Definition: Let f be a continuous function and $b \in \mathbb{R}$ such that $(-\infty, b) \subseteq \text{Dom}(f)$. We define the **improper integral**

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_{a}^{b} f(x)\,dx,$$

provided the limit is defined. If the limit is defined and is not $\pm\infty$, we say that the improper integral **converges**. Otherwise, we say that the improper integral **diverges**.

Let's generalize Problem 6.16:

Problem 6.17: Let r be a real number. Compute

$$\int_{1}^{\infty} \frac{1}{x^r}\,dx.$$

Solution for Problem 6.17: By definition, we compute the improper integral by writing a limit. If $r \neq 1$, then we have:

$$\lim_{b\to\infty} \int_{1}^{b} \frac{1}{x^r}\,dx = \lim_{b\to\infty} -\frac{1}{(r-1)x^{r-1}}\bigg|_{1}^{b}.$$

This equals

$$\lim_{b\to\infty} \frac{1}{r-1}\left(1 - \frac{1}{b^{r-1}}\right).$$

If $r > 1$, then the term $\frac{1}{b^{r-1}}$ approaches 0 as b approaches ∞. Thus, in this case, the improper integral converges to $\frac{1}{r-1}$.

If $r < 1$, then the term $\frac{1}{b^{r-1}}$ grows without bound as b approaches ∞. Thus, the integral diverges. We might also write

$$\int_{1}^{\infty} \frac{1}{x^r}\,dx = \infty \qquad \text{if } r < 1.$$

Our original integration was not valid for $r = 1$, so we have to do that case separately:

$$\lim_{b\to\infty} \int_{1}^{b} \frac{1}{x}\,dx = \lim_{b\to\infty} (\log x)\bigg|_{1}^{b} = \lim_{b\to\infty} (\log b).$$

As b goes towards infinity, this grows without bound, so the integral diverges.

In summary:

$$\int_{1}^{\infty} \frac{1}{x^r}\,dx = \begin{cases} \frac{1}{r-1} & \text{if } r > 1, \\ \text{diverges} & \text{if } r \leq 1. \end{cases}$$

□

The next problem is another common example of an improper integral:

Problem 6.18: Compute $\displaystyle\int_{0}^{\infty} e^{ax}\,dx$, where a is a real number.

Solution for Problem 6.18: We compute, for $a \neq 0$ (we'll investigate $a = 0$ at the end):

$$\lim_{b\to\infty} \int_{0}^{b} e^{ax}\,dx = \lim_{b\to\infty} \frac{1}{a}(e^{ab} - 1) = \frac{1}{a} \lim_{b\to\infty} (e^{ab} - 1).$$

If a is positive, then $\lim_{b\to\infty} e^{ab} = \infty$, so the integral diverges. If a is negative, then $\lim_{b\to\infty} e^{ab} = 0$, so the integral equals $-\frac{1}{a}$. (Note this is a positive number when a is negative, so this answer makes sense.) Finally, if $a = 0$, then the integral is $\int_0^\infty 1\,dx$, which clearly diverges.

Thus, the integral diverges for nonnegative exponents, and converges for negative exponents. □

The result of Problem 6.18 is typically written as follows: if $r > 0$, then

$$\int_0^\infty e^{-rx}\,dx = \frac{1}{r}.$$

Problem 6.19: Suppose f and g are continuous functions on $[a, \infty)$ and $f(x) \le g(x)$ for all $x \ge a$.

(a) Show that, if $\int_a^\infty f$ and $\int_a^\infty g$ both converge, then

$$\int_a^\infty f \le \int_a^\infty g.$$

(b) Show that if both functions are positive, and $\int_a^\infty g$ converges, then $\int_a^\infty f$ converges.

(c) Show that if both functions are positive, and $\int_a^\infty f$ diverges, then $\int_a^\infty g$ diverges.

