

THE EFFECT OF UNTARGETED MOONS ON RESONANT ORBIT STRUCTURE BETWEEN TITAN AND RHEA

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Resonant orbits play a key role in design of outer planet moon tours. While preliminary design has traditionally relied on patched-conic or restricted 3-body models—as in previous studies for future missions to Saturn’s moon Enceladus—these models neglect perturbations from “untargeted” moons on resonant orbits, which can lead to difficulties during ephemeris transition. This study hence characterizes Saturnian orbits resonant with Titan and Rhea using a 4-body model including such perturbations. Studying the effect of each moon on the other’s resonant orbits, the internal geometric structure of a number of resonant orbit families is characterized, thus providing a higher-fidelity dynamical portrait of the Titan-Rhea region for future Enceladus mission design.

INTRODUCTION

Resonant orbits play a key role in design of multiple-flyby outer planet moon tours at both high and low energies. Previous work by Anderson and Lo has shown that ballistic, impulsive, and low-thrust trajectories designed for tours around outer planet moons all closely follow unstable resonant orbits and their associated stable and unstable manifolds.^{1–3} These dynamical structures allow for the design of tours crossing multiple resonances via heteroclinic connections,⁴ thus yielding propellant-free changes of the spacecraft’s orbital semi-major axis. While resonant orbits and their manifolds have been widely studied from a dynamical systems perspective in the Jovian moon system,^{5,6} with the latest US planetary science decadal survey’s⁷ recommendation of a flagship and/or New Frontiers-class missions to Saturn’s moon Enceladus, it is useful to explore unstable resonant orbits and the trajectories they generate for the Saturn system as well.

A number of recent studies^{8,9} have explored some Saturnian tour scenarios to reach Enceladus, with a focus on the “pump-down” phase of the trajectory design where flybys of Saturnian moons are used to reduce the semi-major axis until Enceladus orbit insertion. These studies also leveraged resonant flybys of these moons; however, they used patched-conic techniques for preliminary design. More recent work by Kumar¹⁰ has demonstrated that such patched-conic resonant trajectory arcs largely follow higher-energy unstable resonant periodic orbits when considered in the more accurate planar circular restricted 3-body problem (PCR3BP) model; transfers between different resonances in patched conics correspond to heteroclinic connections between said periodic orbits in the PCR3BP, which in turn largely follow the unstable PCR3BP resonant periodic orbits themselves quite closely as well. However, both patched conics and the PCR3BP suffer from one common deficiency: the problem of untargeted flybys.

Generally, outer planet moon tour design relies on designing and joining together resonant trajectory arcs which (mostly) begin and end at flybys with some particular “targeted” moon of interest. Both patched-conics and PCR3BP models include the gravity of at most this one targeted moon (and/or the planet) at any given time. However, it can occur that a trajectory initially designed in either model may in fact experience significant gravitational perturbations from other “untargeted” moons, not included in the original model, when propagated in a more accurate one. An “untargeted flyby” hence occurs when the spacecraft closely

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approaches an untargeted moon during some such trajectory arc. If such an untargeted moon’s perturbations are strong enough, the preliminary trajectory candidate may fail to converge, or may require a prohibitive amount of Δv , when ephemeris transition is attempted. It would hence be useful to systematically understand the effect that untargeted moons have on trajectory arcs resonant with any given targeted moon.

As described earlier, the resonant trajectory arcs of interest for the just-mentioned perturbative study largely correspond to higher-energy unstable periodic orbits in the PCR3BP model. Understanding the effect of untargeted moons on such orbits thus requires studying what happens to them in a restricted 4-body problem (R4BP) model that accounts for both the targeted moon—with which the orbits are resonant—as well as another moon likely to significantly perturb them. In this vein, to better inform future design of Saturnian moon tours, this study characterizes a number of the Saturnian system orbits resonant with Titan and Rhea using a Saturn-Titan-Rhea concentric circular restricted 4-body problem (CCR4BP) model. Focusing on Titan and Rhea resonances lying between both moons, we compute Titan 1:1 and 3:2 interior unstable resonant orbits as well as Rhea exterior 3:4 and 2:3 orbits in this model, understanding their geometric structure and highlighting *secondary resonant* regions of special interest from a perturbative point of view.

In this paper, we start by briefly reviewing the restricted 4-body model used for the analysis, as well as some background on mean motion and secondary resonances. We then give a brief overview of the computational methodologies used in this study. Next, we briefly review some relevant previous results from Saturn-Titan and Saturn-Rhea PCR3BP-based resonant orbit analysis, followed by the main results of this work on Titan and Rhea resonant orbit families computed under the perturbative effect of the other moon. Finally, we conclude with a discussion of these results’ implications as well as future directions to be explored.

MODELS AND BACKGROUND

Concentric Circular Restricted 4-Body Problem

The concentric circular restricted 4-body problem¹¹ (CCR4BP) is a model for the motion of a spacecraft influenced by the gravity of three large masses m_1 , m_2 , and m_3 with $m_1 \gg m_2, m_3$. We assume m_2 and m_3 revolve around m_1 in coplanar, concentric circles of radii r_{12} and r_{13} , where m_2 has no effect on the motion of m_3 nor vice versa; these assumptions well-approximate the motion of Titan and Rhea around Saturn, given those moons’ near-zero inclinations ($\sim 0.3^\circ$) and eccentricities (~ 0.03 and 0.001). Due to Kepler’s third law, the angular velocities Ω_2 and Ω_3 of the revolution of m_2 and m_3 around m_1 depend on the masses and the orbital radii, as $\Omega_i = \sqrt{\mathcal{G}(m_1 + m_i)r_{1i}^{-3}}$ for $i = 2, 3$ where $\mathcal{G}$ denotes the universal gravitational constant. In the planar CCR4BP used in this study, the orbits of m_2 and m_3 as well as the spacecraft trajectory are assumed to all lie in the same plane.

Now, define mass ratios $\mu = \frac{m_2}{m_1 + m_2}$ and $\mu_3 = \frac{m_3}{m_1 + m_2}$. For m_1 , m_2 , and m_3 being Saturn, Titan, and Rhea, respectively, $\mu = 2.36639 \times 10^{-4}$ and $\mu_3 = 4.05746 \times 10^{-6}$; if instead Rhea is m_2 and Titan m_3 , then $\mu = 4.05841 \times 10^{-6}$ and $\mu_3 = 2.36694 \times 10^{-4}$. Similar to the standard formulation of the planar circular restricted 3-body problem (PCR3BP)—of which the CCR4BP is a perturbation—one can normalize mass, length, and time units so that $\mathcal{G}(m_1 + m_2)$, r_{12} , and Ω_2 are all 1. The planar CCR4BP equations of motion can also then be written in the same rotating coordinate system usually used for the CRTBP, with m_1 and m_2 lying on the rotating frame x -axis, and the m_1 - m_2 barycenter taken as the origin. With these units and coordinate frame, the angle between the position of m_3 and the x -axis at time t will be $\theta_3(t) = (\Omega_3 - 1)t + \theta_{3,0}$; the position of m_3 thus is $(x_3(t), y_3(t)) = (-\mu + r_{13} \cos(\theta_3), r_{13} \sin(\theta_3))$. The equations of motion can then be written in position-momentum space (x, y, p_x, p_y) as (see Blazevski and Ocampo¹¹ for a derivation)

$$\begin{aligned} \dot{x} &= p_x + y & \dot{y} &= p_y - x \\ \dot{p}_x &= p_y - (1 - \mu) \frac{x + \mu}{r_1^3} - \mu \frac{x - (1 - \mu)}{r_2^3} - \mu_3 \frac{x - x_3}{r_3^3} - \mu_3 \frac{\cos \theta_3}{r_{13}^2} \\ \dot{p}_y &= -p_x - (1 - \mu) \frac{y}{r_1^3} - \mu \frac{y}{r_2^3} - \mu_3 \frac{y - y_3}{r_3^3} - \mu_3 \frac{\sin \theta_3}{r_{13}^2} \end{aligned} \tag{1}$$

Here, $r_1 = \sqrt{(x + \mu)^2 + y^2}$, $r_2 = \sqrt{(x - 1 + \mu)^2 + y^2}$, and $r_3 = \sqrt{(x - x_3)^2 + (y - y_3)^2}$ are the distances from the spacecraft to m_1 , m_2 , and m_3 , respectively. Eq. (1) are written in same synodic m_1 - m_2 frame as the PCR3BP, with units such that $r_{12} = \mathcal{G}(m_1 + m_2) = \Omega_2 = 1$. Note that when $\mu_3 = 0$, Eq. (1) is just the m_1 - m_2 PCR3BP. The equations of motion Eq. (1) are Hamiltonian, with time-periodic Hamiltonian

$$H_{\mu_3}(x, y, p_x, p_y, \theta_3) = \frac{p_x^2 + p_y^2}{2} + p_x y - p_y x - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2} - \frac{\mu_3}{r_3} + \mu_3 \frac{x \cos \theta_3}{r_{13}^2} + \mu_3 \frac{y \sin \theta_3}{r_{13}^2} \quad (2)$$

Stroboscopic Maps The CCR4BP flow is defined on a 5D extended phase space, $(x, y, p_x, p_y, \theta_3) \in \mathbb{R}^4 \times \mathbb{T}$. The perturbation phase angle θ_3 increases the phase space dimension by 1 versus the 4D phase space of the PCR3BP (x, y, p_x, p_y) . However, one can instead use a stroboscopic map which reduces the dimension of the CCR4BP back to 4D, which is advantageous for many computations. Define the stroboscopic map $F : \mathbb{R}^4 \times \mathbb{T} \rightarrow \mathbb{R}^4 \times \mathbb{T}$ as the time- $\frac{2\pi}{|\Omega_3 - 1|}$ mapping of extended phase space points by the CCR4BP equations of motion. Since $\dot{\theta}_3 = \Omega_3 - 1$, in time- $\frac{2\pi}{|\Omega_3 - 1|}$, the angle θ_3 will revolve by exactly either $\pm 2\pi$. Thus, the θ_3 component of $F(x, y, p_x, p_y, \theta_{3,f})$ will simply be $\theta_{3,f}$ again, for any $(x, y, p_x, p_y) \in \mathbb{R}^4$. As the θ_3 state component is F -invariant, one can fix its value to any $\theta_{3,f}$ (we take $\theta_{3,f} = 0$) and consider the dynamics of F on the 4D subspace $(x, y, p_x, p_y; \theta_{3,f})$ instead of the CCR4BP flow on its 5D extended phase space.

