

ANALYTIC CONTINUATION METHOD FOR ACCURATE AND EFFICIENT CIS-LUNAR ORBIT PROPAGATION

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Numerical integrators play a crucial role in solving and simulating different fundamental problems in Astrodynamics. To improve computational accuracy and efficiency, there has been a continuous demand for improved integration methods. Hence, a semi-analytic integration method, Analytic Continuation, has been developed to propagate Two-body trajectories incorporating $J_2 - J_6$ as well as Spherical Harmonics gravity and atmospheric drag perturbations. This method has outperformed the widely used Runge-Kutta (RK) methods in both computational accuracy and efficiency. Later on, the Analytic Continuation method is further developed to compute the State Transition Matrix and Higher Order State Transition Tensors for the gravity and atmospheric drag perturbed Two-body problem. Building upon this foundation, this paper presents the application of the Analytic Continuation method in propagating cis-lunar trajectories by solving the Circular Restricted Three-body Problem. The method introduces two vector variables and four scalar variables to simplify the governing equations of the system. By applying the Leibniz product rule, arbitrarily higher-order time derivatives of these defined variables and the states of the dynamical equations are computed recursively. These higher-order time derivatives of the variables are then used in Taylor series expansions to propagate the states over time. Four types of orbits in the Earth-Moon three-body system are considered for numerical simulation: Short Period and Long Period orbits around L_4 libration point, and Lyapunov and Halo orbits around L_1 libration point. The error in the Jacobi constant calculation is evaluated at each integration step using ODE45, ODE87, and the Analytic Continuation method, and the simulation time for each method is also recorded. The results demonstrate that, across all test cases, the Analytic Continuation method maintains a machine precision accuracy level in computing the Jacobi constant and is significantly computationally efficient compared to the other numerical integrators.

INTRODUCTION

Cis-lunar space has long been regarded as a cost-effective and strategic region for advancing human spaceflight missions and exploration beyond Low Earth Orbit (LEO). NASA has identified this region as a critical testing ground for deep space missions, and their studies focus on the deployment of habitable assets and the evaluation of orbital locations for system design and mission planning, offering opportunities to reduce risks and support future missions to destinations such as the Moon, Near-Earth Asteroids, and Mars.^{1,2} As part of this initiative, NASA's Human Spaceflight Architecture Team developed Design Reference Missions to evaluate potential orbits, mission durations, and strategic goals for operations in this region.² A key outcome of this vision is the proposed Gateway,

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a space station planned for a Near-Rectilinear Halo Orbit (NRHO) around the Moon.³ The Gateway has emerged as a cornerstone of both U.S. and international deep space exploration strategies. It is designed to support Artemis lunar missions, serve as a staging point for future Mars expeditions, and facilitate scientific research, while promoting collaboration with global and commercial partners. This initiative was formally articulated through NASA’s Exploration Campaign, which outlines the Gateway as a modular and scalable infrastructure element in cis-lunar space.⁴

The motion of objects in cis-lunar space is primarily governed by the dynamics of the Three-body problem, which describes the gravitational interaction between three celestial bodies. The Three-body problem is too broad for direct application in practical mission design, and therefore, to make it more applicable for real space missions, two assumptions are introduced to simplify the dynamics: the third body is considered to have negligible mass, and the two primary bodies are assumed to move in circular orbits around their common center of mass. These assumptions lead to a simplified dynamical model known as the Circular Restricted Three-Body Problem (CR3BP).⁵

In contrast to the unperturbed Two-body problem, which has a well-known analytical solution, the CR3BP does not have any closed-form solution. As a result, when applying the CR3BP to determine a spacecraft’s motion in cis-lunar space, numerical methods must be used to propagate its trajectory. Various numerical integration techniques are employed to solve the ordinary differential equations (ODEs) that govern the spacecraft’s motion. These include methods such as the Gauss-Jackson predictor-corrector, Runge-Kutta schemes, Multistep approaches, and Extrapolation techniques.^{6,7} Additionally, a range of algorithms and commercial software packages have been developed to handle ODE integration effectively. For instance, MATLAB’s ODE45 employs an algorithm based on the explicit Runge-Kutta (4,5) formula, commonly known as the Dormand-Prince pair.^{8,9} ODE87 is another integrator that is 8th-order accurate, achieving local errors in the order of $O(h^9)$, and requires 13 evaluations of functions per integration step. This method integrates a system of ODEs using 8-7 th order Dormand and Prince formula.^{10,11}

