolate the term with a_0 in it, which gives us

$$\frac{a_0q^n}{p} = -a_np^{n-1} - a_{n-1}p^{n-2}q - a_{n-2}p^{n-3}q^2 - \cdots - a_1q^{n-1}.$$

Each term on the right is an integer, so the entire right side equals an integer. So, the expression a_0q^n/p must equal an integer, which means that p must divide a_0. □

Extra! If $f(n) = n^2 + n + 41$, then $f(1)$, $f(2)$, $f(3)$, and $f(4)$ are all prime. Maybe $f(n)$ is prime for all integers n. Is it? *(Continued on page 220)*

In Problem 7.13, we have proved the **Rational Root Theorem**:

> **Important:** Let $f(x)$ be the polynomial
> $$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0,$$
> where all the a_i are integers, and both a_n and a_0 are nonzero. If p and q are relatively prime integers and $f(p/q) = 0$, then $p \mid a_0$ and $q \mid a_n$.

Problem 7.14: Find all the roots of each of the following polynomials:

(a) $f(x) = 12x^3 - 107x^2 - 15x + 54$

(b) $g(x) = 30x^4 - 133x^3 - 121x^2 + 189x - 45$

Solution for Problem 7.14:

(a) We first look for easy roots by evaluating $f(1)$ and $f(-1)$. We find that $f(1) = 12-107-15+54 = -56$ and $f(-1) = -12 - 107 + 15 + 54 = -50$. Since $f(0) = 54$, we see that there is a root between -1 and 0 (because $f(-1) < 0$ and $f(0) > 0$) and a root between 0 and 1 (because $f(0) > 0$ and $f(1) < 0$).

> **Concept:** We often start our hunt for rational roots by evaluating $f(-1)$, $f(0)$, and $f(1)$, because these are easy to compute and the results may tell us where to continue our search.

We continue our hunt for roots with $\frac{1}{2}$, which may be a root because 2 divides the leading coefficient of $f(x)$. Using synthetic division, we find $f(\frac{1}{2}) = 21\frac{1}{4}$, so there is a root between $\frac{1}{2}$ and 1. We try $\frac{2}{3}$ and it works, giving us

$$f(x) = \left(x - \frac{2}{3}\right)(12x^2 - 99x - 81) = 3\left(x - \frac{2}{3}\right)(4x^2 - 33x - 27).$$

Factoring the quadratic then gives us $f(x) = 3\left(x - \frac{2}{3}\right)(4x+3)(x-9)$. So, the roots of $f(x)$ are $-\frac{3}{4}$, $\frac{2}{3}$, and 9.

(b) We start our hunt for roots by finding $g(0) = -45$, $g(1) = -80$, and $g(-1) = -192$. Unfortunately, that doesn't help too much; we can't immediately tell if there are any roots between -1 and 0, or between 0 and 1. So, we test 3 and -3, knowing that 2 and -2 cannot be roots because the constant term of $g(x)$ is odd. By synthetic division, we find that $g(3) = -1728$. This has the same sign as $g(1)$, so we continue with $g(-3)$. We find that $g(-3) = 4320$, so because $g(-3) > 0$ and $g(-1) < 0$, we know there is a root between -3 and -1. (Note that we didn't need to find the actual value of $g(-3)$. Once it is clear that $g(-3)$ is a large positive number, we know that there is a root between -1 and -3.) With synthetic division, we find that $g(-2)$ is also a large positive number, so there is a root between -2 and -1.

$$\begin{array}{r|rrrr|r} -\frac{3}{2} & 30 & -133 & -121 & 189 & -45 \\ & & -45 & 267 & -219 & 45 \\ \hline & 30 & -178 & 146 & -30 & 0 \end{array}$$

We try $-3/2$. We know that $-3/2$ might be a root of $g(x)$ because -3 divides the constant term and 2 divides the leading coefficient. If $-3/2$ doesn't work, we will at least narrow our search to between -2 and

$-3/2$ or between $-3/2$ and -1. Fortunately, the synthetic division shows us that

$$g(x) = \left(x + \frac{3}{2}\right)(30x^3 - 178x^2 + 146x - 30) = 2\left(x + \frac{3}{2}\right)(15x^3 - 89x^2 + 73x - 15).$$

We now note that $15x^3 - 89x^2 + 73x - 15$ has no negative roots because if x is negative, each term of this polynomial is negative. So, we only have to search for positive roots. We already know that 1 and 3 don't work, and the only two other positive integers that divide 15 are 5 and 15. We find that 5 works, and we have

$$g(x) = 2\left(x + \frac{3}{2}\right)(x - 5)(15x^2 - 14x + 3).$$

Now, we can factor the quadratic or use the quadratic formula to find

$$g(x) = 2\left(x + \frac{3}{2}\right)(x - 5)(3x - 1)(5x - 3) = 30\left(x + \frac{3}{2}\right)(x - 5)\left(x - \frac{1}{3}\right)\left(x - \frac{3}{5}\right).$$

The roots of $g(x)$ are $-3/2$, $1/3$, $3/5$, and 5. Notice that there are roots between 0 and 1. Why doesn't testing $g(0)$ and $g(1)$ reveal the fact that these roots exist? (You'll have a chance to answer this question as an Exercise.)