The stroboscopic map preserves all the dynamical properties of the CCR4BP, including invariant sets and manifolds; each CCR4BP trajectory has a stroboscopic map counterpart and vice versa. From this point onwards, we make a slight abuse of notation and consider $F : \mathbb{R}^4 \rightarrow \mathbb{R}^4$. At times we will use a subscript, such as F_{μ_3} , to signify the dependence of F on that CCR4BP mass parameter. Note that the stroboscopic map definition is valid even for $\mu_3 = 0$, which is just the m_1 - m_2 PCR3BP.

Unstable Resonant Orbits

Resonant motions are ubiquitous in celestial systems. Among the most important resonant phenomena for astrodynamics is that of mean motion resonance (MMR). Roughly speaking[†], an $m:n$ mean motion resonance is a region of a celestial system's phase space where one body (in our case the spacecraft) makes m revolutions around some large central body (here, Saturn) in the same time that another body (e.g., a moon of Saturn) makes n revolutions around the same central body. In the spacecraft-moon MMR case, since this is a relation between orbital periods of two bodies with the the second body's period being fixed, the different MMRs correspond to certain spacecraft semi-major axis values, one for each MMR. MMRs with $m > n$ are called *interior* MMRs, while those with $m < n$ are called *exterior* MMRs.

In multi-body celestial systems, MMR regions contain both stable and unstable resonant orbits with various topological properties and structure. Of these, the unstable resonant orbits are of special interest for tour design and low-cost orbit transfers, as if the unstable manifold of an orbit contained in one MMR intersects the stable manifold of an orbit from another MMR, this creates a zero- Δv *heteroclinic* trajectory from the first MMR to the second. This *MMR overlap* in turn yields a natural change of spacecraft semi-major axis. Due to Chirikov's overlap criterion,¹³ MMR overlap is the key driver of global transport across phase space in celestial systems, and determines the range of semi-major axes a spacecraft can reach without Δv .

PCR3BP case In the PCR3BP, unstable resonant orbits correspond to 1-parameter families of unstable periodic orbits, where the parameter can generally be thought of as Jacobi constant C at least locally.¹⁴ For an $m:n$ MMR, its resonant periodic orbits will have periods of *approximately* but not exactly $2\pi n$; the range of periods in fact will vary throughout the family as a function of Jacobi constant. This variation of period has important implications for unstable resonant orbit structure in the CCR4BP, as explained next.

CCR4BP case In the CCR4BP, the structure of unstable resonant orbits can be significantly more complex. Oftentimes, unstable resonant periodic orbits from the m_1 - m_2 PCR3BP will persist as 2D quasiperiodic orbits (invariant tori) into the 5D m_1 - m_2 - m_3 CCR4BP phase space, at least for small enough m_3 ; this occurs due to the periodic forcing from the angle θ_3 , giving an additional frequency for the orbit compared to the

[†]More rigorously, MMRs are defined through studying the dynamics of certain combinations of angles derived from osculating orbital elements. See for instance the book¹² of Morbidelli for more details.

PCR3BP. If we instead consider the CCR4BP stroboscopic map F , these 2D tori become 1D map-invariant tori in a 4D phase space. For more details we refer the reader to our earlier paper on such tori.¹⁵

However, when transitioning from the PCR3BP to the CCR4BP, a different phenomenon occurs at—and also in a region around—those m_1 - m_2 PCR3BP resonant periodic orbits having a period T such that T_p/T is rational; here, $T_p = \frac{2\pi}{|\Omega_3-1|}$ is the period of the m_3 -induced synodic forcing angle θ_3 . Such orbits are not only in mean motion resonance with m_2 , but are also in *secondary resonance* with the synodic period of revolution of m_3 relative to m_2 . For certain initial values of the phase θ_3 , such secondary resonant periodic orbits transition from the PCR3BP to the CCR4BP as long periodic orbits rather than quasi-periodic orbits; if $T_p/T = p/q$, $p, q \in \mathbb{Z}$, then the equivalent CCR4BP orbits’ periods will be pT .

More significantly, even non-secondary resonant MMR orbits in a region around such secondary resonant orbits will be “consumed” by secondary resonant effects in the CCR4BP, as the dynamics in such regions become qualitatively similar to those of the mathematical pendulum — very different from the simpler, relatively “PCR3BP-like” dynamics of the CCR4BP quasiperiodic orbits described previously. However, for sufficiently small m_3 , these secondary resonance regions will remain small and separated, allowing tori to persist between them. However, as m_3 increases, consecutive secondary resonance regions can begin to overlap *inside* of the $m:n$ unstable orbit family, and thus destroy all the “PCR3BP-like” CCR4BP quasiperiodic orbits over a wider area. If it occurs, this overlap leads to a complete change of the orbit family’s internal structure. For full details on secondary resonant phenomena, the reader is referred to our previous paper.¹⁶

OVERVIEW OF COMPUTATIONAL METHODOLOGIES

In the author’s previous work,¹⁰ several families of unstable resonant periodic orbits were computed and analyzed in the Saturn-Titan and Saturn-Rhea PCR3BP models. In this study, we will seek to determine what happens to these orbits in the CCR4BP model taking both Titan and Rhea into account simultaneously—thereby revealing the effect of untargeted Rhea and Titan approaches on Titan and Rhea resonant orbits, respectively. In particular, we would like to understand whether they persist as invariant tori in the Saturn-Titan-Rhea CCR4BP or not, as well as the strength and extent of secondary resonant phenomena within these orbit families. For this analysis, a number of computational tools are required, which are summarized next.

In the CCR4BP, even for understanding the internal structure of one unstable resonant orbit family, one should compute quasiperiodic orbits (invariant tori), secondary resonant periodic orbits, and “separatrices”. Separatrices are submanifolds of certain secondary resonant periodic orbits’ stable/unstable manifolds that mark boundaries of secondary resonance regions which are contained entirely *inside* the resonant orbit family.

Invariant Tori

As mentioned earlier, using the stroboscopic map F instead of the CCR4BP flow not only reduces the phase space dimension from 5D to 4D, but also reduces quasiperiodic orbits from 2D to 1D. This makes them much easier to compute; thus, most computations of tori are done under the map F . Computing such 1D F -invariant tori involves solving for $K : \mathbb{T} \rightarrow \mathbb{R}^4$ satisfying the torus invariance equation

$$F_{\mu_3}(K(\theta)) = K(\theta + \omega) \quad \omega \in \mathbb{R} \quad (3)$$

Here, since the tori being computed come from PCR3BP periodic orbits with known periods T , the *rotation number* $\omega = \frac{2\pi}{|\Omega_3-1|} \frac{2\pi}{T} = 2\pi \frac{T_p}{T}$ is known as well for each torus to be computed.

To solve Equation (3), the main methodology used in this study was a quasi-Newton method developed by the author and collaborators in previous work.¹⁵ In this method, Equation (3) is solved simultaneously with other equations for stable/unstable/central directions as well as Floquet matrix for the torus; this turns out to have lower computational complexity than solving for K alone. For full details of the torus quasi-Newton method, we refer the reader to our paper.¹⁵ Occasionally, a variant of the same quasi-Newton method that replaces the stroboscopic time- T_p map F with mappings by the periodic orbit period T instead was used; computing tori invariant under this time- T map corresponds to taking a cross section of the R4BP flow’s 2D invariant torus not by fixing θ_3 , but by fixing the other phase angle instead. This method variant is described

in another previous paper;¹⁷ the time- T map-invariant tori it yields can then be converted to stroboscopic map F -invariant ones through simple propagations by appropriately chosen integration times.

Finally, occasionally the torus correction steps were carried out using a GMOS-type¹⁸ Newton method in place of the aforementioned quasi-Newton scheme. This was primarily employed in a few cases of nearly-resonant ω values where the quasi-Newton method struggles to converge due to the presence of small divisors when solving certain “cohomological” equations. However, in all other cases, it was found that the quasi-Newton method yielded significantly better performance as compared to GMOS, the latter of whose requirement to solve a large matrix equation at every Newton step leads to long computation times when the number of discretization points used to represent $K(\theta)$ is large—as was often necessary in this paper.

Assuming an invariant torus exists for a given ω , both quasi-Newton methods as well as GMOS converge to the desired torus given a sufficiently accurate initial guess. Thus, one can use them to numerically continue resonant orbits from the PCR3BP ($\mu_3 = 0$) to the $\mu_3 > 0$ CCR4BP. During continuation, due to changing μ_3 and Kepler’s third law, one must vary r_{13} in order to keep Ω_3 and ω constant (which is needed). Note that ω is just 2π times the quantity T_p/T used earlier to define secondary resonances; thus, if $\frac{\omega}{2\pi}$ is rational, we are actually at a secondary resonance and should thus seek to compute periodic orbits rather than tori. Also, if the situation described earlier occurs, where m_3 is large enough that secondary resonance regions overlap and consume all tori between them, then this continuation will fail at some μ_3 less than the desired final value.¹⁶

Secondary Resonant Periodic Orbits

As explained earlier, overlapping secondary resonance regions drive destruction of invariant tori for large enough m_3 values. To analyze this, one must compute secondary resonant periodic orbits as well as separatrices emanating from them that bound said regions. Recall that if $T_p/T = p/q$ for a PCR3BP periodic orbit of period T , then for appropriate initial θ_3 , the corresponding secondary resonant periodic orbit in the CCR4BP will have period $pT = qT_p$. Thus, it will be a q -iteration stroboscopic map-periodic orbit. If q is large, as is often the case for secondary resonances, single-shooting methods can struggle to solve for such long-periodic orbits. Thus, we use multiple shooting to solve for $X(k) \in \mathbb{R}^4$, $k = 0, 1, \dots, q-1$, satisfying

$$F(X(k)) = X(k+p \bmod q) \quad (4)$$

This is equivalent to the slightly different equation $F(X(k)) = X(k+1 \bmod q)$ often seen in the literature;¹⁹ only the ordering of the solution points $X(k)$ will be different. We in fact use a new multiple shooting method developed by the author as an adaptation of the torus quasi-Newton method¹⁵ to the case of periodic orbits. As in the torus method, the periodic orbit is solved simultaneously with its stable, unstable, and (if they exist) central directions at each $X(k)$ and its Floquet matrix. Like the torus case, this method is used for continuation of PCR3BP orbits and Floquet directions/matrix by μ_3 to the $\mu_3 > 0$ CCR4BP. For more details, the reader is referred to the author’s recently-released preprint²⁰ explaining the method in full.