Although these traditional numerical methods can be used to compute the states of CR3BP by solving the Equations of Motion (EOMs), they often fall short in terms of accuracy and computational performance. To address these challenges, researchers have been exploring new integration techniques aiming to enhance both precision and computational efficiency. For example, Pellegrini and Russell introduced F and G Taylor series solutions to the CR3BP using recursive formulas for the Taylor series coefficients generated using symbolic manipulation software.¹² Tu, Wang, and Tang developed an explicit symmetric integrator for the CR3BP, demonstrating its superior performance compared to both the symplectic Euler method and the Gauss Runge–Kutta methods.¹³ Menges and Scheeres developed a semi-analytical solution using series F and G, where they employed Taylor series expansions and Mathematica to symbolically derive higher-order derivatives of CR3BP EOMs at a known initial state.¹⁴

Analytic continuation is a semi-analytic integration technique that was initially introduced to propagate Two-body trajectories. The accuracy and computational efficiency of this method in propagating Earth’s gravity and atmospheric drag perturbed trajectories have been validated and documented in several studies.^{15–18} This method has been further expanded to compute State Transition Matrix and Higher Order State Transition Tensors of the gravity and atmospheric drag perturbed two-body dynamics, showing machine precision accuracy in the results.^{19–22} Also, the derived State Transition Matrix has been successfully integrated into an Extended Kalman Filter framework for a new perturbed orbit estimation approach based on a network of observer satellites, enhancing its accuracy.²³ Analytic Continuation has also been successfully applied to model hypervelocity breakup

debris trajectories in LEO, enabling high-precision propagation under complex perturbations for use in neural network-based orbit estimation.²⁴

This study further advances the method by applying it to CR3BP, enabling efficient and precise simulation of complex cis-lunar trajectories. The method defines two vector variables $\mathbf{X}$ and $\mathbf{Y}$ and four scalar variables f_1 , f_2 , g_{p1} , and g_{p2} to simplify the dynamical equations of the CR3BP. The Leibniz product rule is then applied to the equations to compute the higher-order time derivatives of these defined variables and the states of the system recursively. Finally, the higher-order time derivatives of the states are implemented in the Taylor series expansion to propagate the solution over time. Four types of orbits in the Earth-Moon three-body system are considered for numerical simulation: Short Period Orbit around $L4$, Long Period Orbit around $L4$, Lyapunov Orbit around $L1$, and Halo Orbit around $L1$. The error in the Jacobi constant calculation is evaluated at each integration step of the simulated orbits using the Analytic Continuation method, along with MATLAB's ODE45 and ODE87 solvers for ten orbital periods. In addition, the simulation time for each method is recorded.

Both ODE45 and ODE87 are based on embedded Runge-Kutta methods using variable step sizes and have round-off errors, which accumulate at each step and degrade the solution's accuracy over time. In contrast, the Analytic Continuation method offers an effective solution to this problem by deriving recursive formulas and computing arbitrarily high-order time derivatives of the state variables. These higher-order time derivatives are then utilized in the Taylor series expansion of the solution, allowing larger time steps and thereby reducing computational costs while maintaining a high accuracy of the solution. This approach offers complete control over both the accuracy of the solution and the computational cost, allowing the user to balance and choose the desired level of precision and computational effort. The numerical results demonstrate that, across all test cases, the Analytic Continuation method consistently maintains a level of machine precision accuracy in computing the Jacobi constant and significantly outperforms ODE45 and ODE87 in computational efficiency.

ANALYTIC CONTINUATION METHOD FOR THE TWO-BODY PROBLEM

In the field of Astrodynamics, the Analytic Continuation method was first introduced by Turner et. al.¹⁵ to generate high-accuracy Two-body problem trajectories for long-term asteroid motion prediction. This method computes arbitrary time derivatives of the acceleration vector of the unperturbed Two-body problem using recursive formulations. This section presents a brief description of the process.

The acceleration vector of the unperturbed Two-body problem is defined as:

$$\mathbf{r}^{(2)}(t) = -\mu \frac{\mathbf{r}(t)}{(\mathbf{r}(t) \cdot \mathbf{r}(t))^{3/2}} \quad (1)$$

where, $\mathbf{r}(t)$ is the position vector at time, t and μ is the standard gravitational parameter.

