

NON-LINEAR PARAMETRIZATION OF COLLISION TUBE FOR LONG-TERM ENCOUNTERS

Tommaso Robbiani*, Zeno Pavanello†, Pierluigi Di Lizia‡, Mauro Massari§, and Roberto Armellin¶

Collision risk assessment is critical for space traffic management. Classical collision tube approaches struggle with long-term encounters due to the statistical ill-posedness of Cartesian formulations and the errors induced by geometry discretization. To overcome these limitations, this work introduces an analytical framework based on Differential Algebra and Automatic Domain Splitting. By mapping encounters into a whitened space, the time-varying covariance becomes a stationary unit sphere, restoring statistical consistency. The probability density function is analytically integrated over Taylor polynomials, avoiding numerical quadrature and eliminating topological gaps. Benchmark assessments confirm that the proposed method matches the accuracy of state-of-the-art methods for both short- and long-term encounters.

INTRODUCTION

The exponential growth of the resident space object population has rendered space traffic management a critical operational challenge. As orbits become increasingly congested, ensuring the safety of space assets requires the accurate and rapid computation of the Probability of Collision (PoC) to execute Collision Avoidance Maneuvers (CAMs),¹ if needed. Traditionally, collision risk assessment is categorized based on the geometry and duration of the event.² In *short-term encounters*, the relative velocity is high, and the relative motion can be approximated as rectilinear, reducing the problem from the computation of a 3D to a 2D integral. On the other hand, in *long-term encounters* the relative velocity is low and the relative trajectory could be highly curved, leading classical linear assumptions to fail. Furthermore, the orbital uncertainties grow significantly over the encounter window, with velocity uncertainties having a non-negligible impact.³

Within the long-term encounter framework, the universal benchmark for collision assessment is the Monte Carlo (MC) method. While it represents the most reliable approach due to its lack of restrictive geometric or statistical assumptions, its computational workload remains high for routine

*PhD Candidate, Department of Aerospace Science and Technology, Politecnico di Milano, via Giuseppe La Masa 34, Milano, 20156, Italy. Email: tommaso.rob主iani@polimi.it

†Post-doctoral Researcher, Department of Aerospace Science and Technology, Politecnico di Milano, via Giuseppe La Masa 34, Milano, 20156, Italy. Email: zeno.pavanello@polimi.it

‡Associate Professor, Department of Aerospace Science and Technology, Politecnico di Milano, via Giuseppe La Masa 34, Milano, 20156, Italy. Email: pierluigi.dilizia@polimi.it

§Associate Professor, Department of Aerospace Science and Technology, Politecnico di Milano, via Giuseppe La Masa 34, Milano, 20156, Italy. Email: mauro.massari@polimi.it

¶Professor, Department of Aerospace Engineering, University of Auckland, 20 Symonds Street, Auckland Central, Auckland, 1010, New Zealand. Email: roberto.armellin@auckland.ac.nz

operations. This limitation has driven research toward enhancing its efficiency, either by improving sampling strategies⁴ or accelerating MC simulations.^{5–7}

Other approaches are based on extensions of the short-term formulation that model a *collision tube* swept by the Hard-Body Sphere (HBS) as it moves through 3D Cartesian space. This was achieved by integrating the relative position probability density along the path,⁸ using a piecewise-linear approximation that divided the trajectory into cylindrical segments to compute local 2D integrals. However, adjoining these discrete segments inevitably introduces geometric gaps and overlaps, leading to inaccuracies in the collision probability. To address this issue, the parallelepipeds method was proposed,⁹ which is based on discretizing each cylindrical segment. This method reduces geometric discontinuities, improving the accuracy of the result, but it presents theoretical issues. While conceptually intuitive, evaluating the PoC in Cartesian space via collision-tube discretization is statistically ill-posed for time-evolving covariances.¹⁰ As demonstrated by Chan, integrating the flux of a time-dependent Probability Density Function (PDF) over a swept volume fundamentally contradicts probability theory, as Cartesian coordinates lose their validity as Random Variables (RVs) in the integral formulation.

A theoretically robust alternative is the Voxel Method,⁹ where the region in the proximity of the encounter is split into volume pixels. Within this framework, the tube geometry is mapped into the whitened space,¹¹ where the position covariance becomes a stationary identity matrix, casting the problem into a proper RV space and resolving the statistical ill-posedness of Cartesian formulations. If the tube crosses a voxel, its probability contribution is added to the total estimate. While this approach yields highly precise solutions,¹² it incurs high computational cost. Further efforts have sought to map the collision geometry into a proper RV space, in which the PDF is independent of time. Chan defined an analytical framework applicable to linear systems with a specific State Transition Matrix (STM), for which the integration volume is not trivial to define. To generalize this concept, Wen and Qiao¹³ proposed a back-mapping to the initial conditions based on reachable-domains theory, presenting some limitations: it requires the inversion of the STM position sub-matrix, which can become singular, and its formulation is restricted to scenarios involving only position or only velocity uncertainties. Another approach applies the same back-mapping principle by bounding the swept volume at the initial time via a polynomial approximation using Polynomial Superlevel Sets,¹⁴ thereby reducing the double-counting problem but requiring substantial computational effort.

Alternative approaches to guarantee the well-posedness of the problem are based on different paradigms, retaining the Cartesian space but dropping the concept of the collision tube. Among the available options, the most used is a 3D flux integration method,³ where the probability flux is integrated over the fixed cross-section of the HBS. The results of this method are comparable to those obtained via the Voxel method; however, it suffers from a non-trivial definition of the encounter duration and lacks guaranteed convergence.¹⁵ While its applicability is limited by restrictive assumptions, this method remains one of the most accurate and computationally efficient approaches developed to date.

Faced with the computational complexities and theoretical inaccuracies of the methods listed before, some authors rely on the Instantaneous Probability of Collision (IPoC) as a surrogate metric for computing CAMs.^{16–18} The underlying assumption is that an increase in the cumulative PoC is linked with a high IPoC, meaning that controlling IPoC bounds the overall collision risk. However, relying on IPoC introduces critical theoretical flaws, as it leads to significant conservatism and fails to condition on the absence of prior collisions.¹⁹

To overcome these limitations, this paper combines a formulation of the collision tube in the whitened space with a Differential Algebra (DA)-based algorithm.²⁰ By mapping the dynamics into a space where the covariance ellipsoid is a stationary unit sphere, this formulation guarantees a statistically well-posed problem, defining the state variables as proper RVs. The collision tube is parametrized exactly via a DA formulation eliminating the classical geometrical limitations,^{10,13} without relying on the discretization of the geometry. Furthermore, the probability integration is performed analytically, bypassing the need for numerical cubature methods. The computational accuracy is rigorously controlled by the Automatic Domain Splitting (ADS)²¹ technique, which also enables an efficient domain-pruning strategy based on interval bounds, automatically identifying high-probability regions over any arbitrary time span.