Secondary Resonance Separatrices

Once a solution to Equation (4) and the Floquet directions/matrix have been found for a secondary resonance $p:q$, we not only have the corresponding periodic orbit, but also its stability information. In particular, all secondary resonant periodic orbits will inherit a stable-unstable pair of directions and Floquet multipliers from the PCR3BP orbit they were continued from. However, certain secondary resonant periodic orbits will also gain a *second* pair of “weak” stable/unstable directions and multipliers (weak meaning that these multipliers are closer to 1 than the “strong” pair inherited from the PCR3BP). Under the map F , where the periodic orbit consists of q discrete points, this second pair of directions will correspond to 1D weak stable/unstable submanifolds of the full 2D stable/unstable manifolds of the secondary resonant periodic orbits. These 1D submanifolds mark boundaries of the secondary resonance region *inside* of the $m:n$ unstable resonant orbit family,¹⁶ and are called *separatrices*. They allow us to see the extent of each secondary resonance region and whether or not consecutive regions overlap, which is important for understanding the orbit family structure.

Given a CCR3BP secondary resonant periodic orbit with weak stable/unstable directions, we use a parameterization method to compute its separatrices as nonlinear polynomial curves. We solve for a function

$W(k, s) : \{0, 1, \dots, q - 1\} \times \mathbb{R} \rightarrow \mathbb{R}^4$ satisfying

$$F(W(k, s)) = W(k + p \mod q, \lambda s) \quad (5)$$

where λ is the weak stable or unstable Floquet multiplier found during the computation of the periodic orbit. Equation (5) can be solved recursively by expressing W as a set of q Taylor series with degree- d coefficients $W_d(k)$ depending on the integer k ; here, $W_0(k)$ will be the known periodic orbit points $X(k)$, and $W_1(k)$ is taken from the orbit's weak stable/unstable directions computed during the multiple shooting step. For full details, we again refer the reader to the preprint²⁰ on computing secondary resonant orbits.

SATURN-TITAN AND SATURN-RHEA PCR3BP RESONANT ORBIT PROPERTIES

In this paper, the main analysis will apply the above computational methodologies to various unstable resonant periodic orbit families from the Saturn-Titan and Saturn-Rhea PCR3BPs, to understand their fates in the Saturn-Titan-Rhea CCR4BP. However, before presenting these results, it is worth briefly reviewing the structure of such unstable resonant periodic orbit families each PCR3BP, their relationship with traditional patched-conic trajectories, and orbit family regions of particular interest for transfers as well. As this paper focuses on Titan 1:1 and 3:2 and Rhea 3:4 and 2:3 resonances, we will restrict the following summary—based on the author's previous relevant paper¹⁰—to discussion of periodic orbits from those MMRs.

PCR3BP Resonant Periodic Orbit Families

For any given mean motion resonance (MMR) in the PCR3BP, the unstable periodic orbits often exist as two distinct families based on their geometry and energy levels: a low/mid-eccentricity (e) family and a high- e one. In the Saturn-Titan PCR3BP, the 3:2 interior resonance contains both families. The low/mid- e family consists of lower-energy (high Jacobi constant, C) orbits most of whom reside entirely inside Titan's orbital radius, while the high- e family occurs at much higher energies (lower C) and consists of orbits whose apogees extend beyond Titan's orbit; these are shown on the left and center of Figure 1, respectively. The 1:1 Titan resonance, at least at the high energies characteristic of most Titan tours, is characterized by high-energy planar Lyapunov orbits generated by the collinear libration points; in this study, we focus on those corresponding to L1, shown on the right of Figure 1. Due to Titan's large mass ratio, these unstable periodic orbits exist over a relatively wide range of Jacobi constants without colliding with Titan's surface.

In contrast, due to Rhea's much smaller mass, the Saturn-Rhea system presents much more restrictive bounds on unstable non-collision orbits. The Rhea 2:3 and 3:4 exterior resonances also possess both low/mid- e and high- e families, but their useful non-colliding energy ranges are much smaller. Mid/low- e orbits, though they exist, are mostly not unstable enough to be of utility for tour design. On the other hand, high- e orbits for Rhea MMRs remain stable until a much lower eccentricity is reached (as compared to Titan high- e families), but once they do become unstable, decreasing e much more to further increase instability soon leads to collision with Rhea's surface. For example, non-colliding high- e 2:3 Rhea orbits exist only for $C \in (2.9810, 2.9865)$, while such low/mid- e 2:3 orbits exist for $C \in (2.9879, 3.0524]$. This creates a gap for

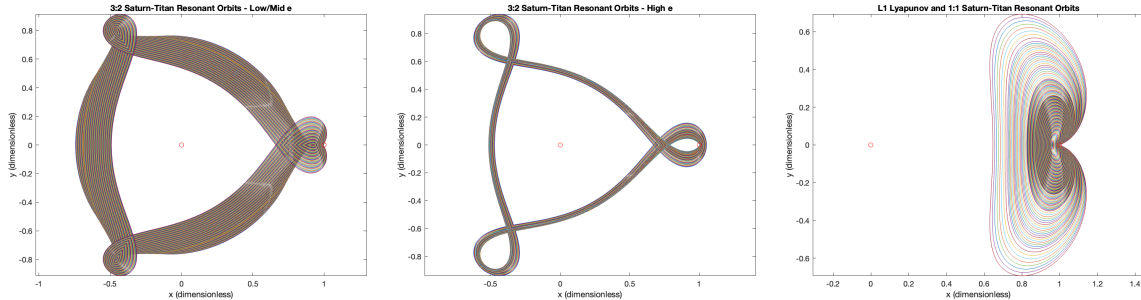


Figure 1. Saturn-Titan PCRTBP periodic orbit families. From left to right: 3:2 low/mid- e , 3:2 high- e , 1:1 L1 Lyapunov. Saturn and Titan as red circles, not to scale

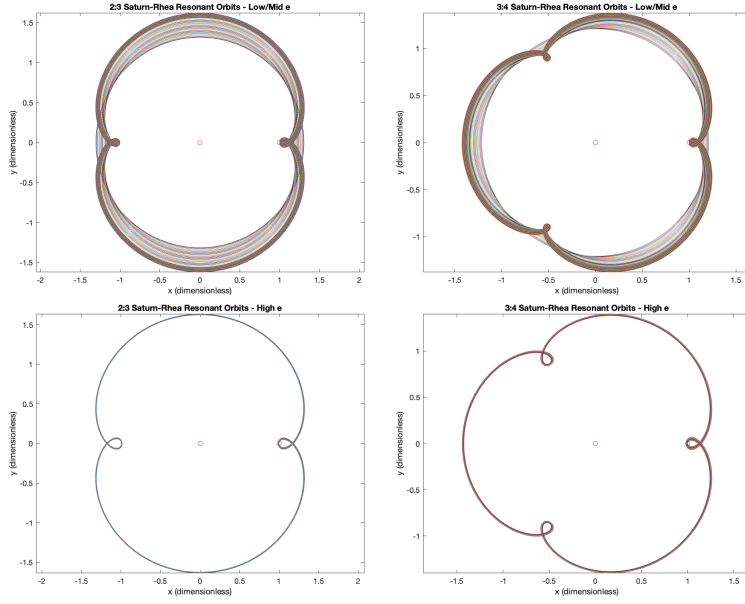


Figure 2. Saturn-Rhea PCRTBP resonant periodic orbit families. Left to right columns: 2:3, 3:4 resonances; top row: low/medium e orbits, bottom row: high e orbits. Saturn and Rhea as red circles, not to scale

$C \in [2.9865, 2.9879]$ where no non-colliding unstable 2:3 orbits exist. Similarly, the 3:4 exterior orbits are restricted to $C \in (2.9931, 3.0266]$ for the low/mid- e family and $C \in (2.9861, 2.9921)$ for the high- e case. All these non-colliding low/mid- e and high- e 2:3 and 3:4 unstable resonant orbits are plotted in Figure 2.

While the PCR3BP provides a higher-fidelity dynamical model, the computed unstable resonant orbits—primarily at higher energies (lower C)—can also be viewed from the perspective of traditional tour design techniques due to a close correspondence with patched-conic trajectory arcs. When viewed in an inertial reference frame, the high-energy, high- e PCR3BP resonant orbits can be clearly described using classical patched-conic vocabulary. For instance, trajectories along high-energy Titan 1:1 Lyapunov orbits can also be understood as a largely-elliptical segment, starting just after a Titan flyby, such that the spacecraft velocity vector points in an “outbound” direction at the initial post-flyby time, but which eventually encounters Titan in an “inbound” direction—only for the velocity to be bent outbound again post-encounter. Similarly, high- e Titan 3:2 interior orbits map well to “inbound-to-outbound” trajectory arcs, while high- e Rhea 3:4 and 2:3 orbits correspond to “outbound-to-inbound” ones. This suggests that understanding what happens to these PCR3BP periodic orbits in the R4BP should shed light on what happens to the patched conics arcs as well.

Transfers Between Resonances and Energy Constraints

Heteroclinic connections created by intersections of these periodic orbits’ stable and unstable manifolds enable natural transport between different MMRs. In the Titan PCR3BP, heteroclinic connections between resonances (e.g., transferring from a 1:2 exterior orbit down to a 3:2 interior orbit via the 1:1 resonance, as shown in Figure 3) are quite abundant over a wide range of Jacobi constants. Furthermore, these heteroclinic trajectories between resonant periodic orbits very closely shadow the periodic orbits themselves; for instance, on the right of Figure 3, a plot of the L1 Lyapunov orbit at the same energy is overlaid on the red heteroclinic 1:2-to-1:1-to-3:2 transfer. A close resemblance is immediately visible between the L1 Lyapunov orbit and the 1:1 portion of the heteroclinic; although not shown here, the same is true for the 1:2 and 3:2 portions of the heteroclinic and the corresponding periodic orbits as well.