In order to simplify this equation, two Lagrange-like time dependent invariants f and g_p are defined as:

$$\begin{aligned} f(t) &= \mathbf{r}(t) \cdot \mathbf{r}(t) \\ g_p(t) &= f(t)^{-p/2} \end{aligned} \quad (2)$$

where, p is an integer. Using these two variables, and setting $p = 3$, the acceleration vector defined in Eq. (1) can be simplified as,

$$\mathbf{r}^{(2)}(t) = -\mu \mathbf{r}(t) f(t)^{-3/2} = -\mu \mathbf{r}(t) g_3(t) \quad (3)$$

The higher order time derivatives for $f(t)$, $g_p(t)$ and $\mathbf{r}(t)$ are computed recursively using the Leibniz product rule as follows:

$$f^{(n)}(t) = \sum_{m=0}^n \binom{n}{m} \mathbf{r}^{(m)}(t) \cdot \mathbf{r}^{(n-m)}(t) \quad (4)$$

$$g_p^{(n+1)}(t) = -\frac{1}{f(t)} \left\{ \frac{p}{2} f^{(1)}(t) g_p^{(n)}(t) + \sum_{m=1}^n \binom{n}{m} \left(\frac{p}{2} f^{(m+1)}(t) g_p^{(n-m)}(t) + f^{(m)}(t) g_p^{(n-m+1)}(t) \right) \right\} \quad (5)$$

$$\mathbf{r}^{(n+2)}(t) = -\mu \sum_{m=0}^n \binom{n}{m} \mathbf{r}^{(m)}(t) g_3^{(n-m)}(t) \quad (6)$$

To compute the position and velocity at the next time step, the arbitrarily higher order time derivatives of the position vector, $\mathbf{r}^{(n+2)}(t)$ are substituted into the Taylor series expansions as presented in Eq. (7) and Eq. (8):

$$\mathbf{r}(t + dT) = \mathbf{r}(t) + \sum_{m=1}^n \mathbf{r}^{(m)}(t) \frac{dT^{(m)}}{m!} \quad (7)$$

$$\mathbf{r}^{(1)}(t + dT) = \mathbf{r}^{(1)}(t) + \sum_{m=2}^n \mathbf{r}^{(m)}(t) \frac{dT^{(m-1)}}{(m-1)!} \quad (8)$$

where, dT is the size of the time step.

ANALYTIC CONTINUATION METHOD FOR THE CR3BP

The previous section outlined the formulation of recursive equations for applying the Analytic Continuation method to propagate the Two-body problem trajectories. This section builds on that framework to develop the required recursive formulas to implement the Analytic Continuation method in solving the CR3BP.

The non-dimensional dynamical EOMs for the CR3BP are as follows:⁵

$$x^{(2)}(t) - 2y^{(1)}(t) - x(t) = -\frac{(1 - \mu_{EM})(x(t) + \mu_{EM})}{d_1^3} - \frac{\mu_{EM}(x(t) - 1 + \mu_{EM})}{d_2^3} \quad (9)$$

$$y^{(2)}(t) + 2x^{(1)}(t) - y(t) = -\frac{(1 - \mu_{EM})y(t)}{d_1^3} - \frac{\mu_{EM}y(t)}{d_2^3} \quad (10)$$

$$z^{(2)}(t) = -\frac{(1 - \mu_{EM})z(t)}{d_1^3} - \frac{\mu_{EM}z(t)}{d_2^3} \quad (11)$$

where, $d_1 = \sqrt{(x(t) + \mu_{EM})^2 + y(t)^2 + z(t)^2}$, $d_2 = \sqrt{(x(t) + \mu_{EM} - 1)^2 + y(t)^2 + z(t)^2}$ and $\mu_{EM} = \frac{m_2}{m_1 + m_2}$.

In order to apply the Analytic Continuation method on CR3BP, 6 new variables are defined as follows:

$$\mathbf{X}(t) = \mathbf{r}(t) + \begin{bmatrix} \mu_{EM} \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{Y}(t) = \mathbf{r}(t) + \begin{bmatrix} \mu_{EM} - 1 \\ 0 \\ 0 \end{bmatrix} \quad (12)$$

$$f_1(t) = \mathbf{X}(t) \cdot \mathbf{X}(t) \quad \text{and} \quad f_2(t) = \mathbf{Y}(t) \cdot \mathbf{Y}(t) \quad (13)$$

$$g_{p_1}(t) = f_1(t)^{-p/2} \quad \text{and} \quad g_{p_2}(t) = f_2(t)^{-p/2} \quad (14)$$

The dynamical equations of CR3BP can be simplified using g_{p_1} and g_{p_2} as follows:

$$x^{(2)}(t) = 2y^{(1)}(t) + x(t) - (1 - \mu_{EM})x(t)g_{p_1}(t) - (1 - \mu_{EM})\mu_{EM}g_{p_1}(t) \\ - \mu_{EM}x(t)g_{p_2}(t) + (1 - \mu_{EM})\mu_{EM}g_{p_2}(t) \quad (15)$$

$$y^{(2)}(t) = -2x^{(1)}(t) + y(t) - (1 - \mu_{EM})y(t)g_{p_1}(t) - \mu_{EM}y(t)g_{p_2}(t) \quad (16)$$

$$z^{(2)}(t) = -(1 - \mu_{EM})z(t)g_{p_1}(t) - \mu_{EM}z(t)g_{p_2}(t) \quad (17)$$

where, $p = 3$.

