The term **related rates** refers to two quantities that are dependent on each other and that are changing over time. We can use the dependent relationship between the quantities to determine a relationship between their rates of change.

Here is a basic example:

**Problem 4.30:** A balloon in the shape of a sphere is being inflated at the rate of 2 cm$^3$/sec. At the time at which the radius of the balloon is 3 cm, how fast is its radius increasing?

*Solution for Problem 4.30:* Let $r$ denote the radius of the balloon (in cm) and $V$ be its volume (in cm$^3$). This already is a little bit deceptive, since both of the quantities are changing over time. So what we should really do is let $r(t)$ denote the radius and $V(t)$ denote the volume, where $r$ and $V$ are functions of $t$, and where $t$ denotes the time (in seconds). The relationship between the quantities is

$$V(t) = \frac{4}{3}\pi(r(t))^3.$$

What we want is an expression that relates the *rates of change* of these quantities. For that, we take the derivative of the entire equation with respect to $t$; that is, we take

$$\frac{d}{dt}V(t) = \frac{d}{dt}\left(\frac{4}{3}\pi(r(t))^3\right).$$

The left side of this equation is just $\frac{dV}{dt}$. On the right side, we must use the Chain Rule:

$$\frac{d}{dt}\left(\frac{4}{3}\pi(r(t))^3\right) = \frac{4}{3}\pi\frac{d}{dt}((r(t))^3) = 4\pi(r(t))^2\frac{dr}{dt}.$$

Thus $\frac{dV}{dt} = 4\pi(r(t))^2\frac{dr}{dt}$. Now we can plug in our given data. The rate of change of the volume is given as a constant 2 cm$^3$/sec, so $\frac{dV}{dt} = 2$. Also, at the time $t$ we are interested in, we are given $r(t) = 3$. So we plug these in:

$$2 = 4\pi(3)^2\frac{dr}{dt}.$$

Thus, we solve to get $\frac{dr}{dt} = \frac{1}{18\pi}$. So the radius is increasing by $\frac{1}{18\pi}$ cm/sec. □

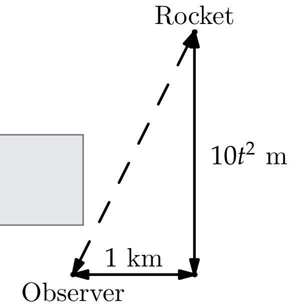
This is the basic premise behind all related rates problems. We have two quantities, both of which are a function of the same independent variable (usually time). If there is an equation relating the two quantities, then we can take the implicitly differentiate that equation with respect to the independent variable to get an equation relating the variables and their derivatives.

In related rates problems, it is customary not to write explicitly the $(t)$ part of the notation. For example, in Problem 4.30, we would typically write the equation relating the quantities as simply $V = \frac{4}{3}\pi r^3$, omitting the mention of the variable $t$. Then, the derivative of this equation is $V' = 4\pi r^2 r'$. However, it's important to remember that in this expression, $V$ and $r$ (and $r'$) are quantities that depend on $t$.

**Problem 4.31:** An observer is 1 kilometer away from a rocket launch pad. At time $t = 0$, the rocket lifts off straight upwards, and at time $t$ seconds has achieved an altitude of $10t^2$ meters. At 5 seconds after takeoff, how fast is the distance between the rocket and the observer increasing?

*Solution for Problem 4.31:* Even though there is only one moving object—the rocket—there are two quantities that we are interested in that are changing with respect to time: the altitude of the rocket (which we'll call $h$) and the distance from the rocket to the observer (which we'll call $x$).

The picture at right shows how these quantities are related. But don't make the following mistake:




**Bogus Solution:**

$$x^2 = h^2 + 1.$$

We have to be more careful about the units! The observer is 1 *kilometer* from the launch pad, but the height of the rocket is given in *meters*. So, if we want to express everything in meters, we have the equation

$$x^2 = h^2 + (1000)^2.$$

Taking the derivative of both sides with respect to $t$, we get

$$2x\frac{dx}{dt} = 2h\frac{dh}{dt}.$$

We are given $t = 5$. But $t$ doesn't directly appear as a term in the above expression. So we need to figure out what the quantities in the above equation are in terms of $t$.