In contrast, in the Rhea PCR3BP, achieving zero- Δv heteroclinic transfers with reasonable times-of-flight requires operating within very narrow ranges of energy values. For example, to successfully transfer from a Rhea 2:3 exterior to a 3:4 exterior resonance without impacting Rhea, one such range exists near $C \approx 2.988$,

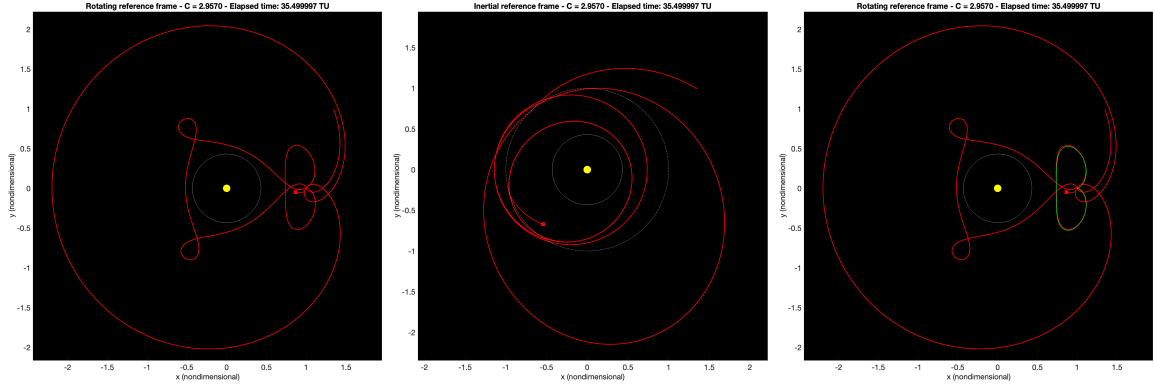


Figure 3. Titan 1:2 to 1:1: to 3:2 PCRTBP heteroclinic for $C = 2.957$ in rotating (left) and inertial (right) frames. Center: L1 Lyapunov 1:1 orbit overlaid. Saturn (yellow) to scale, Titan orbit as white dashed circle, Rhea orbit as white solid circle

where heteroclinic connections link mid- e 2:3 orbits to high- e 3:4 orbits via an intermediate 7:10 resonant trajectory; such a connection does not exist even at the only-slightly higher $C = 2.9885$, while on the other hand trying to decrease C leads to impact just at $C < 2.9879$. The only other (highly constrained) window without collisions exists between $2.9861 \leq C \leq 2.9865$, where one can transfer from the high- e 2:3 orbits to high- e 3:4 orbits utilizing the 5:7 resonance as an intermediate stepping stone.

TITAN RESONANT ORBITS + RHEA

With computational tools for transitioning PCR3BP orbits into the R4BP described, and the Saturn-Titan and Saturn-Rhea PCR3BP unstable resonant periodic orbits of greatest interest identified, we can now begin our study of what happens to the latter orbits in the Saturn-Titan-Rhea CCR4BP. We begin with an analysis of the effect of Rhea on L1 1:1 and interior 3:2 Titan resonant orbits; the former resonance was the final Titan resonance used by Takubo et al⁸ in their patched conics-based Saturn tour study, while the latter could potentially be used to further decrease the spacecraft semimajor axis from 1:1 before switching to Rhea flybys. Moreover, both resonances come closer to Rhea’s orbit than any other Titan resonance used by Takubo et al,⁸ so it is worth exploring whether Rhea has any significant effects on any part of either unstable orbit family.

As a note, recall that when computing CCR4BP orbits, we find invariant objects (tori, periodic orbits, separatrices) of the stroboscopic map F rather than continuous-time orbits of the flow; even when using the time- T period map method, the resulting tori are converted to F -invariant ones for the sake of visualization. Unless otherwise specified, in the rest of this paper, all plots will display objects invariant under F rather than under the flow or period map. Also, though many plots will be in Cartesian coordinates, it is also often illustrative to plot the map-invariant orbits in “adapted” coordinates derived from linear symplectic transformations of the synodic Delaunay coordinates* (L, G, ℓ, g). In particular, except for the 1:1 case, we will also plot the CCR4BP unstable orbit families for the $m:n$ mean motion resonance in $(\ell + g, -mL + nG)$ space. This is because when m and n differ by 1—as is true for all non-1:1 resonances considered in this paper—then one can show that in these coordinates, for $\mu_3 = 0$, trajectories will lie largely on horizontal lines. Secondary resonant phenomena then become very easy to detect visually for $\mu_3 > 0$. For an explanation of why this occurs, we refer readers to our previous paper¹⁶ which used such coordinates with $m = 4, n = 3$.

Titan 1:1 resonant L1 Lyapunov orbits

We start with the Titan L1 planar Lyapunov orbit family, which is part of the 1:1 resonance and indeed contains orbits closely matching 1:1 patched-conic trajectory arcs within it at high energies. Primarily using the period map-based torus computation method, a wide range of such orbits were numerically continued

*For the unfamiliar reader, synodic Delaunay coordinates²¹ are related to basic osculating orbital elements through the equations $L = \sqrt{(1 - \mu)a}$, $G = L\sqrt{1 - e^2}$, ℓ = osculating mean anomaly, and g = osculating argument of periapsis with respect to the m_1 - m_2 direction vector (i.e., the x -axis of the PCRTBP and CCR4BP). This means that g is not constant over the course of an orbit.

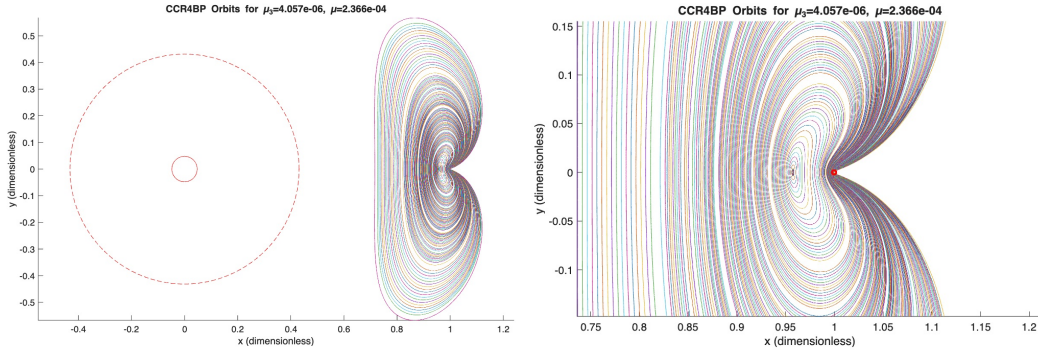


Figure 4. 1:1 L1 Lyapunov tori in Saturn-Titan-Rhea CCR4BP, with Titan shown as small red circle and Saturn as large red circle. Rhea orbit shown as red dashed circle.

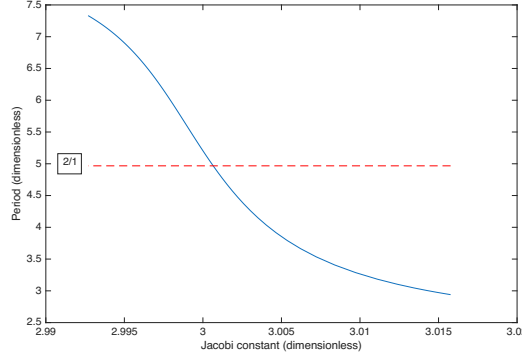


Figure 5. 1:1 Titan L1 Lyapunov orbit periods vs Jacobi constant in the PCR3BP, with 1:2 Rhea secondary resonance period marked as dashed red line.

successfully as tori into the physical-mass Saturn-Titan-Rhea CCR4BP. These are shown in Figure 4 and correspond to Titan PCR3BP orbits with Jacobi constants from that of Saturn-Titan L1 ($C = 3.0158$) all the way until $C = 2.95$. This last, very-high-energy Jacobi constant is even past the value below which the Lyapunov orbits collide with Titan in the PCR3BP, which occurs around $C = 2.992$ (although note that non-colliding heteroclinics may still shadow such orbits at $C < 2.992$, e.g. the one shown in Figure 3). The R4BP tori can be seen to remain visually very similar to the PCR3BP L1 Lyapunov orbits shown in Figure 1.

Given that this wide range of tori were successfully computed in the R4BP, it can be concluded that Rhea-induced secondary resonant phenomena do not play a dominant role in shaping this Titan L1 Lyapunov 1:1 resonant orbit family even when Rhea’s gravity is accounted for. This is not surprising, given Rhea’s small size and the visibly-large distance between even the highest-eccentricity Lyapunov orbits and Rhea’s orbit (see Figure 4). One particular secondary resonance does play a part in the numerical computation process, though. Plotting the PCR3BP orbit periods vs Jacobi constants in Figure 5 (only shown for non-collision orbits), and given that the synodic period of Rhea with respect to Titan is about $T_p \approx 2.48376$, it can be seen that the $T_p/T = p/q = 1/2$ secondary resonance lies in the middle of the orbit family. The presence of this 1:2 secondary resonance does make it difficult to compute tori corresponding to PCR3BP orbits with periods in a small but non-trivial interval surrounding $2T_p$, suggesting that the secondary resonance region surrounding that orbit may have a small-but-noticeable size as well.

Titan interior 3:2 resonant orbits

We next studied the Titan interior 3:2 resonant orbits across both low and high energy values. As described earlier, the 3:2 Titan orbits are divided into two families: a low/mid- e one and a high- e family. Again mostly using the period map-based torus computation method, a wide range of low/mid- e orbits were numerically continued successfully as tori into the physical-mass Saturn-Titan-Rhea CCR4BP. These are shown by themselves in the center of Figure 6 and alongside high- e orbits in the rightmost plot of the same figure; they correspond to Titan PCR3BP low/mid- e 3:2 orbits with Jacobi constants from $C = 3.0547$ to about 2.960.