The remainder of this paper is organized as follows. Section 2 presents the formulation of the PoC problem, establishes the dynamical models, and highlights the hypotheses underlying tube-based formulations. Section 3 details the proposed methodology, describing the high-order polynomial parameterization of the collision tube, the whitening transformation strategies, and the adaptive integration algorithm via ADS. Section 4 evaluates the performance of the proposed approach by comparing it with numerical benchmarks. Finally, Section 5 summarizes the main findings and outlines potential future developments.

PROBLEM STATEMENT

This section establishes the physical and mathematical foundations of the collision risk assessment problem. First, the dynamical models governing trajectory evolution and uncertainty propagation are formalized. Subsequently, the classical tube-based formulation of the PoC is introduced, highlighting its inherent theoretical and computational limitations and motivating the transition to the proposed DA-based framework.

In the subsequent mathematical development, scalar quantities are denoted by standard type (e.g., t), while vectors and matrices are represented in boldface (e.g., $\mathbf{x}$). Continuous polynomial maps defined over a set of independent DA variables, $\boldsymbol{\xi}$, are denoted by the operator $\mathcal{T}$. The subscript identifies the specific physical quantity mapped: standard type for scalar-valued maps $\mathcal{T}_s(\boldsymbol{\xi})$, and boldface for vector- or matrix-valued maps $\mathcal{T}_v(\boldsymbol{\xi})$. The evaluation of a DA map at a specific domain point entails the direct substitution of numerical values for the variables $\boldsymbol{\xi}$ within the polynomial expansion.

Dynamical Models

Let the absolute states of the primary (p) and secondary (s) spacecraft in the Earth-Centered Inertial (ECI) frame at a reference epoch t_e be modeled as uncorrelated multivariate Gaussian random variables:

$$\mathbf{x}_i(t_e) \sim \mathcal{N}(\boldsymbol{\mu}_i(t_e), \boldsymbol{\Sigma}_i(t_e)), \quad i \in \{p, s\}$$

where $\mathbf{x}_i \in \mathbb{R}^6$ represents the Cartesian state vector (concatenating position and velocity), $\boldsymbol{\mu}_i \in \mathbb{R}^6$ is the expected mean state, and $\boldsymbol{\Sigma}_i \in \mathbb{R}^{6 \times 6}$ is the initial covariance matrix. The nominal state evolution of each spacecraft is governed by Keplerian dynamics. Although the DA framework inherently supports high-order, non-linear uncertainty propagation,²⁰ this study restricts the uncertainty propagation to the linear regime to maintain standard operational assumptions and computational efficiency. Consequently, the covariance matrices evolve according to:

$$\boldsymbol{\Sigma}_i(t) = \boldsymbol{\Phi}_i(t, t_e) \boldsymbol{\Sigma}_i(t_e) \boldsymbol{\Phi}_i^\top(t, t_e) \quad (1)$$

where $\Phi_i(t, t_e) \in \mathbb{R}^{6 \times 6}$ is the STM derived from the variational equations of the Keplerian dynamics. Under this linear assumption, the absolute states remain normally distributed at any time t , with the nominal trajectory tracking the mean of the distribution ($\bar{\mathbf{x}}_i(t) = \boldsymbol{\mu}_i(t)$). Since collision assessment inherently depends on the proximity of the two bodies, the problem must be mapped into the relative state space. The mean relative state in the ECI frame is simply the difference between the absolute mean states: $\bar{\mathbf{x}}_{rel}(t) = \bar{\mathbf{x}}_s(t) - \bar{\mathbf{x}}_p(t)$. Due to the statistical independence of the two space objects, the joint relative covariance is the algebraic sum of their individual covariances:

$$\boldsymbol{\Sigma}_{rel}(t) = \boldsymbol{\Sigma}_p(t) + \boldsymbol{\Sigma}_s(t)$$

To properly evaluate the encounter geometry, this relative dynamics is projected into the Local-Vertical Local-Horizontal (LVLH) frame centered on the primary spacecraft. Let $\mathcal{R}_{LVLH}(t) \in \mathbb{R}^{6 \times 6}$ be the block-diagonal rotation matrix from the ECI to the LVLH frame. The full relative covariance is rotated as $\boldsymbol{\Sigma}_{LVLH}(t) = \mathcal{R}_{LVLH}(t) \boldsymbol{\Sigma}_{rel}(t) \mathcal{R}_{LVLH}^\top(t)$. By extracting the upper-left 3×3 sub-matrix, the relative position covariance $\boldsymbol{\Sigma}_r(t) \in \mathbb{R}^{3 \times 3}$ is obtained. Consequently, the relative position vector $\mathbf{r}(t)$ of the secondary spacecraft is fully characterized by a time-varying 3D marginal Gaussian distribution:

$$\mathbf{r}(t) \sim \mathcal{N}_3(\bar{\mathbf{r}}(t), \boldsymbol{\Sigma}_r(t)) \quad (2)$$

where $\bar{\mathbf{r}}(t)$ denotes the nominal relative trajectory in the LVLH frame.

Probability of Collision Problem Formulation

The PoC over a time interval $t \in [t_e, t_f]$ is formally defined as the integral of the joint initial probability density function over the subset of all initial conditions that result in a collision.³ Assuming a spherical geometry for both resident space objects, a collision event is triggered if their relative distance breaches the combined Hard-Body Radius (HBR), defined as $R = R_p + R_s$.² Defining the 12-dimensional joint initial state as $\mathbf{X}_e = [\mathbf{x}_p^\top(t_e) \ \mathbf{x}_s^\top(t_e)]^\top \in \mathbb{R}^{12}$, and leveraging the statistical independence assumption, the joint initial PDF is the product of the marginal densities: $p_e(\mathbf{X}_e; t_e) = p_p(\mathbf{x}_p; t_e)p_s(\mathbf{x}_s; t_e)$. The PoC is thus computed as:

$$P = \int_{\mathcal{V}_e} p_e(\mathbf{X}_e; t_e) d\mathbf{X}_e$$

where $\mathcal{V}_e \subseteq \mathbb{R}^{12}$ encompasses the set of all initial states for which $\exists t \in [t_e, t_f]$ such that $\|\mathbf{r}(t)\| \leq R$. While mathematically rigorous, explicitly characterizing the boundary of the integration domain $\mathcal{V}_e$ under non-linear orbital dynamics remains a computationally intractable task for routine conjunction assessments.¹⁴

To bypass this multi-dimensional complexity, the standard methodology recasts the problem from the 12-dimensional state space into the 3D spatial relative domain.⁸ By shifting the total positional uncertainty onto the primary spacecraft, the PoC is approximated as the volumetric integral of the relative position PDF over the collision tube V swept by the combined HBS moving along the nominal trajectory:²

$$P = \iiint_V p(\mathbf{r}; t) d\mathbf{r} \quad (3)$$

where $p(\mathbf{r}; t)$ represents the time-varying PDF of the relative position. It is important to emphasize that in standard operational practices, this volumetric integration is not executed across the entire orbital lifespan from the reference epoch t_e , but is restricted to a narrow interval of interest typically centered around the estimated Time of Closest Approach (TCA): defining the exact lower and

upper temporal bounds, t_0 and t_f , for the integration window remains a non-trivial task, as will be discussed in subsequent sections.

