2.8: Sticks and Stones

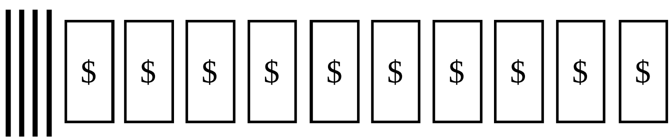
Another great counting trick best explained with an example:

Example: Uncle Henry has ten one-dollar bills to distribute among his five youngest nieces and nephews. How many ways are there to distribute his money?

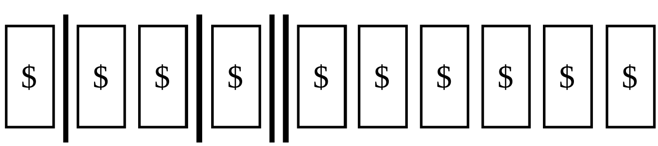
Reasoning:

He could give all of the money to one child (five ways).
He could give two dollars to each child (one way).
He could give \$0.00, \$1.00, \$2.00, \$3.00, and \$4.00 5! ways.

This is getting us nowhere, there must be a trick!
Consider the following diagram:



Henry will arrange the dividers on the left with the bills on the right. Money to the left of the first divider goes to the youngest, money between the first and second divider goes to the next child, etc. For example, in the arrangement below, the first child gets \$1, the second gets \$2, the third gets \$1, the fourth gets nothing, and the fifth gets \$6.



Using this model, it is simply a matter of choosing the number of arrangements of ten bills and four dividers:

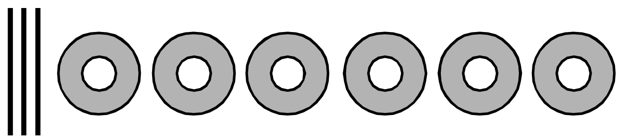
$$\frac{14!}{(10!)(4!)} = 1{,}001, \quad \text{or:} \quad {}_{14}C_4 = 1{,}001.$$

Think of this as the number of ways to place 4 dividers into 14 blanks, filling the remaining 10 blanks with bills.

Sticks and Stones: Placing dividers (sticks) to separate indistinguishable items (stones) into categories.

Example: You are ordering a half-dozen doughnuts (6), and need to choose from among four flavors: glazed, powdered, cream-filled, and jelly-filled. How many different doughnut orders are possible?

Reasoning: Once again, even with just six doughnuts and four flavors the casework is virtually impossible to do quickly and get right. We will use our divider trick again. This is the same as finding the number of arrangements of dividers and doughnuts below.



The math is the same as finding the number of letter arrangements of AAABBBBBB, or placing 3 dividers (make sure you know why there are 3 not 4 dividers) in 9 blanks (doughnuts fill the other 6):

$$\frac{9!}{(3!)(6!)} = 84 \text{ or } \binom{9}{3} = 84.$$

Practice: Sticks and Stones.

2.051 There are six different choices on the dollar menu at Burgerboy: fries, apple pies, onion rings, chicken nuggets, hamburgers, and cheeseburgers. You decide to purchase four of these items. How many different combinations of four items can you buy from the dollar menu?

2.151 You are purchasing a dozen roses for Valentine’s day. The roses come in red, white, and pink and you want at least one of each color. How many different bouquets of one dozen roses are possible?

Sticks and Stones with restrictions.

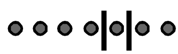
Sometimes it is useful to use a sticks-and-stones approach in some unexpected situations:

Example: How many ways are there to roll a sum of 7 with three standard six-faced dice?

Reasoning:

This is typically solved using casework: There are $6^3 = 216$ ways to roll three standard die, and we can list the ways to roll a 7 and count the ways to roll each: 5-1-1 (3 ways), 4-2-1 (6 ways), 3-2-2 (3 ways), or 3-3-1 (3 ways) for a total of 15 ways.

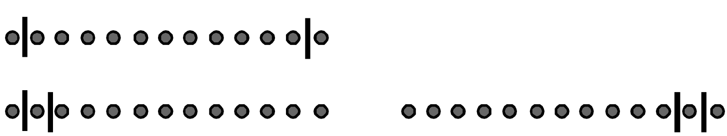
Consider the problem with dividers placed between 7 dots. This time, we cannot place 2 dividers adjacent to each other, so there are 6 available gaps between dots in which we can place 2 dividers in ${}_6C_2 = 15$ ways to represent rolling a 7 with 3 dice. Below is an example of rolling 4-1-2.



Example: Each of the numbers 1 through 10 is placed in a bag and drawn at random with replacement. How many ways can three numbers be drawn whose sum is 13?

Reasoning:

There are many more ways to do this, and counting cases is not practical. Consider a diagram similar to the one above, only with 13 dots. There are three ways that the dividers *cannot* be placed which are shown below (because you cannot draw an 11):



This leaves ${}_{12}C_2$ minus 3 = 63 ways for the sum to equal 13.

Practice: Sticks and Stones.

2.061 How many ways can a dozen hundred-dollar bills be distributed among five numbered briefcases?

2.161 You are playing a racing video game. To begin, you get to adjust the tuning of your race car by adding a combined total of ten points to three categories. You can adjust your speed, handling, and acceleration by adding anywhere from 0 to 10 points to each category. How many tuning options are there for the car's initial setup?

2.261 Uncle Henry is feeling generous and has decided to distribute four $1 bills, three $5 bills, two $20 bills, and a $100 bill to three nephews. How many different ways can he distribute the money?

2.361 A standard six-faced die is rolled four times. How many different ways are there to roll a 9?

2.461 Three friends: Al, Bill, and Carl, are painting their chests with the word CAVALIERS for a basketball game, with at least two letters painted on each friend's chest. For example, Al might get "CA", Bill "VAL", and Carl "IERS", or Carl could get "CAVA" while Bill gets "LI" and Al gets "ERS". How many ways can the three friends get their chests painted so that when they stand in the correct order, they spell-out the word CAVALIERS?

2.561 An elaborate code consists of left and right arrows which are placed together to represent words. There are spaces between words. For example, the code of three left and four right arrows arranged with two spaces: << >>> <> represents a three-word code phrase. How many different three-word code phrases in this system use 8 letters?