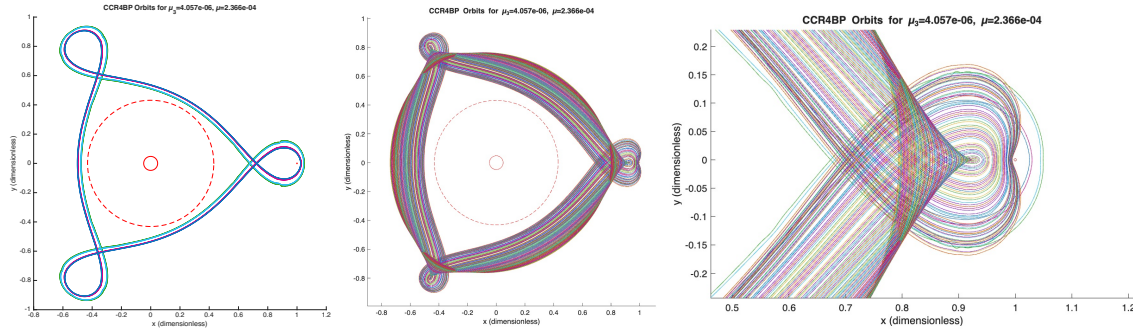


Figure 6. 3:2 unstable resonant tori in CCR4BP, with Titan, Saturn, and Rhea orbit shown as before. (L) high e , (C) low/mid- e , (R) all 3:2 orbits, zoomed in near Titan

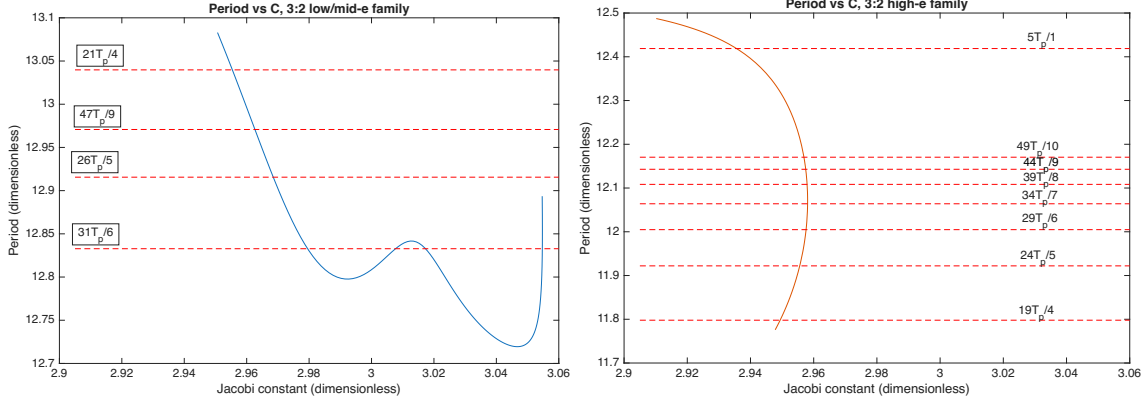


Figure 7. Low/mid- e (left) and high- e (right) 3:2 Titan resonant orbit periods vs Jacobi constant in the PCR3BP, with prominent secondary resonances indicated

A smaller number of high- e orbits, shown on the left-most plot of Figure 6 by themselves and on the rightmost plot alongside the low/mid- e family, were also continued as invariant tori into the CCR4BP. These successfully-computed high- e 3:2 CCR4BP tori include an orbit corresponding to one of the highest-energy/eccentricity PCR3BP unstable 3:2 orbits, at $C = 2.9257$, as well as one continued from a PCR3BP $C = 2.949$ orbit. The much lower number of computed high- e tori here is not necessarily due to their non-existence; indeed, the highest- e PCR3BP orbits—which approach Rhea’s orbit most closely—in fact were successfully continued into the R4BP. However, many of the slightly-less eccentric high- e Titan orbits are very unstable and thus difficult to transition into the R4BP using the single-shooting torus computation methods employed in this study. Attempted computation of more high- e orbits is currently work in progress.

It is clear from Figure 6 that the 3:2 Titan family in the R4BP is largely comprised of invariant tori that persist under the perturbation from Rhea. Nevertheless, plotting the periods T of the 3:2 unstable PCR3BP orbit families in Figure 7, a number of lower-order (denominator less than 50) secondary resonances were identified between the varying orbit periods T and the Rhea forcing period of $T_p \approx 2.48376$ —a Farey sequence of rational numbers is useful for this. Lowest-order is a 1:5 secondary resonance inside the high- e family, but many others are present as well in both families; a subset of their corresponding PCR3BP orbit periods are marked as red dashed lines in the Figure 7 plots as well. We discuss these in more detail next.

3:2 Titan mid/low- e orbits Secondary resonant (SR) periodic orbits and separatrices were successfully computed in the R4BP for all four SRs marked in the low/mid- e (left) plot of Figure 7. As SR effects are likely to be strongest at higher energies/eccentricities (where the orbit periapse comes closest to Rhea’s orbital radius), we only continued the highest-energy 6:31 SR orbit (near $C = 2.98$) rather than all four 6:31 orbits visible in Figure 7. Plotting the computed low/mid- e tori, secondary resonant periodic orbits, and separatrices in the previously-described resonance-adapted combination of synodic Delaunay coordinates $(\ell + g, -3L + 2G)$ then yields Figure 8 and its zoomed-in version Figure 9. In this coordinate system, the orbits’ energy and eccentricity increase as one moves downwards in the plot.

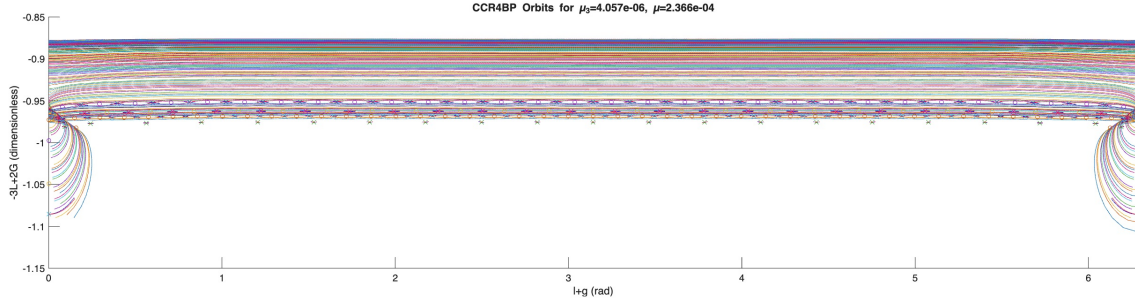


Figure 8. Low/mid- e 3:2 Titan orbits in Saturn-Titan-Rhea CCR4BP, adapted coordinates. Secondary resonant periodic orbits with weak stable/unstable directions shown as x's (with stable/unstable separatrices in blue/red), otherwise as o's.

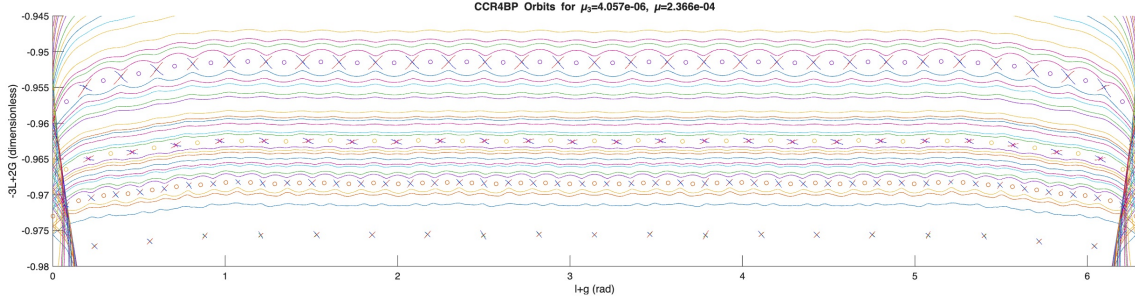


Figure 9. Zoomed in view of low/mid- e 3:2 Titan orbits in Saturn-Titan-Rhea CCR4BP, adapted coordinates. 6:31, 5:26, 9:47, and 4:21 secondary resonant periodic orbits and separatrices visible (from top to bottom)

From the adapted-coordinate plots, a few things become clear. First of all is once again the lack of secondary resonance overlap in the low/mid- e orbit family; all four SR regions are separated from each other by invariant tori which cannot be crossed by separatrices. However, some localized effects at and near the SR regions are nevertheless clearly visible in the zoomed-in Figure 9, which focuses on the highest- e part of the low/mid- e family. There, one can clearly see pendulum phase portrait-shaped gaps corresponding to secondary resonances (from top to bottom) 6:31, 5:26, 9:47, and 4:21. Interestingly enough, the 6:31 SR near $-3L + 2G = -0.953$, which is the least eccentric one computed in the low/mid- e 3:2 family, visually seems to have the largest vertical width and extent. Also, we have not yet successfully computed any tori below the 4:21 SR, so it is not yet clear whether there could be secondary resonance overlap occurring in the (highest-eccentricity) low/mid- e orbit region below the last successfully computed torus.

While some secondary resonant effects are seen in this 3:2 Titan low/mid- e family, overall the persistence of tori into the R4BP all the way until PCR3BP orbits with $C = 2.96$ does restrict the widths of any secondary resonant motions which might occur within the orbit family at all but the highest possible orbital energies. Thus, secondary resonances are not the dominant driver of the dynamics inside the low/mid- e 3:2 family here, which remains mostly on “PCR3BP-like” tori with fairly small Titan perturbations.

3:2 Titan high- e orbits Similarly to the low/mid- e case, the 3:2 high- e tori as well as periodic orbits and separatrices corresponding to the secondary resonances marked in the right plot of Figure 7 (except 7:34) are plotted in 3:2 resonance-adapted coordinates in Figure 10. In this case, the only SR region which clearly displays a significant extent and influence is the 1:5 SR near $-3L + 2G = -0.995$. This secondary resonance very visibly perturbs the tori below it in the plot of Figure 10, and gives rise to stable/unstable separatrices that form a significant angle between each other. However, this SR too is bracketed on both sides by invariant tori, thus preventing it from interacting with any of the other secondary resonances that occur in the region above those tori.