This formulation is equivalent to the 12-dimensional one if and only if three hypotheses hold:

1. **H1** - *No self-intersection*: The collision tube does not intersect itself.³
2. **H2** - *Constant covariance*: The relative position covariance matrix Σ_r is constant.¹⁰
3. **H3** - *Deterministic velocity*: The relative velocity vector is perfectly known.

Hypothesis H1 is a constraint shared by all tube-based methods in the literature, including the flux-based formulations proposed by Coppola.³ It requires that the collision tube does not overlap with itself within the encounter window; if self-intersection occurs, the same physical region of the state space is integrated multiple times, leading to an overestimation of the PoC due to double counting. Hypothesis H2 constitutes the most critical theoretical issue for classical tube-based algorithms operating in a Cartesian framework. As demonstrated by Chan,¹⁰ integrating an explicitly time-dependent PDF over a spatial Cartesian volume violates basic probability theory, requiring the formulation of the problem within a proper, stationary RV space. Hypothesis H3 is a necessary condition to uniquely define the collision tube’s centerline as a deterministic function of spatial geometry. When velocity uncertainties are non-negligible, a single nominal spatial tube can no longer bound the uncertainty envelope, and the 3D tube formulation introduces strong approximations.

Beyond these constraints, standard tube-based methods are limited by the need for spatial discretization. The piecewise-linear approximation of highly curved tubes introduces topological gaps and overlaps,⁹ while the required numerical quadrature methods introduce truncation errors and impose a computational cost that increases significantly as the accuracy requirements grow.

METHODOLOGY

The proposed methodology introduces a mathematically rigorous framework designed to overcome the theoretical and numerical flaws of standard Cartesian tube approaches while preserving their volumetric computational advantages. The core of the algorithm relies on DA,²² a mathematical tool based on the algebra of Taylor polynomials that enables the exact representation of complex geometries up to a specified expansion order k . By manipulating non-linear relative dynamics, time-varying covariances, and spatial boundaries as continuous multivariate polynomial maps, the framework bypasses traditional spatial discretizations. Consequently, since every element is modeled through polynomial algebra, both the probability density function and the integration volume boundaries from Eq. (3) are mapped into continuous polynomial maps. This specific formulation allows the final multidimensional integral to be evaluated entirely through exact analytical integration, without resorting to inefficient numerical cubature methods. To ensure accuracy control over polynomial maps, the ADS algorithm²¹ is embedded within the architecture. The ADS monitors the polynomial truncation error by estimating the $(k + 1)$ -th order coefficients via exponential fitting. If a user-defined accuracy tolerance is exceeded, the polynomial domain is adaptively split and re-expanded around new local centers, maintaining a low expansion order k even under highly non-linear dynamics.

Tube Parameterization via Differential Algebra

Consider a non-linear relative dynamical system governed by $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t)$, where t is time and the relative state vector $\mathbf{x}(t) = [\mathbf{r}(t)^\top \mathbf{v}(t)^\top]^\top$ concatenates relative position and velocity. The collision tube is defined as the geometric envelope swept by the combined HBS of radius R moving along the nominal relative trajectory. To analytically parameterize this continuous 3D volume over an arbitrary temporal window $[t_0, t_f]$, a set of curvilinear-cylindrical coordinates, representing time t , azimuth angle θ , and radial displacement ρ , is mapped into three independent DA variables $\boldsymbol{\xi} = [\xi_1 \ \xi_2 \ \xi_3]^\top \in [-1, 1]^3$. These variables linearly scale the physical boundaries according to:

$$\mathcal{T}_t(\xi_1) = \frac{t_f + t_0}{2} + \xi_1 \frac{t_f - t_0}{2}, \quad \mathcal{T}_\theta(\xi_2) = \pi + \xi_2 \pi, \quad \mathcal{T}_\rho(\xi_3) = \frac{R}{2} + \xi_3 \frac{R}{2} \quad (4)$$

By expanding the local flow in the temporal variable via Picard-Lindelöf iterations,²³ the nominal relative state translates into a continuous polynomial map $\mathcal{T}_{\bar{\mathbf{x}}}(\xi_1)$, which isolates the position map $\mathcal{T}_{\bar{\mathbf{r}}}(\xi_1)$ and the velocity map $\mathcal{T}_{\bar{\mathbf{v}}}(\xi_1)$. To construct the tube geometry, a local orthonormal moving reference frame is built along the trajectory's centerline. The tangent unit vector $\hat{\mathbf{T}}$ aligns with the relative velocity map, while the bi-normal $\hat{\mathbf{B}}$ and normal $\hat{\mathbf{N}}$ unit vectors are generated by introducing an auxiliary Cartesian basis vector $\hat{\boldsymbol{\alpha}}_{ref}$ that maximizes linear independence with respect to $\hat{\mathbf{T}}$:

$$\mathcal{T}_{\hat{\mathbf{T}}}(\xi_1) = \frac{\mathcal{T}_{\bar{\mathbf{v}}}(\xi_1)}{\|\mathcal{T}_{\bar{\mathbf{v}}}(\xi_1)\|} \quad (5a)$$

$$\mathcal{T}_{\hat{\mathbf{B}}}(\xi_1) = \frac{\mathcal{T}_{\hat{\mathbf{T}}}(\xi_1) \times \hat{\boldsymbol{\alpha}}_{ref}}{\|\mathcal{T}_{\hat{\mathbf{T}}}(\xi_1) \times \hat{\boldsymbol{\alpha}}_{ref}\|} \quad (5b)$$

$$\mathcal{T}_{\hat{\mathbf{N}}}(\xi_1) = \mathcal{T}_{\hat{\mathbf{B}}}(\xi_1) \times \mathcal{T}_{\hat{\mathbf{T}}}(\xi_1) \quad (5c)$$