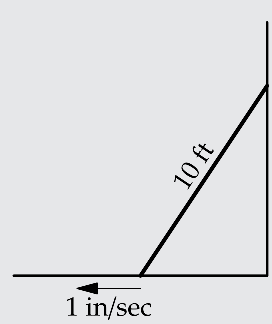
When $t = 5$, we have $h = 10t^2 = 250$. This means that $x^2 = (250)^2 + (1000)^2 = 17(250)^2$, so $x = 250\sqrt{17}$. Also, $\frac{dh}{dt} = 20t = 100$. Thus, we have

$$\begin{aligned}
& & 2x\frac{dx}{dt} &= 2h\frac{dh}{dt} \\
&\Rightarrow & x\frac{dx}{dt} &= h\frac{dh}{dt} \\
&\Rightarrow & (250\sqrt{17})\frac{dx}{dt} &= (250)(100) \\
&\Rightarrow & \frac{dx}{dt} &= \frac{100}{\sqrt{17}} = \frac{100\sqrt{17}}{17}.
\end{aligned}$$

So the distance between the rocket and the observer is increasing at a rate of $\frac{100\sqrt{17}}{17}$ m/sec. □

The next problem is a classic related rates problem:

**Problem 4.32:** A 10-foot-tall ladder rests against a wall, but the foot of the ladder is slipping away from the wall at a rate of 1 in/sec. When the ladder forms a 60° degree angle to the ground, how fast is the top of the ladder sliding down?



*Solution for Problem 4.32:* We should set this up to have as our two "variables" the functions whose rates we care about. Therefore, we should use $w$ (the distance from the wall) and $h$ (the height of the top of the ladder). We can then write an equation relating these quantities:

$$w^2 + h^2 = 100.$$

But this uses feet as the units on the right side. Since the rate is in in/sec, maybe we should write the whole thing in inches:

$$w^2 + h^2 = 14400.$$

Although it is always good practice to keep our units consistent, it doesn't really matter here, because the next step is to take the derivative with respect to time:

$$2ww' + 2hh' = 0.$$

Now we need to plug in the values of these quantities so that we can solve for $h'$. We know that $w' = 1$(in/sec). We need to figure out $w$ and $h$. But the other bit of information that we haven't used yet is the $60^\circ$ angle. This gives us $w = 5$ (feet) and $h = 5\sqrt{3}$ (feet). But the rate of change is in inches per second! So we need to use $w = 60$ (inches) and $h = 60\sqrt{3}$ (inches). We then plug these values in to our related rates expression:

$$2(60)(1) + 2(60\sqrt{3})h' = 0.$$

Solving for $h'$ gives $h' = -\frac{1}{\sqrt{3}}$. So the top of the ladder is falling at a rate of $\frac{1}{\sqrt{3}}$ inches per second. □

By the way, in Problem 4.32, what does the function $h'(t)$ look like in general? We can solve for $h'$ in our related rates equation:

$$h' = -w'\frac{w}{h} = -\frac{w}{h},$$

since $w' = 1$. But we know what the quantity $w/h$ is: it's the cotangent of the angle that the ladder makes with the ground. So we have

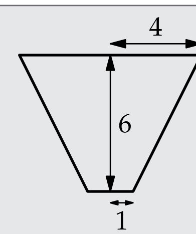
$$h' = -\cot\theta,$$

where $\theta$ is the angle that the ladder makes with the ground.

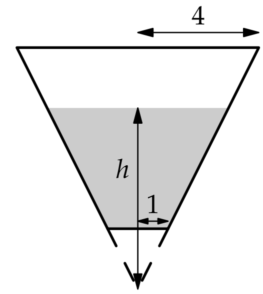
This makes "real world" sense. At $\theta = \pi/2$, the ladder is vertical, and $\cot(\pi/2) = 0$, meaning that the top of the ladder is stationary. As $\theta$ decreases, $\cot\theta$ increases, so the top of the ladder moves faster and faster downwards.

Let's look at one more related rates problem that's a bit harder.

**Problem 4.33:** A cone-shaped filter has a hole at the top of radius 4 cm and a hole at the bottom of radius 1 cm, and is 6 cm in height. Water flows out of the bottom at a rate of 2 cm$^3$/sec. If the filter begins completely filled at time $t = 0$, how fast is the water level decreasing after 30 seconds?




*Solution for Problem 4.33:* We start by identifying the relevant quantities that are changing with respect to time. These are the height of the water, which we'll call $h$, and the volume of water in the filter, which we'll call $V$. We need to determine how they are related.




We see that the filter is a **frustum**, which is a cone with a smaller cone chopped off. The volume of a frustum is most easily computed as the difference in volume between the two cones. To compute the volumes of the cones, we need their heights. So we will be much better off if our height is measured from the vertex of the imaginary cones, not from the bottom of the filter.

Next, the formula for the volume of a cone is $\frac{1}{3}\pi r^2 h$, where $r$ is the radius and $h$ is the height. So we need to determine how the height is related the radius. They are directly proportional, as we can see from our picture at right. In particular, decreasing the height by 6 causes a decrease of 3 in the radius, so we conclude that $h/r = 2$, and thus the radius at height $h$ is $h/2$. Note that the top of our frustum is at $h = 8$ (not $h = 6$), and at the top we have radius $8/2 = 4$, as expected. The bottom of the frustum is at $h = 2 \cdot 1 = 2$.