Given the small number of tori computed in the high- e family, it is not immediately clear whether there might be significant secondary resonance overlap and destruction of tori in any part of the family. However,

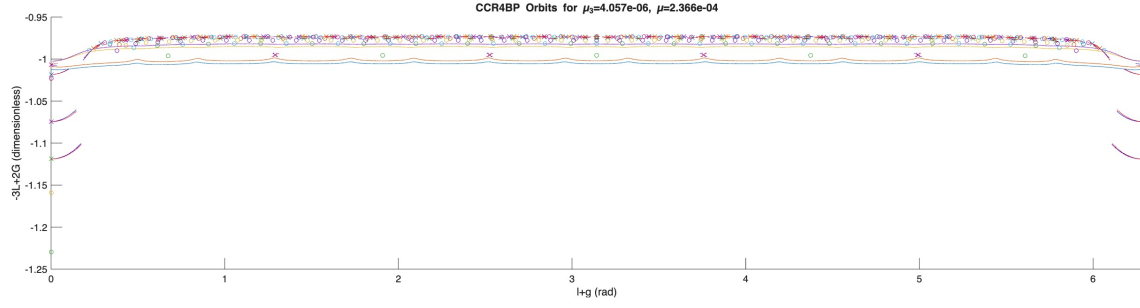


Figure 10. high- e 3:2 Titan orbits in Saturn-Titan-Rhea CCR4BP, adapted coordinates. Secondary resonant periodic orbits with weak stable/unstable directions shown as x's (with stable/unstable separatrices in blue/red), otherwise as o's.

looking closely at the other secondary resonances of Figure 9 (between $-3L + 2G = -0.98$ and -0.97), one finds stable/unstable separatrices which lie almost right on top of each other—only forming very small angles between stable/unstable separatrix directions. This is an indication that tori likely do exist on either side of these secondary resonant periodic orbits, instead of overlap with other secondary resonances. This part of the 3:2 high- e orbit family, lying above the last currently-computed torus, also displays an interesting behavior wherein the eccentricity, after having decreased as one moved up the plot starting from the bottom tori to the 1:5 SR to other tori and certain SRs above them, then starts increasing again. The last few SR orbits thus move back down along the plot's vertical axis; this may be related to the fold bifurcation visible in the period vs Jacobi constant plot of Figure 7, where C takes a maximum along the family and then starts decreasing as the orbit period decreases.

Summary of 3:2 Titan resonance In summary, the most likely Rhea-induced secondary resonances which could potentially have some utility when using the 3:2 Titan resonance in future mission design are the 1:5 secondary resonance in the high- e family, and the highest-energy/eccentricity 6:31 region in the low/mid- e family. However, even these 1:5 and 6:31 regions will only allow movement in their relatively-close vicinities, rather than connecting with other secondary resonances to enable larger-scale transport along the 3:2 orbit family. Overall, the results of this and the previous 1:1 analysis indicate that the very vast majority of 3:2 and 1:1 Titan trajectory arcs will not be irrecoverably perturbed by untargeted Rhea flybys; the corresponding unstable PCR3BP periodic orbits persist as very “PCR3BP-like” invariant tori which are in 1-to-1 correspondence with those periodic orbits and patched conic arcs as well.

RHEA RESONANT ORBITS + TITAN

With the most interesting Titan resonant orbits largely fully characterized in the Saturn-Titan-Rhea CCR4BP (except a part of the high- e 3:2 orbits), we now shift focus to orbits in exterior mean motion resonance with Rhea. While it was fairly foreseeable that the effect of Rhea on 1:1 and 3:2 Titan resonant orbits would be small, given Rhea's small size, Titan's effect on many Rhea resonant orbits is less easy to predict. Titan is two orders of magnitude larger than Rhea, and even one order of magnitude larger than Jovian and Uranian moons that have been shown^{16,22} to generate overlapping secondary resonances in those systems. However, Titan is also relatively far from Rhea's orbit, with a semimajor axis over 2.3 times larger than Rhea's. It is thus not clear for which Titan resonances the effect of untargeted Titan approaches will be significant. Hence, in this section, we investigate 3:4 and 2:3 unstable Rhea resonant orbits in the Saturn-Titan-Rhea CCR4BP.

Rhea exterior 3:4 resonant orbits

We start our investigation of Rhea's exterior resonances with the 3:4 unstable resonant orbits. Applying our numerical continuation tools to transition the Saturn-Rhea PCR3BP periodic orbit families into the Saturn-Titan-Rhea CCR4BP once again yields many invariant tori, which are displayed in Cartesian coordinates in Figure 11. The left and center panels show the successfully continued high- e and low/mid- e invariant tori, respectively, while the rightmost panel displays both families in a zoomed-in view near Rhea. The computed low/mid- e 3:4 tori correspond to Rhea PCR3BP 3:4 unstable resonant orbits having Jacobi constants between

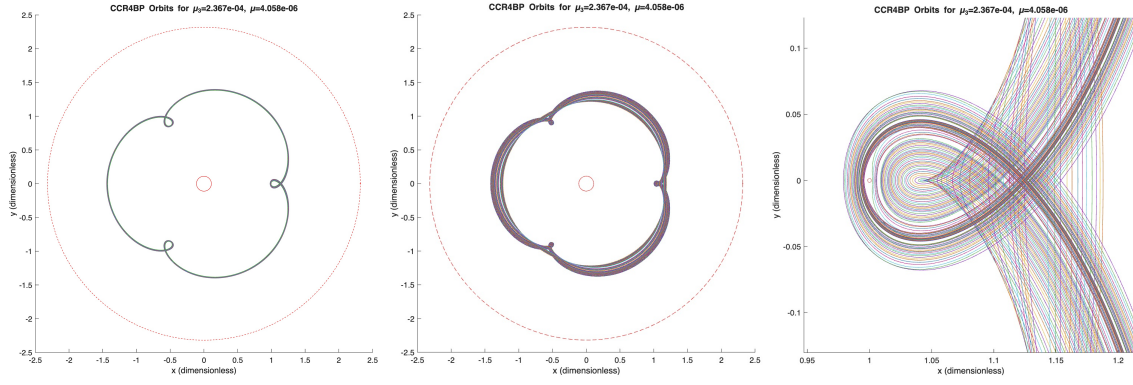


Figure 11. 3:4 unstable Rhea resonant tori in CCR4BP, with Titan, Rhea, and Titan orbit overlaid. (L) high e , (C) low/mid- e , (R) all 3:4 orbits, zoomed in near Rhea

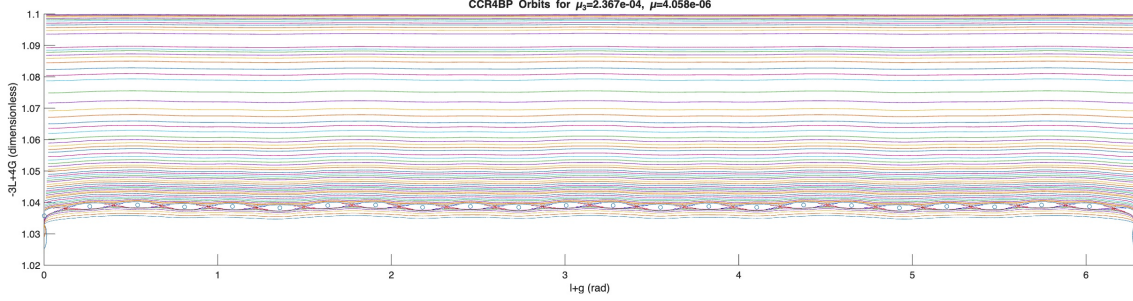


Figure 12. Low/mid- e 3:4 Rhea orbits in Saturn-Titan-Rhea CCR4BP, adapted coordinates. Secondary resonant periodic orbits with weak stable/unstable directions shown as x's (with stable/unstable separatrices in blue/red), otherwise as o's.

3.0264 and 2.994, the latter being just above the collision value of $C = 2.993$ for the mid- e Rhea 3:4 orbits. While these low/mid- e family energy values are not well-suited to natural heteroclinic transfers to 3:4 from resonances like 2:3—recall the PCR3BP results summarized in the background section—it may be possible to use a small propulsive maneuver to transfer to such a lower-energy 3:4 orbit instead. Thus, it is worth analyzing the low/mid- e family nevertheless. The successfully-continued high- e R4BP 3:4 tori on the other hand correspond to C between 2.986 and 2.9918, which does include both ranges of C values at which heteroclinic transfers to 3:4 from 2:3 orbits are possible in the Saturn-Rhea PCR3BP.

As in the 3:2 Titan case, the lowest-order secondary resonances between the Titan forcing period $T_p \approx 8.7669$ and the Rhea PCR3BP 3:4 orbit periods T can be identified using a Farey sequence of rational numbers; one simply finds the minimum and maximum values of T along each continuous orbit family, computes the two corresponding minimum and maximum ratios T_p/T , and finally checks which rational numbers with denominator less than a chosen threshold (given in order from 0 to 1 by the Farey sequence) lie between those two T_p/T values. In the low/mid- e family, only considering non-collision orbits, the lowest-order SR is 8:23. Looking at the high- e non-collision orbits, the only SRs with denominator less than 50 were 7:20 and 13:37. Secondary resonant periodic orbits and separatrices were computed successfully in the R4BP for all of these SRs; we discuss these as well as both R4BP 3:4 orbit families next.