Because the unit vectors in Eqs. (5) are continuous DA polynomials driven by the temporal variable ξ_1 , the frame evolves along the curved path, preserving its orthogonality properties at every instant and eliminating the precision loss typical of pointwise double-precision arithmetic. With the moving frame established, the position vector $\mathbf{r}$ of any generic point within the continuous collision tube is mapped as a function of the three DA variables:

$$\mathcal{T}_{\mathbf{r}}(\xi_1, \xi_2, \xi_3) = \mathcal{T}_{\bar{\mathbf{r}}}(\xi_1) + \mathcal{T}_\rho(\xi_3) [\mathcal{T}_{\hat{\mathbf{N}}}(\xi_1) \cos(\mathcal{T}_\theta(\xi_2)) + \mathcal{T}_{\hat{\mathbf{B}}}(\xi_1) \sin(\mathcal{T}_\theta(\xi_2))] \quad (6)$$

This formulation captures the intrinsic curvature of the trajectory without resorting to discrete piecewise-linear segments,^{8,9} thereby preserving continuity.

Whitened Space Formulation and Transformation Strategies

The critical theoretical violation of hypothesis H2 is resolved by shifting the integration from Cartesian coordinates into the whitened space.¹¹ Given a zero-mean multivariate Gaussian random variable $\mathbf{y} \sim \mathcal{N}_m(\mathbf{0}_m, \boldsymbol{\Sigma})$, a linear whitening transformation is defined via a matrix $\mathbf{W} \in \mathbb{R}^{m \times m}$ such that:

$$\mathbf{z} = \mathbf{W} \mathbf{y}, \quad \mathbf{W}^\top \mathbf{W} = \boldsymbol{\Sigma}^{-1} \quad (7)$$

In this transformed domain, the covariance reduces to a stationary identity matrix $\mathbf{I}_m$, rendering the joint PDF isotropic ($\mathbf{z} \sim \mathcal{N}_m(\mathbf{0}_m, \mathbf{I}_m)$). This maps the collision geometry into a proper RV space,

in which the probability distribution becomes independent of time, thereby satisfying the statistical conditions for spatial integration highlighted by Chan.¹⁰

For the problem at hand, the whitening transformation is constructed using the relative position covariance sub-matrix $\Sigma_r(t)$. This stems directly from the spatial formulation established in Eq. (3), since the PoC computation is restricted to a volumetric 3D integral over the relative position domain. In long-term encounters, the position covariance matrix $\Sigma_r(t)$ evolves, making the whitening matrix time-dependent. While evaluating a time-varying matrix transformation over discrete spatial segments is computationally prohibitive, the DA framework naturally handles this complexity by propagating the continuous covariance map $\mathcal{T}_{\Sigma_r}(\xi_1)$ through the DA version of Eq. (1). The whitening matrix map $\mathcal{T}_W(\xi_1)$ is derived analytically, allowing the Cartesian collision tube from Eq. (6) to be directly projected into the whitened domain:

$$\mathcal{T}_u(\xi_1, \xi_2, \xi_3) = \mathcal{T}_W(\xi_1)\mathcal{T}_r(\xi_1, \xi_2, \xi_3) \quad (8)$$

where $\mathbf{u}$ represents the position vector in the transformed space.

Because an infinite set of matrices satisfies Eq. (7), three linear transformations, Zero-phase Component Analysis (ZCA), ZCA-Correlated (ZCA-Cor), and Cholesky decomposition,¹¹ were tested to identify the formulation that minimizes artificial topological distortions:

$$\mathbf{W}_{ZCA} = \Sigma_r^{-1/2}, \quad \mathbf{W}_{ZCA-cor} = \mathbf{P}^{-1/2}\mathbf{V}^{-1/2}, \quad \mathbf{W}_{Chol} = \mathbf{L}^\top \quad (9)$$

where $\mathbf{P}$ is the correlation matrix, $\mathbf{V}$ is the diagonal variance matrix, and $\mathbf{L}\mathbf{L}^\top = \Sigma_r^{-1}$.^{*} To quantify the volumetric distortion induced by these transformations, the tube's geometrical volume V was integrated over Alfano's benchmark Case 1,¹² leveraging the absolute value of the Jacobian determinant:

$$V = \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} |\mathcal{T}_{\det \mathbf{J}}(\xi_1, \xi_2, \xi_3)| d\xi_1 d\xi_2 d\xi_3 \quad (10)$$

The term $\mathcal{T}_{\det \mathbf{J}}$ represents the polynomial map of the Jacobian determinant associated with the mapping from the DA coordinate space ξ to the transformed whitened space variables $\mathbf{u}$, analytically computed via the DA `deriv()` operator.²⁴

Preserving the topological simplicity of the tube is fundamental: any artificial folding introduces regions where the spatial map is no longer injective. Since the baseline trajectory in Cartesian space does not have physical self-intersections for the analyzed scenario, any topological folding detected in the transformed space is induced by the specific whitening matrix rotation. These geometric foldings can be systematically monitored by assessing the overestimations in the integrated volume.

The relative volume errors reported in Table 1 are computed by comparing the evaluation of Eq. (10) inside the DA framework, where the accuracy of the map is guaranteed by the ADS algorithm, against a high-fidelity numerical benchmark obtained via a fine spatial discretization in MATLAB. As demonstrated by the results, Cholesky and ZCA-Cor strategies introduce volume overestimations as the combined spacecraft radius increases, proving that these transformations induce topological folding near the encounter region. Conversely, ZCA consistently preserves the original topological features across all scales, minimizing artificial distortion, as theoretically expected.¹¹ Consequently, ZCA is adopted as the baseline whitening strategy.

^{*}Here the animations of the tubes obtained with different whitening strategies are available.

Table 1. Relative Volume Error Induced by Whitening Transformations.

HBR [m]	Chol (%)	ZCA (%)	ZCA-Cor (%)
1	-0.007	-0.002	-0.002
10	+0.142	-0.002	+0.160
30	+1.540	+0.209	+1.857

To compute $\mathbf{W}_{ZCA}$ within the DA environment, the Denman-Beavers square root iteration²⁵ is implemented as:

$$\begin{aligned} \mathbf{Y}_0 &= \frac{1}{c} \mathcal{T}_{\Sigma_r}(\xi_1), & \mathbf{Z}_0 &= \mathbf{I}_3, \\ \mathbf{Y}_{h+1} &= \frac{1}{2} (\mathbf{Y}_h + \mathbf{Z}_h^{-1}), & \mathbf{Z}_{h+1} &= \frac{1}{2} (\mathbf{Z}_h + \mathbf{Y}_h^{-1}). \end{aligned} \quad (11)$$

where $c = \text{tr}(\mathcal{T}_{\Sigma_r}(0))$ represents the trace of the constant component of the position covariance matrix.