Solution for Problem 6.19:

(a) For any $b \ge a$, we have $(g - f)(x) \ge 0$ for all $x \in [a, b]$, thus

$$\int_a^b (g - f)(x)\,dx \ge 0.$$

Therefore,

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx,$$

and since limits preserve non-strict inequalities, we conclude that

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx \le \lim_{b\to\infty} \int_a^b g(x)\,dx = \int_a^\infty g(x)\,dx.$$

(b) Define a function

$$F(x) = \int_a^x f(t)\,dt.$$

Note that F is an increasing function (since $f(x) \ge 0$ for all $x \ge a$), and that $\int_a^\infty f(x)\,dx = \lim_{x\to\infty} F(x)$, if this limit exists. Also, since $0 \le f(x) \le g(x)$ for all $x \ge a$, we have

$$0 \le F(x) = \int_a^x f(t)\,dt \le \int_a^x g(t)\,dt \le \int_a^\infty g(t)\,dt.$$

Thus F is increasing and has an upper bound (namely, $\int_a^\infty g(t)\,dt$, which by assumption converges), so by the result of Problem 6.5, the limit

$$\lim_{x\to\infty} F(x) = \int_a^\infty f(t)\,dt$$

exists, so the integral converges.

(c) This is just the contrapositive statement to part (b), so there is nothing additional to prove.

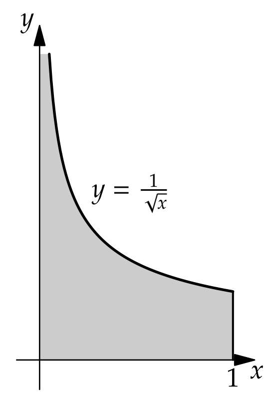
□

WARNING!! We can only use the comparison tests in parts (b) and (c) of Problem 6.19 if both functions are positive. As a trivial example, if $f(x) = -1$ and $g(x) = 0$, then for any $a \in \mathbb{R}$, $\int_a^\infty g = 0$, so $\int_a^\infty g$ converges, but $\int_a^\infty f$ diverges.

Thus far in this section, we have looked at improper integrals that compute areas of regions that are unbounded in the x-direction. There is another type of improper integral that occurs when the region that we are examining is unbounded in the y-direction, as in the following example:

Problem 6.20: Compute $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$.

Solution for Problem 6.20: Sketching the graph will immediately show the issue. We have $\displaystyle\lim_{x\to 0^+} \frac{1}{\sqrt{x}} = \infty$. So the area under $y = \frac{1}{\sqrt{x}}$ is potentially infinite (and in fact the function is not even defined at 0).



We can do essentially the same thing we did for improper integrals with a limit of integration of $\pm\infty$. We define

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0^+} \int_c^1 \frac{1}{\sqrt{x}}\,dx.$$

Note the "0^+"—since we only care about the interval $(0,1]$, we only care about what happens to the right of 0.

This integral is now easy to compute:

$$\int_c^1 \frac{1}{\sqrt{x}}\,dx = 2\sqrt{x}\Big|_c^1 = 2 - 2\sqrt{c}.$$

As $c \to 0^+$, this approaches 2. Hence

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.$$

Once again, a seemingly infinite area turns out to be finite. □

We can generalize the definition from Problem 6.20:

Definition: Suppose f is a function, continuous on $(a, b]$, such that $\displaystyle\lim_{x\to a^+} f(x) = \pm\infty$. We define the **improper integral**

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx,$$

provided this limit is defined. If the limit is defined, we say that this improper integral **converges**, and if it is undefined, we say that the improper integral **diverges**.

Of course, we can do the same thing if the function has a limit of $\pm\infty$ at the "b" end of $[a, b)$. (We will omit writing out the formal definition.)