3:4 Rhea mid/low- e orbits Again, to better analyze the internal geometric structure and check for secondary resonant phenomena induced by Titan's periodic perturbation, we transform these invariant tori and the computed 8:23 secondary resonant orbits into the resonance-adapted synodic Delaunay coordinates $(\ell + g, -3L + 4G)$. For the low/mid- e family orbits, the result is shown in Figures 12-13, with the latter zooming into the highest-eccentricity portion of the low/mid- e Rhea 3:4 R4BP family. The successful continuation of the invariant tori visible in these figures means that despite Titan's much larger mass, significant secondary resonance overlap does not occur in the 3:4 Rhea low/mid- e orbit family either; once again, the

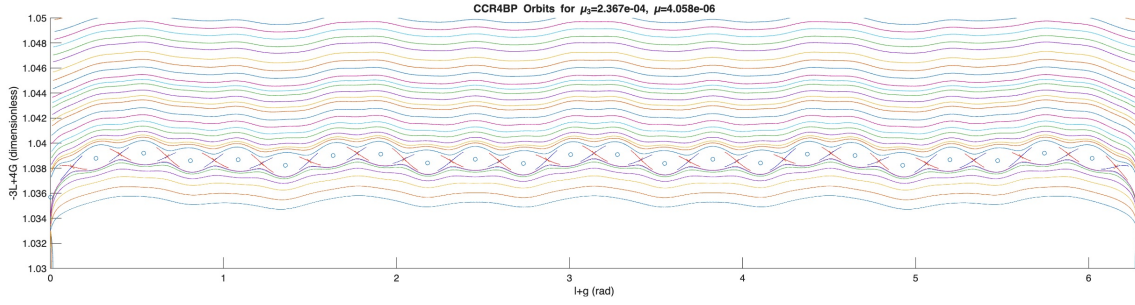


Figure 13. Zoomed-in view of higher-eccentricity portion of low/mid- e 3:4 Rhea orbits in CCR4BP, adapted coordinates. 8:23 secondary resonance prominently visible.

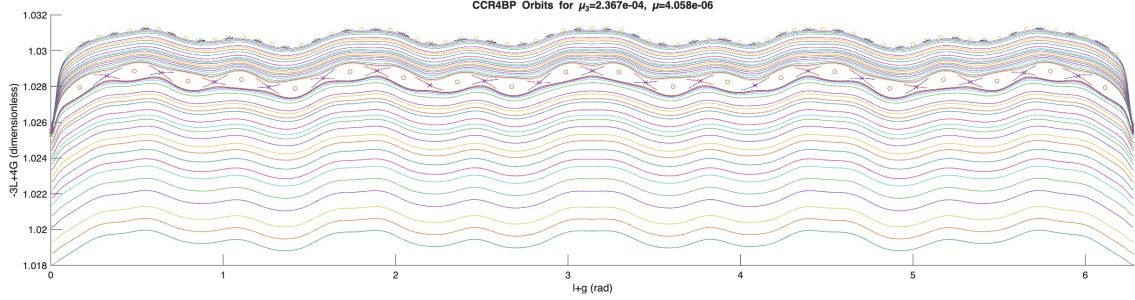


Figure 14. High- e 3:4 Rhea orbits in Saturn-Titan-Rhea CCR4BP, adapted coordinates. Secondary resonant periodic orbits with weak stable/unstable directions shown as x's (with stable/unstable separatrices in blue/red), otherwise as o's.

family is primarily dominated by persisting, “PCR3BP-like” invariant tori. The primary exception to this occurs among its highest-eccentricity orbits around $-3L + 4G \approx 1.039$. Here, a noticeable pendulum phase portrait-shaped island appears corresponding to the 8:23 secondary resonance with Titan’s synodic forcing period T_p , as is clearly visible in Figure 13. However, similar to the SR regions in the Titan 3:2 case, this 8:23 SR region is bracketed by invariant tori on both sides as well, isolating and preventing any overlap with other secondary resonances; nevertheless, the dynamics induced therein and on nearby tori may be of utility for (small) Jacobi constant or v_∞ changes.

3:4 Rhea high- e orbits A similar structural persistence of most orbits as tori is observed in the high- e family of 3:4 orbits, which is displayed in $(\ell + g, -3L + 4G)$ coordinates in Figure 14. Due to their higher eccentricities, the apogees of these orbits extend further outward towards Titan as compared to the low/mid- e family, thereby leading to stronger perturbations, but tori continue to nevertheless prevent secondary resonant islands from overlapping. Near $-3L + 4G \approx 1.029$, one can clearly see the 7:20 secondary resonance has a significant but localized effect; while its stable and unstable separatrices form prominent islands within the family, the SR region once again remain bounded by tori on either side. The second-lowest denominator SR, namely 13:37, can be seen in Figure 14 just above all the tori; the very small angle between its stable and unstable separatrices provides strong evidence of the presence of tori above it as well.

In summary, untargted Titan approaches do not induce widespread transport within the unstable 3:4 Rhea orbits. Except for significant-but-local changes near 8:23 (in low/mid- e) and 7:20 (in high- e) secondary resonances, the overarching 3:4 orbital structures mostly persist with only relatively-small deformations as tori. This means that PCR3BP or even patched-conics versions of most 3:4 orbits (except perhaps those near the 7:20 and 8:23 secondary resonances) can likely be reliably leveraged in moon tour design without large unexpected effects from Titan’s gravity. These two secondary resonances, on the other hand, may allow for small changes in Rhea-relative orbital energy or v_∞ if targeted in mission design, but the magnitude of this effect will be limited by the presence of tori above and below.

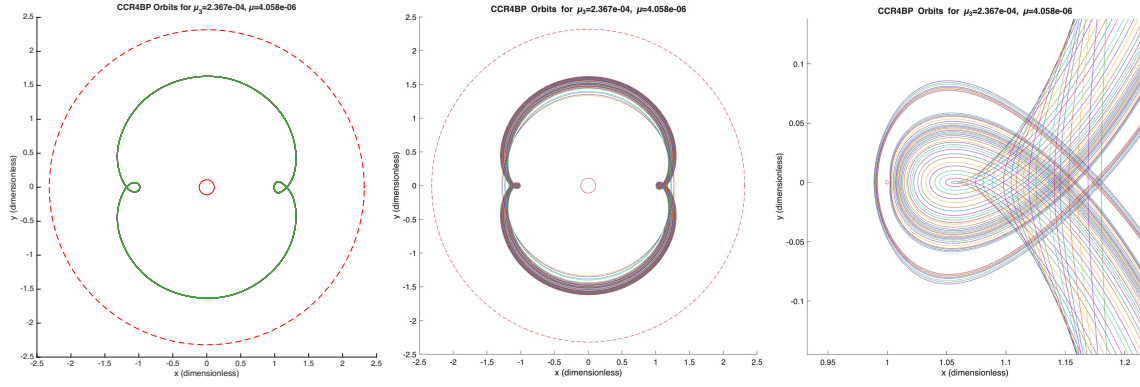


Figure 15. 2:3 unstable Rhea resonant tori in CCR4BP, with Titan, Rhea, and Titan orbit overlaid. (L) high e, (C) low/mid-e, (R) all 2:3 orbits, zoomed near Rhea

Rhea exterior 2:3 resonant orbits

We finally turn our attention to the Rhea exterior 2:3 resonance. Having a semi-major axis larger than that of the 3:4 MMR, the 2:3 resonance brings a spacecraft closer to Titan’s orbit than the 3:4 resonance—especially at apogee. The Rhea 2:3 orbits which were successfully continued into the Saturn-Titan-Rhea CCR4BP as tori are displayed in Cartesian coordinates in Figure 15, which shows the high- e (left), low/mid- e (center), and combined families zoomed in near Rhea (right).

Here, the low/mid- e tori that were successfully computed correspond to Rhea PCR3BP 2:3 orbits having Jacobi constant values of 3.0522 to 2.9885; note that the latter is just slightly above the mid/low- e family Rhea collision value of $C = 2.9879$ as well as just above the value $C = 2.988$ where Rhea PCR3BP 2:3 mid- e to 3:4 heteroclinics were found.¹⁰ As for the high- e family, the R4BP tori found so far were continued from PCR3BP orbits ranging from $C = 2.9823$ to 2.9836; thus, it is not yet known whether the 2:3 high- e PCR3BP orbits which enabled 2:3-to-3:4 heteroclinics for C near 2.9861-2.9865 would persist as tori or not. The investigation of 2:3 orbits in this last range of energy values is ongoing.

Again using the Farey sequence-based method to find secondary resonances among the non-colliding Rhea 2:3 orbits, the lowest-denominator SRs for the low/mid- e 2:3 family were 13:28, corresponding to a Rhea PCR3BP orbit with $C = 2.9919$, and 6:13, continued from a $C = 2.9881$ PCR3BP orbit. Note that the 6:13 occurs at a Jacobi constant value found to be useful for 2:3-to-3:4 heteroclinic transfers in previous PCR3BP work,¹⁰ as well as just slightly above the $C = 2.9879$ value where the low/mid- e family collides with Rhea. In the high- e family, the only secondary resonance with denominator less than 30 (among non-colliding orbits) is 7:15, corresponding to a $C = 2.985$ PCR3BP orbit. Again, secondary resonant periodic orbits and separatrices were computed in the CCR4BP for all these SRs, and are discussed next.

2:3 Rhea low/mid- e orbits Transforming the computed low/mid- e tori and 13:28 and 6:13 secondary resonant periodic orbits/separatrices into the adapted Delaunay coordinates $(\ell + g, -2L + 3G)$ yields Figure 16. In this figure, eccentricity increases from top to bottom; Figure 17 then focuses on the highest-eccentricity/energy portion of the 2:3 low/mid- e family, corresponding to orbits near the bottom of Figure 16. Despite the relatively closer approaches of the 2:3 Rhea orbits to Titan, the overall dynamical portrait of the 2:3 low/mid- e orbits nevertheless remains quite torus-dominated as in the Rhea 3:4 case, largely retaining a PCR3BP-like structure; we observe no significant secondary resonance overlap in this family either, at least in the area computed (the torus on bottom corresponds to a $C = 2.9885$ Rhea PCR3BP orbit).