So the volume of the water at height $h$ is:

$$V = \frac{1}{3}\pi r^2 h - \frac{1}{3}\pi(1)^2(2) = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h - \frac{1}{3}\pi(1)^2(2) = \frac{1}{12}\pi h^3 - \frac{2}{3}\pi.$$

We can differentiate this equation to get an equation relating the rates:

$$V' = \frac{1}{4}\pi h^2 h'.$$

We are told that $V' = -2$, as the rate of decrease of volume is constant. But we also need to find the value of $h$ in order to solve for $h'$. So we need to find the volume at $t = 30$ and compute its height.

We can start by finding the initial volume (at $t = 0$):

$$V(0) = \frac{1}{12}\pi(8)^3 - \frac{2}{3}\pi.$$

This simplifies to $42\pi$.

When $t = 30$, we will have lost $2 \cdot 30 = 60$ cm$^3$ of water, so the volume will be $V(30) = 42\pi - 60$. We can plug this in to our equation for volume to find $h$ at time $t = 30$:

$$42\pi - 60 = \frac{1}{12}\pi h^3 - \frac{2}{3}\pi.$$

This gives $h^3 = \dfrac{512\pi - 720}{\pi}$, and thus $h = \sqrt[3]{\dfrac{512\pi - 720}{\pi}} \approx 6.564$.

Finally, we can go back to our related rates equation:

$$-2 = \frac{1}{4}\pi h^2 h',$$

and plug in the found value of $h$. This gives:

$$-2 = \frac{1}{4}\pi\left(\frac{512\pi - 720}{\pi}\right)^{\frac{2}{3}} h'$$

We solve for $h'$ to get our answer:

$$h' = -\frac{8}{\sqrt[3]{\pi}(512\pi - 720)^{\frac{2}{3}}} = -\frac{2}{\sqrt[3]{\pi}(64\pi - 90)^{\frac{2}{3}}}.$$

This is approximately $-0.0591$, and thus at time $t = 30$ the water level is decreasing at a rate of 0.0591 cm/sec. □

## Exercises

**4.6.1** A plane is flying overhead at an altitude of 10 km and a speed of 200 m/sec. The plane is 20 km away from a camera on the ground, and the plane is flying in a direction that will take it directly over the camera. The camera continuously rotates to keep the plane centered in its lens. When the plane is 15 km away from the camera, how fast is the camera rotating?

**4.6.2** **Boyle's Law** states (assuming that the temperature is constant) that the pressure and volume of a gas are inversely proportional. Suppose Gas X starts at 1 atm (atmosphere) of pressure and takes up 200 $\text{cm}^3$. If the pressure is increased by 0.1 atm/min, then after 10 minutes, at what rate is the volume decreasing? **Hints:** 64

**4.6.3** A spherical snowball is 10 cm in diameter, but is melting at the rate of 0.5 $\text{cm}^3$ per second. When the snowball is reduced to half its original volume, at what rate is its diameter decreasing?

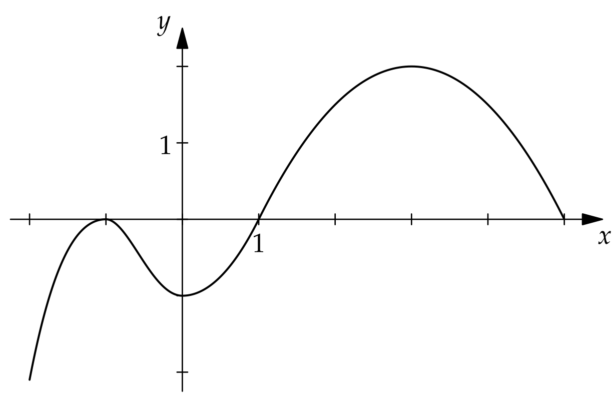
**4.6.4** A meteorite has entered the earth's atmosphere and is burning up at a rate that is proportional to the meteorite's surface area. What can you determine about the rate that the meteorite's radius is decreasing?

**4.6.5** Two cars start at an intersection at time $t = 0$ with velocity 0. At $t = 0$, one car accelerates due north at 8 $\text{m/sec}^2$, and the other car accelerates due east at 5 $\text{m/sec}^2$. After 6 seconds, by what rate is the distance between the cars increasing?

**4.6.6★** The clock on the wall at my office has an hour hand that's 6 cm long and a minute hand that's 10 cm long. At exactly 2:00, at what rate is the distance between the tips of the two hands changing? **Hints:** 148, 67

## Review Problems

**4.34** The following is the graph of the derivative $f'$ for some function $f$. Sketch possible graphs of $f$ and $f''$. Take into account where the function is increasing, decreasing, its concavity, and any inflection points.




**4.35** Suppose that $f$ is a function with domain $\mathbb{R}$ such that $f'(x) = x^2 - 3x + 3$ for all $x \in \mathbb{R}$. Prove that $f$ has an inverse.

**4.36**

(a) Show that if $f$ is a quadratic polynomial, then the graph of $f$ has no inflection points.

(b) Show that if $f$ is a cubic polynomial, then the graph of $f$ has exactly one inflection point.

(c) Can you generalize?