□

Problem 7.15: Find all r such that $12r^4 - 16r^3 > 41r^2 - 69r + 18$.

Solution for Problem 7.15: First, we move all the terms to the left, which gives us

$$12r^4 - 16r^3 - 41r^2 + 69r - 18 > 0.$$

Now, we must factor the polynomial on the left. Let this polynomial be $f(r)$. Because $f(0) = -18$ and $f(1) = 6$, there is a root between 0 and 1. We find that $f(1/2) = 5$, so the root is between 0 and $1/2$. Trying $1/3$ gives us

$$f(r) = \left(r - \frac{1}{3}\right)(12r^3 - 12r^2 - 45r + 54) = 3\left(r - \frac{1}{3}\right)(4r^3 - 4r^2 - 15r + 18).$$

Continuing with $4r^3 - 4r^2 - 15r + 18$, we find that -1 and 2 are not roots, but -2 is, and we have

$$f(2) = 3\left(r - \frac{1}{3}\right)(r + 2)(4r^2 - 12r + 9) = 3\left(r - \frac{1}{3}\right)(r + 2)(2r - 3)^2.$$

So, our inequality is $3\left(r - \frac{1}{3}\right)(r + 2)(2r - 3)^2 > 0$.

The expression on the left side of the inequality equals 0 when $r = -2$, $1/3$, or $3/2$. We consider the intervals between (and beyond) these values to build the table at right. We see that $f(r) > 0$ for $r \in (-\infty, -2) \cup (\frac{1}{3}, \frac{3}{2}) \cup (\frac{3}{2}, +\infty)$. The inequality is strict, so the roots of $f(r)$ do not satisfy the inequality. □

	$r + 2$	$r - \frac{1}{3}$	$(2r - 3)^2$	$f(r)$
$r < -2$	$-$	$-$	$+$	$+$
$-2 < r < \frac{1}{3}$	$+$	$-$	$+$	$-$
$\frac{1}{3} < r < \frac{3}{2}$	$+$	$+$	$+$	$+$
$r > \frac{3}{2}$	$+$	$+$	$+$	$+$

Exercises

7.3.1 Find all roots of the following polynomials:

(a) $g(y) = 12y^3 - 28y^2 - 9y + 10$ (b) $f(x) = 45x^3 + 48x^2 + 17x + 2$

7.3.2 In part (b) of Problem 7.14, we saw that $g(0)$ and $g(1)$ are both negative, and yet we still found roots between 0 and 1. How can that be? Shouldn't $g(0)$ and $g(1)$ have different signs if $g(x) = 0$ for some x between 0 and 1?

7.3.3 Suppose $f(x) = x^3 + \frac{3}{4}x^2 - 4x - 3$. Notice that $f(2) = 0$, so 2 is a root of $f(x)$. But 2 doesn't evenly divide the constant term of $f(x)$. This seems to violate the Rational Root Theorem! Does it really?

7.3.4 In Problem 7.15, we built a table to find the values of r for which $3(r - \frac{1}{3})(r + 2)(2r - 3)^2 > 0$. How could we have solved this inequality by thinking about the graph of $y = 3(x - \frac{1}{3})(x + 2)(2x - 3)^2$?

7.3.5 Solve the two inequalities below:

(a) $24x^3 + 26x^2 \geq 21x + 9$ (b)★ $6s^4 + 13s^3 - 2s^2 + 35s - 12 < 0$

7.4 Bounds

In this section, we learn how to use synthetic division to determine bounds on the roots of a polynomial.

Problems

Problem 7.16:

(a) Let $f(x) = x^3 + 13x^2 + 39x + 27$. Without even finding the roots, how can we tell that there are no positive values of x for which $f(x) = 0$?

(b) Let $f(x) = 3x^3 - 28x^2 + 51x - 14$. Without finding the roots, how can we tell that there are no negative values of x for which $f(x) = 0$?

Problem 7.17: Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$.

(a) Show that if all the coefficients of $f(x)$ have the same sign (positive or negative), then $f(x)$ has no positive roots.

(b) Show that if $a_i > 0$ for all odd i and $a_i < 0$ for all even i, then $f(x)$ has no negative roots.

(c) Is it possible for $f(x)$ to have negative roots if a_i is negative for all odd i and positive for all even i?

Problem 7.18: Let $g(x) = x^4 - 3x^3 - 12x^2 + 52x - 48$.

(a) Use synthetic division to divide $g(x)$ by $x - 6$.

(b) What do you notice about the coefficients in the quotient and the remainder from part (a)?

(c) How can you use your answer to part (b) to deduce that there are no roots of $g(x)$ greater than 6? (In other words, how do you know you don't have to test 8, 12, 16, etc.?)