Phase 1: Geometry Accuracy Control and Domain Reduction

Representing the final probability integrand over the entire encounter window via a single high-order polynomial can lead to local inaccuracies due to severe non-linearities. To maintain rigorous error control and computational efficiency, a two-phase architecture is implemented. This section details the first phase, which is divided into two distinct sequential steps: the generation and validation of the temporal maps, followed by the mapping of the Squared Mahalanobis Distance (SMD) with domain pruning.

Step 1: Temporal Map Generation and Validation The first step focuses on generating accurate temporal maps for all time-dependent quantities. Starting from the midpoint of the encounter window $t_m = (t_f + t_0)/2$, the absolute Keplerian flow is expanded using double-precision nominal states $\bar{\mathbf{x}}_i^{ECI}(t_m)$ as initial guesses: $\mathcal{T}_{\bar{\mathbf{x}}_i^{ECI},0}(\xi_1) = \bar{\mathbf{x}}_i^{ECI}(t_m)$ for $i \in \{p, s\}$. The k -th order Taylor polynomials of the trajectories are obtained via Picard-Lindelöf iterations:²³

$$\mathcal{T}_{\bar{\mathbf{x}}_i,j+1}(\xi_1) = \bar{\mathbf{x}}_i(t_m) + \frac{t_f - t_0}{2} \int_{-1}^{\xi_1} f_{kep}(\mathcal{T}_{\bar{\mathbf{x}}_i,j}(\tau)) d\tau, \quad j = 0, \dots, k-1 \quad (12)$$

where $f_{kep}(\cdot) : \mathbb{R}^6 \rightarrow \mathbb{R}^6$ defines the absolute Keplerian dynamics. After convergence, the relative state map is derived algebraically as done in Eq. (2). To ensure mapping fidelity, the temporal polynomials are evaluated at the domain boundaries ($\xi_1 = \pm 1$), and their accuracy is assessed by comparison with an exact numerical integration. If the deviation exceeds a prescribed tolerance, the temporal domain is automatically bisected, and the Picard-Lindelöf expansion is re-applied at the new local midpoints. This iteration runs until full convergence is achieved across the entire window, and the same approach is applied to define the STMs $\Phi_i(t, t_e)$ time-dependent maps. All subsequent time-dependent maps, including the moving reference frame of Eq. (5), and the whitening transformation of Eq. (8), are directly computed from these highly accurate polynomial maps.

This step is fundamental for the overall efficiency of the algorithm as it provides a library of accurate temporal sub-domains. Consequently, whenever further refinements (in Step 2 and Phase 2) split the integration domain along the time direction ξ_1 , the framework evaluates these pre-computed polynomials at the new local expansion points. This eliminates the need for further numerical integrations, dramatically reducing the computational effort.

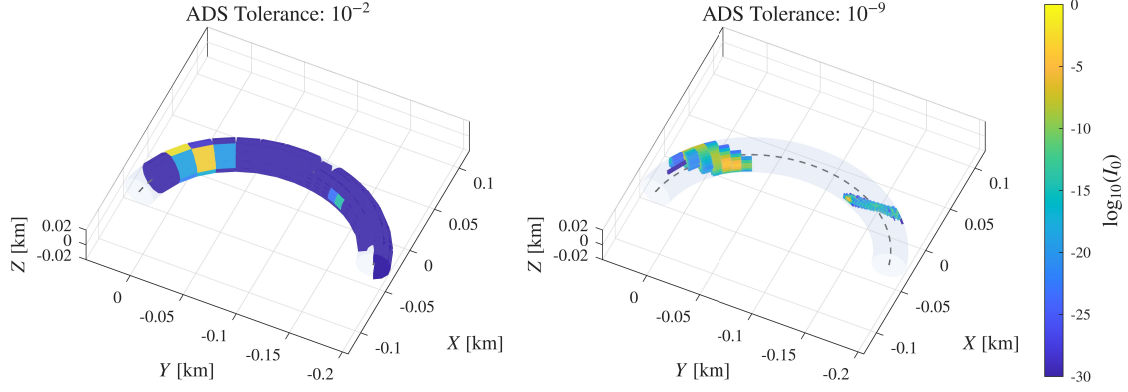


Figure 1. Phase 1 Domain Reduction for different tolerances in LVLH frame.

Step 2: SMD Mapping and Domain Pruning Once the temporal baseline is established, the spatial expansion incorporating the full variable set $\xi = [\xi_1, \xi_2, \xi_3]^\top$ is initialized. In the isotropic whitened space, the coordinates are defined by the Taylor polynomial map $\mathcal{T}_u(\xi)$ as introduced in Eq. (8) and the spatial Gaussian PDF is uniquely driven by the Squared Mahalanobis Distance (SMD) map, defined as $\mathcal{T}_Q(\xi) = \mathcal{T}_u(\xi)^\top \mathcal{T}_u(\xi)$:

$$p(\mathcal{T}_u(\xi_1, \xi_2, \xi_3)) = \frac{1}{\sqrt{(2\pi)^3}} \exp\left(-\frac{1}{2}\mathcal{T}_Q(\xi_1, \xi_2, \xi_3)\right) \quad (13)$$

Directly performing accuracy control via ADS on the full integrand $\mathcal{T}_{int}$ is numerically inefficient due to the exponential operator appearing in the PDF. If the SMD map $\mathcal{T}_Q$ is inaccurate, local errors are greatly amplified by the exponential transformation, producing artificial and non-physical density spikes that hinder convergence. Consequently, a preliminary ADS splitting stage is applied to ensure the accuracy of the $\mathcal{T}_Q$ map before evaluating the full integrand.

Refining the SMD map also enables a robust domain reduction strategy. Utilizing interval arithmetic, the algorithm evaluates the lower bound of $\mathcal{T}_Q(\xi)$ for each individual sub-domain. If this minimum value is larger than a defined statistical significance limit (set to 30.66, corresponding to a local PDF contribution below 10^{-6}), the sub-domain's statistical weight is assumed to be negligible, and the interval is safely pruned from the integration tree. As illustrated in Figure 1, this pruning mechanism is conditioned on the domains being well-approximated. Loose ADS tolerances yield poorly estimated bounds that conservatively avoid pruning of intervals with small nominal integrand values $I_0 = |\mathcal{T}_{int}(\mathbf{0})|$. As the ADS tolerance is tightened, the sub-domains shrink, truncation errors decay, and exact pruning tightly isolates the high-probability encounter region.