Sidenote: Note that the above definition is consistent with our usual (non-improper) integrals. In particular, if $\int_a^b f$ is defined, then by the Fundamental Theorem of Calculus, the function

$$g(x) = \int_x^b f(t)\,dt$$

is differentiable, hence continuous, and thus

$$\int_a^b f(t)\,dt = g(a) = \lim_{x\to a^+} g(x) = \lim_{x\to a^+} \int_x^b f(t)\,dt.$$

We know that for regular (not improper) integrals, we can break them apart at any point into two separate integrals. Specifically, if $c \in (a,b)$, then

$$\int_a^b f = \int_a^c f + \int_c^b f.$$

This is also how we evaluate integrals that are improper at both ends, as in the following example:

Problem 6.21: Compute $\displaystyle\int_0^\infty \frac{1}{x^r}\,dx$ for all $r > 0$ (or determine when it diverges).

Solution for Problem 6.21: The correct thing to do with an integral that is improper at both ends is to split it somewhere in the middle. For example, we can write

$$\int_0^\infty \frac{1}{x^r}\,dx = \int_0^1 \frac{1}{x^r}\,dx + \int_1^\infty \frac{1}{x^r}\,dx.$$

(We didn't have to pick $x = 1$ as the point at which to split them, but it seems convenient since x^r is nicely behaved at $x = 1$.) We already know by Problem 6.17 that $\int_1^\infty \frac{1}{x^r}\,dx$ converges if and only if $r > 1$. The other integral is

$$\int_0^1 \frac{1}{x^r}\,dx = \lim_{a\to 0+} \int_a^1 \frac{1}{x^r}\,dx = \lim_{a\to 0+} \left(-\frac{1}{(r-1)x^{r-1}} \bigg|_a^1 \right) = \frac{1}{r-1} \lim_{a\to 0+} \left(\frac{1}{a^{r-1}} - 1 \right).$$

If $r > 1$, then the fraction gets arbitrarily large, so the limit is infinite. Thus $\int_0^1 \frac{1}{x^r}\,dx$ diverges for $r > 1$.

Hence our original doubly-improper integral is never convergent: the integral on $(0,1]$ diverges for $r > 1$, and the integral on $[1,\infty)$ diverges for $r \le 1$. □

Important: If $(a,b) \subseteq \text{Dom}(f)$ and $\displaystyle\int_a^b f(t)\,dt$ is improper at both ends of (a,b), then

$$\int_a^b f(t)\,dt = \lim_{x\to a^+} \int_x^c f(t)\,dt + \lim_{x\to b^-} \int_c^x f(t)\,dt,$$

for any $c \in (a,b)$.

As noted in the solution to Problem 6.21, it doesn't matter at which point we break up the doubly-improper integral.

Concept: We can break an integral apart as

$$\int_a^b f = \int_a^c f + \int_c^b f$$

at any $c \in (a, b)$ that we choose. Thus, choose c to be as convenient as possible.

We will leave it as an exercise to prove this. Also, it is not correct to try to take a shortcut and deal with both ends of a double-improper integral at once. In particular:

WARNING!! $\int_{-\infty}^{\infty} f(x)\,dx$ is not the same as $\lim_{a\to\infty} \int_{-a}^{a} f(x)\,dx$.

The correct way to evaluate an integral over all of $\mathbb{R}$ is to choose $c \in \mathbb{R}$, and then compute

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx = \lim_{a\to-\infty} \int_{a}^{c} f(x)\,dx + \lim_{b\to+\infty} \int_{c}^{b} f(x)\,dx.$$

We will leave it as an exercise to explore this further.

We also have to be a bit cautious when dealing with functions with domains that are not all of $\mathbb{R}$. Integrals of such functions might be improper but not immediately appear so. For example:

Problem 6.22: Compute $\displaystyle\int_{-2}^{3} \frac{1}{x^2}\,dx$.

Solution for Problem 6.22: If you weren't paying close attention, you might do this:

Bogus Solution:

$$\int_{-2}^{3} \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_{-2}^{3} = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}.$$

We can't do this, because the function is not defined at 0! To be a little more precise, the function $\frac{1}{x^2}$ does not have an antiderivative on the interval $[-2, 3]$, because it is not defined at $x = 0$, so we cannot apply the Fundamental Theorem of Calculus.