In this low/mid- e 2:3 family, again secondary resonant islands become noticeable only towards the higher-eccentricity portions of the orbits. The fairly-thin 13:28 secondary resonance region occurs near $-2L + 3G \approx 1.053$, and seems to only have minimal effect on the tori which tightly bracket it from both above and below—isolating it from other SR regions. The 6:13 secondary resonance near $-2L + 3G \approx 1.048$ is more noticeable, although only a bounding torus on top has been computed so far; the lack of tori below the 6:13 island is not yet known to be due to any secondary resonance overlap, and is currently under investigation to see

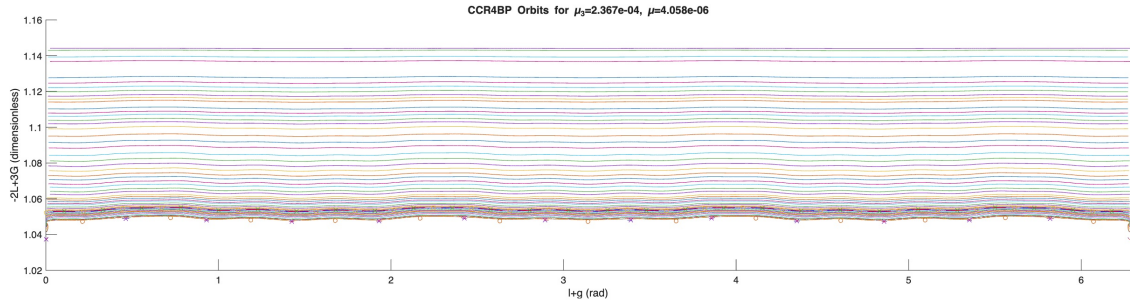


Figure 16. Low/mid- e 2:3 Rhea orbits in CCR4BP, adapted coordinates. Secondary resonant periodic orbits with weak stable/unstable directions shown as x's (with stable/unstable separatrices in blue/red), otherwise as o's.

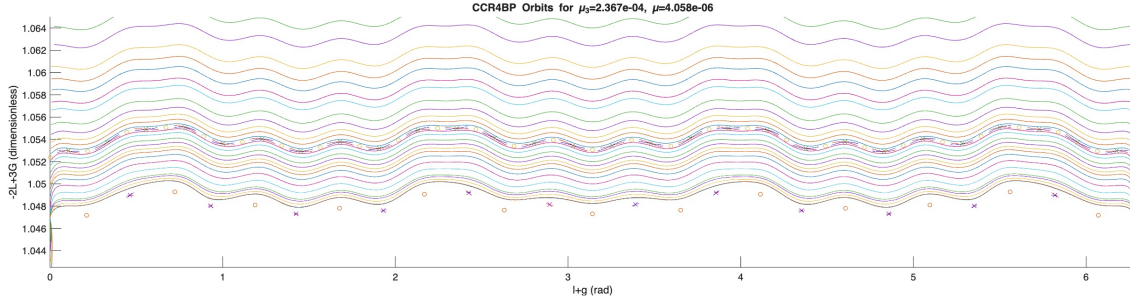


Figure 17. Zoomed-in view of higher-eccentricity portion of low/mid- e 2:3 Rhea orbits in CCR4BP, adapted coordinates. 13:28 and 6:13 SRs shown from top to bottom.

if any tori might exist there as well. Also, recall that the center of this 6:13 SR region corresponds to a $C = 2.9881$ PCR3BP orbit useful for heteroclinic transfers, but also lying just slightly above Rhea collision at $C = 2.9879$. Thus, it is possible that the effect of this secondary resonance with Titan could lead to Rhea collision if, for example, one started on a separatrix leading towards higher-eccentricity/energy orbits.

2:3 Rhea high- e orbits In the high- e family, where spacecraft apogees reach their closest proximity to Titan, secondary resonance phenomena become more prominent. This can be seen clearly when plotting the successfully-computed high- e 2:3 tori and 7:15 SR orbits in the adapted coordinate plane of Figure 18. In this plot, the 7:15 secondary resonance becomes very clearly visible, with stable/unstable separatrices that form a wide angle with each other and a strong distorting effect on the tori nearby it. The tori below the 7:15 SR correspond to PCR3BP orbits near $C = 2.9823$, which are the most energetic and eccentric orbits of the high- e unstable 2:3 family. Inside the 2:3 high- e family, these tori bound and isolate this 7:15 secondary resonance from other SRs on one side—in fact, on the higher- e side, which would have the closest apogees to Titan's orbit and thus be *more* strongly perturbed by Titan. Despite this stronger Titan forcing, though, these

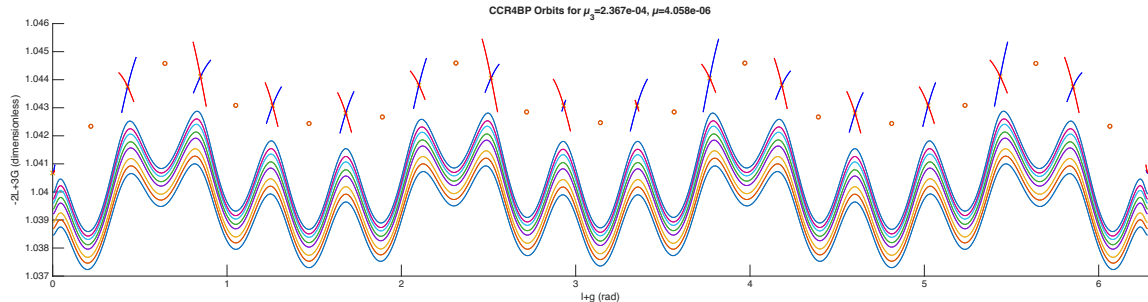


Figure 18. High- e 2:3 Rhea resonant orbits in Saturn-Titan-Rhea CCR4BP, adapted coordinates. Secondary resonant periodic orbits with weak stable/unstable directions shown as x's (with stable/unstable separatrices in blue/red), otherwise as o's.

tori below the 7:15 SR survive into the R4BP. Determining the existence or breakdown of bounding tori on the lower- e side (above the 7:15 SR) is currently work in progress as well, though the persistence of the most strongly-perturbed orbits would suggest that these lower- e , less perturbed ones may likely survive as well.

Summary In summary, the widespread persistence of invariant tori across wide ranges of energy and eccentricity in both 3:4 and 2:3 exterior resonances carries significant implications for trajectory design in the Saturnian system. Most notably, it confirms that these two Rhea resonant orbit families—and almost certainly all others with semi-major axis below the 2:3 MMR—do not completely change their internal topological structure when Titan’s gravitational perturbation is introduced. While some prominent secondary resonant islands appear towards higher eccentricities in both cases, tori persist between and separate these secondary resonant islands, so the underlying PCR3BP orbits (and likely their patched-conic equivalents) largely survive in the CCR4BP mostly without major changes. This indicates that with the exception of localized secondary resonances which are topologically restricted in amplitude, Titan perturbations likely can neither be leveraged nor cause major problems during trajectory design in these 2:3 and 3:4 MMR regions.

CONCLUSION

In this paper, we investigated the dynamical persistence and internal structural evolution of a wide range of Saturn-Titan and Saturn-Rhea unstable resonant flyby orbits when transitioned from the planar CR3BP into a more accurate restricted 4-body model of the Saturn-Titan-Rhea system. Utilizing numerical continuation, we evaluated the fates of key mission-relevant resonant families in this higher-fidelity R4BP: the Titan 1:1 L1 Lyapunov and 3:2 interior resonances under Rhea’s perturbation, as well as the Rhea 3:4 and 2:3 exterior resonances under Titan’s perturbation. The overarching finding of this study is the broad absence of topology-changing secondary resonance overlap across all four investigated mean motion resonances. Even when subjecting the Rhea resonant orbits to the gravitational perturbations of the much-larger Titan, the orbits within both families largely persist as quasiperiodic invariant tori. The dynamical architecture of these resonances in the CCR4BP remains mostly dominated by these tori, which maintain a direct correspondence with both PCR3BP periodic orbit families and traditional patched-conic trajectory arcs.

Across all Titan and Rhea resonant orbit families of this paper, secondary resonant phenomena induced by the untargeted moon’s forcing appear as localized, dynamically isolated islands. In the Titan 3:2 interior resonance, secondary resonances such as the 6:31 island in the low/mid- e family and the prominent 1:5 island in the high- e family distort nearby trajectories, but remain strictly bracketed by invariant tori. Similarly, in the Rhea 3:4 and 2:3 exterior resonances, untargeted approaches by Titan generate noticeable secondary resonant islands at higher eccentricities—including 8:23 and 7:20 islands in the 3:4 family and 6:13 and 7:15 islands in the 2:3 families. Nevertheless, even where perturbations are strongest, such as in the highest- e Rhea 2:3 orbits next to the 7:15 Titan secondary resonance, invariant tori still persist and act as dynamical barriers constraining the motions induced by such regions. This structural bracketing prevents interaction between adjacent secondary resonances and prevents global chaotic transport within each orbit family.

These dynamical results carry significant implications for Saturnian tour design in this Titan-to-Rhea region. Because the internal topological structure of these PCR3BP resonant orbit families is largely preserved in the fuller Saturn-Titan-Rhea model, preliminary trajectories involving these MMRs and designed using simpler 3-body models (or likely even patched-conic approximations) should have equivalents in higher-fidelity models as well. Thus, higher-fidelity ephemeris transition can generally be expected to converge without prohibitive Δv penalties, except potentially in the case of operating exactly within a secondary resonance region. A note of caution though is that such secondary resonant effects, while localized, can happen in higher- e regions where the perturbation could potentially send a spacecraft on a collision course with the targeted moon—as was seen, for example, in the Rhea 2:3 MMR family with the 6:13 secondary resonance. In such cases, using resonant trajectory arcs that take the secondary resonant effects into account, such as the long SR periodic orbits or separatrices of this work, could be beneficial.

Looking forward, the primary avenue for future work is to extend this numerical investigation to Rhea exterior mean motion resonances with semi-major axes larger than that of the 2:3 resonance. As larger orbits will more closely approach Titan, the gravitational perturbations from Titan will inevitably grow stronger,

increasing the likelihood that secondary resonance islands will eventually grow wide enough to overlap and destroy the invariant tori preventing natural motion between them. Determining the resonance thresholds at which this structural breakdown occurs will provide important understanding for multi-moon tour design in the Titan-Rhea region, on one side of which orbits computed in simpler models remain largely valid, and on the other side of which more complicated analysis could be beneficial. Additionally, while most orbits found in this paper were tori, it could be worth exploring whether trajectory segments along these persisting invariant tori—or within the bounded secondary resonant islands—can nevertheless be leveraged to alter a spacecraft’s energy state between encounters. Because the 3-body Jacobi constant C , and consequently the asymptotic velocity v_∞ relative to a flyby moon, is no longer conserved along trajectories in the 4-body problem, traversing specific segments of these 4-body invariant objects may allow a spacecraft to leverage the untargeted moon’s gravity to systematically achieve desired changes in C or v_∞ between consecutive flybys of the targeted moon, thereby expanding the trade space for zero or low- Δv trajectory design.

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