This bounding strategy enables autonomous identification of statistically relevant intervals over an arbitrary time span $[t_0, t_f]$, effectively resolving the non-trivial challenge of selecting a tightly bounded integration window around TCA. By pruning regions with negligible statistical weight, the proposed framework efficiently handles wide temporal windows without incurring additional computational cost or underestimating the collision probability, even in the absence of prior knowledge of the encounter's exact duration.

Phase 2: Adaptive Integration

Following domain reduction, a further ADS accuracy control on the integrand maps is performed leading to $\mathcal{Z}$ active sub-domains on which PoC integration is performed. By formulating the problem within the stationary whitened space, the volumetric integral from Eq. (3) is strictly reformulated in DA framework as:

$$P = \sum_{l \in \mathcal{Z}} \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} p(\mathcal{T}_{u,l}(\xi)) |\mathcal{T}_{\det J,l}(\xi)| d\xi_1 d\xi_2 d\xi_3 \quad (14)$$

where $p(\mathcal{T}_{u,l}(\xi))$ represents the isotropic Gaussian probability density function evaluated over the whitened state map of the l -th sub-domain and the term $\mathcal{T}_{\det J,l}(\xi)$ denotes the Taylor polynomial expansion of the Jacobian determinant associated with the mapping from the normalized DA coordinate space ξ to the transformed whitened domain. The resulting integrand within each sub-domain results into a single high-order polynomial map, $\mathcal{T}_{int,l}(\xi)$.

A critical topological challenge arises from the absolute value operator applied to the Jacobian determinant, $|\mathcal{T}_{\det J,l}(\xi)|$. To preserve spatial injectivity, any coordinate transformation must operate as a C^1 -diffeomorphism, requiring the determinant to maintain a constant positive sign. However, for the scenario at hand, if the trajectory's local radius of curvature is smaller than the combined spacecraft radius, the inner boundary of the collision tube folds upon itself, locally reversing spatial orientation and creating a geometric singularity. Taking the absolute value is, therefore, mandatory to guarantee a non-negative differential volume dV , but applying a non-differentiable absolute value operator to a continuous DA polynomial compromises its algebraic properties.

To solve this issue without introducing numerical integration across the entire domain, a hybrid analytical-numerical integration architecture is implemented. The active set of $\mathcal{Z}$ domains is partitioned into two distinct sets: $\mathcal{Z}_A$ (analytical) and $\mathcal{Z}_N$ (numerical fallback). Relying on interval bounds, the maximum M_l and minimum m_l values of the determinant polynomial are evaluated for each domain. If the interval does not cross zero ($m_l > 0$ or $M_l < 0$), the transformation is locally non-singular; the absolute value is resolved algebraically by multiplying the polynomial by its constant sign, exactly preserving the DA structure. The probability contribution is then computed analytically on each domain by evaluating the exact primitive maps $\mathcal{T}_{Int,l}(\xi)$ of $\mathcal{T}_{int,l}(\xi)$.

Conversely, if the bounds cross zero ($m_l \cdot M_l \leq 0$), a spatial singularity or a centerline boundary condition ($\rho = 0$) is detected. The domain is transferred to the numerical fallback set $\mathcal{Z}_N$, and the integral is resolved via a standard numerical cubature to accurately capture the positive mass of the folded topology. Clearly, this approach increases the computational effort with respect to DA analytical integration and can lead to PoC overestimation. The total cumulative PoC is thus computed as:

$$P = \sum_{i \in \mathcal{Z}_A} [\mathcal{T}_{Int,i}(\xi)]_{-1}^1 + \sum_{j \in \mathcal{Z}_N} \sum_{q=1}^{N_q} w_q |\mathcal{T}_{\det J,j}(\xi_q)| p(\mathcal{T}_{u,j}(\xi_q)) \quad (15)$$

where w_q and ξ_q represent the weights and nodes of an N_q -point numerical cubature scheme.

NUMERICAL ASSESSMENT

This section assesses the validity and performance of the proposed architecture. The assessment is structured in three parts: first, the simulation setup and benchmarks are defined; second, the

framework is compared against state-of-the-art methods in terms of accuracy and computational cost; finally, the methodological boundaries and limits of the underlying hypotheses are explored.

Simulation Setup and Benchmarks

The framework is evaluated using the benchmark test cases proposed by Alfano,¹² which serve as a standard reference in the literature for collision avoidance methods, including both short-term [S] and long-term [L] encounters. While the proposed method specifically targets the long-term ones, short-term scenarios are included to demonstrate that the framework remains statistically well-posed and accurate across all conjunction geometries. To assess the accuracy of the proposed method, high-fidelity MC simulations are used as the reference solution. The baseline MC results reported by Alfano¹² were independently reproduced by the authors and extended to more scenarios.

For the performance assessment, the proposed DA method is compared against the flux-based formulation proposed by Coppola,³ representing a highly accurate state-of-the-art approach. Coppola's results were obtained by discretizing the time-span used in¹² with 500 time nodes and using a Lebedev integration order of 131. Conversely, the proposed DA method operates with an expansion order of $k = 8$. The adaptive domain splitting relies on a Phase 1 nominal time-map tolerance of 10^{-6} , an ADS tolerance on the SMD of 10^{-7} , and a Phase 2 integration tolerance of 10^{-12} .

Accuracy and Computational Performance

The first numerical assessment is reported in Table 2, where relative errors of DA architectures and Coppola's method, used as the benchmark approach, with respect to the MC reference are shown. The test cases are classified as long-term (L) or short-term (S), as indicated in the table headings.

Table 2. Relative Probability Error (%) vs Monte Carlo Reference Values.

Method	1 [L]	2 [L]	3 [S]	4 [L]	5 [S]	6 [L]	7 [L]	8 [L]	10 [L]
Coppola	+0.02	-0.02	+0.01	-0.01	+0.07	-0.17	+0.05	+0.02	-0.01
DA-Whitened	-0.06	-0.16	+0.19	-0.05	+0.07	-0.19	-0.41	-5.09	-4.29
DA-Cartesian	+2.07	+3.20	-0.32	-2.09	-0.01	+0.75	+0.11	-5.02	+47.05

The DA-Whitened method performs comparably to Coppola's method across Cases 1 through 7, maintaining relative errors well below 0.5% for both short- and long-term encounters. This confirms that the exact DA polynomial parameterization successfully resolves the volumetric integral. Note that for both DA and Coppola's method, the absence of self-intersection (Hypothesis H1) was verified *a posteriori*, and no self-intersections were detected. Cases 8 and 10, where Case 10 is equivalent to Case 9 with a corrected propagation time span, present anomalous errors and will be addressed in detail in the following sections.