In order to evaluate the integral, we need to break it up into a sum of two improper integrals at the point at which the function is undefined:

$$\int_{-2}^{3} \frac{1}{x^2}\,dx = \int_{-2}^{0} \frac{1}{x^2}\,dx + \int_{0}^{3} \frac{1}{x^2}\,dx.$$

As we saw in Problem 6.21, both of these diverge. Thus, the original integral itself diverges. □

More generally, when computing something like $\displaystyle\int_{-1}^{1} \frac{dx}{x}$, it might be tempting to say "$\frac{1}{x}$ is an odd function, so the integral from -1 to 0 will cancel out the integral from 0 to 1, and thus the overall integral is 0." This is also the result that naive calculation will give:

Bogus Solution:

$$\int_{-1}^{1} \frac{dx}{x} = \log|x|\bigg|_{-1}^{1} = \log(1) - \log(1) = 0.$$

But this is not correct! The only way legally to evaluate this integral is to break it up into its improper parts.

$$\int_{-1}^{1}\frac{dx}{x}=\int_{-1}^{0}\frac{dx}{x}+\int_{0}^{1}\frac{dx}{x}.$$

Neither part converges, so the integral diverges.

Exercises

6.5.1 Compute the following improper integrals:

(a) $\displaystyle\int_3^\infty \frac{1}{(2x-1)^2}\,dx$ (b) $\displaystyle\int_2^\infty \frac{1}{x(\log x)^2}\,dx$ (c) $\displaystyle\int_0^\infty xe^{-x^2}\,dx$ (d) $\displaystyle\int_0^2 \frac{1}{4-x^2}\,dx$

6.5.2

(a) Compute $\displaystyle\int_0^\infty \frac{1}{1+x^2}\,dx$.

(b) Compute $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$.

6.5.3 Compute $\displaystyle\int_0^\infty x^2e^{-x}\,dx$.

6.5.4 Show that it doesn't matter at which point we break up a doubly-improper integral. Specifically, show that, for any $c,d\in(a,b)$, if $\int_a^c f$ and $\int_c^b f$ converge, then $\int_a^d f$ and $\int_d^b f$ also converge, and

$$\int_a^c f+\int_c^b f=\int_a^d f+\int_d^b f.$$

Hints: 230, 97

6.5.5★

(a) Show that if $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ converges, then $\displaystyle\int_{-\infty}^\infty f(x)\,dx=\lim_{a\to\infty}\int_{-a}^a f(x)\,dx$. **Hints:** 165

(b) Show that the converse of part (a) is not true; that is, it is possible that $\displaystyle\lim_{a\to\infty}\int_{-a}^a f(x)\,dx$ converges but that $\displaystyle\int_{-\infty}^\infty f(x)\,dx$ diverges. **Hints:** 82, 225

Review Problems

6.23 Compute the following:

(a) $\displaystyle\lim_{x\to1}\frac{x^2-1}{\log x}$ (b) $\displaystyle\lim_{x\to0}\frac{\cos^2 x-1}{x^2}$ (c) $\displaystyle\lim_{x\to0}\frac{10x^2-\frac{1}{2}x^3}{e^{4x^2}-1}$ *(Source: Rice)*

6.24 Suppose a and b are nonzero real numbers. Find $\displaystyle\lim_{t\to0}\frac{\sin at}{\sin bt}$ and $\displaystyle\lim_{t\to0}\frac{\tan at}{\tan bt}$. **Hints:** 32

6.25 Compute

(a) $\displaystyle\int_1^\infty e^{-2x}\,dx$ (b) $\displaystyle\int_0^2\frac{1}{x^3}\,dx$ (c) $\displaystyle\int_{-\infty}^\infty\frac{1}{x^2+2x+2}\,dx$ **Hints:** 72