In order to demonstrate the theoretical necessity of the shift from Cartesian to whitened space, a parallel architecture was implemented in Cartesian space, where the integral in Eq. (14) is replaced by:

$$P = \int_{-1}^{+1} \int_{-1}^{+1} \int_{-1}^{+1} p(\mathcal{T}_r(\xi_1, \xi_2, \xi_3)) |\mathcal{T}_{\det} \mathbf{J}_r(\xi_1, \xi_2, \xi_3)| d\xi_1 d\xi_2 d\xi_3 \quad (16)$$

and Jacobian map $\mathcal{T}_{\det} J_r$ is obtained from the derivatives of Eq. (6) with respect to the DA variables. The rest of the method is implemented in the same way, with the integration bounds remaining unchanged due to the scaling of the DA variables. As shown in Table 2, the DA-Cartesian approach exhibits large errors for long-term encounters, confirming that integrating a time-dependent PDF over a Cartesian volume violates basic probability theory (Hypothesis H2), numerically validating Chan’s¹⁰ theoretical statements. In short-term encounters, the covariance matrix does not evolve over the event duration, allowing the Cartesian states to act as accurate RVs.

In addition to its high accuracy, the proposed framework is computationally efficient, as demonstrated by the CPU execution times reported in Table 3. All simulations were implemented in C++ and executed on a workstation equipped with a 13th Gen Intel Core i7-1370P processor (1.90 GHz) and 16 GB of RAM, running a 64-bit Windows operating system. The reported CPU times correspond to the average over 10 runs.

Table 3. CPU Time Comparison: Mean execution time (ms) and speedup ratio.

Method	1 [L]	2 [L]	3 [S]	4 [L]	5 [S]	6 [L]	7 [L]	8 [L]	10 [L]
Coppola	212	213	211	206	191	213	291	352	271
DA-Whitened	712	156	135	340	242	124	42	324	2689
Ratio	0.30	1.37	1.56	0.61	0.79	1.72	6.93	1.09	0.10

Despite the polynomial expansions and tight ADS tolerances, the computational times of the DA-Whitened method are low, often on the order of a few hundred milliseconds. While Coppola’s approach maintains a more uniform execution time across different geometries due to the similar time and spatial discretization, the DA-Whitened method adapts to the non-linearity of the problem. In highly non-linear scenarios (e.g., Case 1 and Case 10), the DA method requires more domain splits across Phase 1 and Phase 2, increasing the CPU time. However, in scenarios like Case 7, the efficient pruning strategy drastically reduces the integration domain, making the DA method nearly 7 times faster than Coppola’s.

Methodological Limits

While the DA-Whitened method proves extremely robust for Cases 1 through 7, Table 2 highlights a drop in accuracy for Case 8 and Case 10. To explain this behavior, Hypothesis H3 must be examined.

A proper tube-based PoC formulation relies on the assumption of negligible velocity uncertainty, which is required to uniquely define the tube centerline. For Cases 1–7, the velocity uncertainties are sufficiently small relative to the nominal relative velocity for this assumption to hold, making the resulting tube geometry a valid approximation. In contrast, Cases 8 and 10 represent extreme conditions under which this assumption no longer applies. In Case 8, involving two spacecraft in MEO, the relative velocity at the TCA is only 0.9 mm/s, whereas the square root of the trace of the velocity covariance is 11.6 mm/s. Similarly, in Case 10 (HEO orbits), the relative velocity at the TCA is 2.1 mm/s with a velocity uncertainty of 1.5 mm/s. Under these conditions, the tube centerline can no longer be regarded as deterministic, rendering any tube-based formulation theoretically invalid.

Finally, it is worth mentioning that Alfano’s Cases 11 and 12 are excluded from the current frame-

work. Because they are characterized by an absence of relative motion (zero nominal velocity), the fundamental moving reference frame and curvilinear tube definition formulated in Eq. (5) mathematically encounter a singularity. Although virtually never encountered in real space operations, these static encounter scenarios, together with extreme low-velocity cases such as Cases 8 and 10, highlight the limitations of all tube-based methodologies.

CONCLUSIONS

This paper presented an analytical framework based on Differential Algebra (DA) for robust Probability of Collision (PoC) evaluation in both short- and long-term space encounters. By mapping the collision tube geometry into a stationary whitened space, the formulation resolves the statistical ill-posedness inherent to integrating a time-varying Cartesian PDF over a swept spatial volume. The combination of high-order polynomial parameterization and Automatic Domain Splitting (ADS) enables an exact representation of the continuous collision topology, bypassing the geometric inaccuracies of traditional piecewise spatial discretizations while allowing the integration to be performed analytically.

Numerical assessments on standard literature benchmarks demonstrate that the DA-Whitened approach achieves high accuracy, matching state-of-the-art methods, while maintaining a low computational effort compatible with operational space traffic management. Conversely, comparative testing in Cartesian space confirms the critical necessity of a proper random-variable domain.

The analysis also successfully isolated the encounters conditions that limit tube-based paradigms. When the nominal relative velocity drops to values comparable to the velocity uncertainty, the physical assumption of a deterministic centerline breaks down, and the accuracy of any tube-based methodology inevitably degrades. Similarly, static encounters featuring zero nominal relative velocity remain outside the mathematical domain of the current curvilinear-cylindrical parameterization.

Future developments will focus on reformulating the mathematical framework to directly incorporate velocity uncertainties, moving beyond the deterministic centerline assumption that limits current approach. Additionally, extending the architecture to incorporate non-linear uncertainty propagation will allow the framework to move beyond standard Gaussian assumptions. By leveraging the flexibility of the presented DA-based framework, these developments could pave the way for a computationally efficient tool for advanced collision risk mitigation and optimal maneuver design.

REFERENCES

- [1] Z. Pavanello, L. De Maria, A. De Vittori, M. Maestrini, P. Di Lizia, and R. Armellin, “CAMmary: A review of spacecraft collision avoidance manoeuvre design methods,” *Acta Astronautica*, Vol. 236, Nov. 2025, pp. 770–789, 10.1016/j.actaastro.2025.07.027.
- [2] K. Chan, *Spacecraft Collision Probability*. El Segundo, USA: The Aerospace Press, 2008, 10.1017/S0001924000087546.
- [3] V. T. Coppola, “Including Velocity Uncertainty in the Probability of Collision Between Space Objects,” *AAS/AIAA Astrodynamics Specialist Conference*, Vol. 145 of *Advances in the Astronautical Sciences*, Minneapolis, MN, USA, Aug. 2012. AAS Paper 12-247.
- [4] Y. Sztamfater-Garcia, M. Sanjurjo-Rivo, G. Escibano, H. Molina-Bulla, and J. Miguez, “An approximate model for the computation of in-orbit collision probabilities using importance sampling,” *Advances in Space Research*, Vol. 75, Feb. 2025, pp. 3791–3805, 10.1016/j.asr.2024.12.074.
- [5] B. A. Jones and A. Doostan, “Satellite collision probability estimation using polynomial chaos expansions,” *Advances in Space Research*, Vol. 52, Dec. 2013, pp. 1860–1875, 10.1016/j.asr.2013.08.027.
- [6] A. Morselli, R. Armellin, P. Di Lizia, and F. Bernelli Zazzera, “A high order method for orbital conjunctions analysis: Monte Carlo collision probability computation,” *Advances in Space Research*, Vol. 55, Jan. 2015, pp. 311–333, 10.1016/j.asr.2014.09.003.

- [7] A. Zollo, C. Parigini, R. Armellin, J. F. S. J. Díaz, A. Trombetta, and R. Kahle, "A polynomial-based Monte Carlo approach for estimating long-term collision probabilities," *Acta Astronautica*, Vol. 242, May 2026, pp. 178–192, 10.1016/j.actaastro.2026.01.039.
- [8] R. P. Patera, "Satellite Collision Probability for Nonlinear Relative Motion," *Journal of Guidance, Control, and Dynamics*, Vol. 26, Sept. 2003, pp. 728–733, 10.2514/2.5127.
- [9] S. Alfano, "Eliminating Assumptions Regarding Satellite Conjunction Analysis," *The Journal of the Astronautical Sciences*, Vol. 59, Dec. 2012, pp. 676–705, 10.1007/s40295-014-0002-4.
- [10] K. Chan, "Formulation of Collision Probability with Time-Dependent Probability Density Functions," *AAS/AIAA Space Flight Mechanics Meeting*, Advances in the Astronautical Sciences, Williamsburg, VA, USA, Jan. 2015. AAS Paper 15-233.
- [11] A. Kessy, A. Lewin, and K. Strimmer, "Optimal whitening and decorrelation," *The American Statistician*, Vol. 72, Oct. 2018, pp. 309–314. arXiv:1512.00809 [stat], 10.1080/00031305.2016.1277159.
- [12] S. Alfano, "Satellite Conjunction Monte Carlo Analysis," *AAS Spaceflight Mechanics Meeting*, Advances in the Astronautical Sciences, Pittsburgh, PA, USA, Feb. 2009. AAS Paper 09-233.
- [13] C. Wen and D. Qiao, "Calculating collision probability for long-term satellite encounters through the reachable domain method," *Astrodynamics*, Vol. 6, June 2022, pp. 141–159, 10.1007/s42064-021-0119-8.
- [14] D. Arzelier, F. Brehard, M. Joldes, J.-B. Lasserre, S. Laurens, and A. Rondepierre, "Polynomial superlevel set approximation of swept-volume for computing collision probability in space encounters," *2021 60th IEEE Conference on Decision and Control (CDC)*, Austin, TX, USA, IEEE, Dec. 2021, pp. 6300–6305, 10.1109/CDC45484.2021.9683445.
- [15] V. Schaeffer, S. Laurens, P. Seimandi, and F. Delmas, "Collision Probability through Time Integration - Implementation and Operational Results," *2018 SpaceOps Conference*, Marseille, France, American Institute of Aeronautics and Astronautics, May 2018, 10.2514/6.2018-2720.
- [16] Z. Pavanello, L. Pirovano, and R. Armellin, "Long-Term Fuel-Optimal Collision Avoidance Maneuvers with Station-Keeping Constraints," *Journal of Guidance, Control, and Dynamics*, Vol. 47, Sept. 2024, pp. 1855–1871, 10.2514/1.G007839.
- [17] A. De Vittori, Z. Pavanello, P. Di Lizia, J. McMahon, and R. Armellin, "Combined Long-Term Collision Avoidance and Stochastic Station-Keeping in Geostationary Earth Orbit," *Journal of Guidance, Control, and Dynamics*, Vol. 48, Apr. 2025, pp. 840–854, 10.2514/1.G008629.
- [18] S. Dutta and A. K. Misra, "Convex optimization of collision avoidance maneuvers in the presence of uncertainty," *Acta Astronautica*, Vol. 197, Aug. 2022, pp. 257–268, 10.1016/j.actaastro.2022.05.038.
- [19] U. E. Núñez Garzón and E. G. Lightsey, "Relating Collision Probability and Separation Indicators in Spacecraft Formation Collision Risk Analysis," *Journal of Guidance, Control, and Dynamics*, Vol. 45, Mar. 2022, pp. 517–532, 10.2514/1.G005744.
- [20] M. Valli, R. Armellin, P. Di Lizia, and M. R. Lavagna, "Nonlinear Mapping of Uncertainties in Celestial Mechanics," *Journal of Guidance, Control, and Dynamics*, Vol. 36, No. 1, 2013, pp. 48–63, 10.2514/1.58068.
- [21] A. Wittig, P. Di Lizia, R. Armellin, K. Makino, F. Bernelli-Zazzera, and M. Berz, "Propagation of large uncertainty sets in orbital dynamics by automatic domain splitting," *Celestial Mechanics and Dynamical Astronomy*, Vol. 122, July 2015, pp. 239–261, 10.1007/s10569-015-9618-3.
- [22] M. Massari, P. Di Lizia, and M. Rasotto, "Nonlinear Uncertainty Propagation in Astrodynamics Using Differential Algebra and Graphics Processing Units," *Journal of Aerospace Information Systems*, Vol. 14, Sept. 2017, pp. 493–503, 10.2514/1.I010535.
- [23] A. Evans, R. Armellin, L. Pirovano, and N. Baresi, "High-Order Guidance for Time-Optimal Low-Thrust Trajectories with Accuracy Control," *Journal of Guidance, Control, and Dynamics*, Vol. 47, No. 2, 2024, pp. 279–290, 10.2514/1.G007540.
- [24] M. Rasotto, A. Morselli, A. Wittig, M. Massari, P. D. Lizia, R. Armellin, C. Valles, and G. Ortega, "Differential algebra space toolbox for nonlinear uncertainty propagation in space dynamics," *The 6th International Conference on Astrodynamics Tools and Techniques (ICATT) (14/03/16 - 17/03/16)*, March 2016.
- [25] E. D. Denman and A. N. Beavers, "The matrix sign function and computations in systems," *Applied Mathematics and Computation*, Vol. 2, No. 1, 1976, pp. 63–94, [https://doi.org/10.1016/0096-3003\(76\)90020-5](https://doi.org/10.1016/0096-3003(76)90020-5).