The higher order time derivatives of $X(t)$ and $Y(t)$ are as follows:

$$\mathbf{X}^{(n)}(t) = \mathbf{Y}^{(n)}(t) = \mathbf{r}^{(n)}(t) \quad (18)$$

Applying Leibniz product rule on Eq. (13) and Eq. (14), the higher order time derivatives of f_1 , f_2 , g_{p_1} , and g_{p_2} can be calculated as:

$$f_1^{(n)}(t) = \sum_{m=0}^n \binom{n}{m} \mathbf{X}^{(m)}(t) \cdot \mathbf{X}^{(n-m)}(t) \quad (19)$$

$$f_2^{(n)}(t) = \sum_{m=0}^n \binom{n}{m} \mathbf{Y}^{(m)}(t) \cdot \mathbf{Y}^{(n-m)}(t) \quad (20)$$

$$g_{p_1}^{(n+1)}(t) = -\frac{1}{f_1(t)} \left\{ \frac{p}{2} f_1^{(1)}(t) g_{p_1}^{(n)}(t) + \sum_{m=1}^n \binom{n}{m} \left(\frac{p}{2} f_1^{(m+1)}(t) g_{p_1}^{(n-m)}(t) + \right. \right. \\ \left. \left. f_1^{(m)}(t) g_{p_1}^{(n-m+1)}(t) \right) \right\} \quad (21)$$

$$g_{p_2}^{(n+1)}(t) = -\frac{1}{f_2(t)} \left\{ \frac{p}{2} f_2^{(1)}(t) g_{p_2}^{(n)}(t) + \sum_{m=1}^n \binom{n}{m} \left(\frac{p}{2} f_2^{(m+1)}(t) g_{p_2}^{(n-m)}(t) + \right. \right. \\ \left. \left. f_2^{(m)}(t) g_{p_2}^{(n-m+1)}(t) \right) \right\} \quad (22)$$

Using the values of $g_{p_1}^{(n+1)}(t)$ and $g_{p_2}^{(n+1)}(t)$ and implementing Leibniz product rule on Eq. (15) - Eq. (17), the arbitrarily higher order time derivatives of x , y , and z can be calculated as:

$$x^{(n+2)}(t) = 2y^{(n+1)}(t) + x^{(n)}(t) - (1 - \mu_{EM}) \sum_{m=0}^n \binom{n}{m} x^{(m)}(t) g_{p_1}^{(n-m)}(t) \\ - (1 - \mu_{EM})\mu_{EM}g_{p_1}^{(n)}(t) - \mu_{EM} \sum_{m=0}^n \binom{n}{m} x^{(m)}(t) g_{p_2}^{(n-m)}(t) + (1 - \mu_{EM})\mu_{EM}g_{p_2}^{(n)}(t) \quad (23)$$

$$\begin{aligned}
y^{(n+2)}(t) = & -2x^{(n+1)}(t) + y^{(n)}(t) - (1 - \mu_{EM}) \sum_{m=0}^n \binom{n}{m} y^{(m)}(t) g_{p_1}^{(n-m)}(t) \\
& - \mu_{EM} \sum_{m=0}^n \binom{n}{m} y^{(m)}(t) g_{p_2}^{(n-m)}(t)
\end{aligned} \tag{24}$$

$$z^{(n+2)}(t) = -(1 - \mu_{EM}) \sum_{m=0}^n \binom{n}{m} \left(z^{(m)}(t) g_{p_1}^{(n-m)}(t) \right) - \mu_{EM} \sum_{m=0}^n \binom{n}{m} \left(z^{(m)}(t) g_{p_2}^{(n-m)}(t) \right) \tag{25}$$

After computing $x^{(n+2)}(t)$, $y^{(n+2)}(t)$, $z^{(n+2)}(t)$, they can be used in Taylor series expansions to compute $\mathbf{r}(t + dt)$ and $\mathbf{r}^{(1)}(t + dt)$ for the CR3BP as follows:

$$\mathbf{r}(t + dt) = \mathbf{r}(t) + \sum_{m=1}^n \begin{bmatrix} x^{(m)}(t) \\ y^{(m)}(t) \\ z^{(m)}(t) \end{bmatrix} \frac{dt^{(m)}}{m!} \tag{26}$$

$$\mathbf{r}^{(1)}(t + dt) = \mathbf{r}^{(1)}(t) + \sum_{m=2}^n \begin{bmatrix} x^{(m)}(t) \\ y^{(m)}(t) \\ z^{(m)}(t) \end{bmatrix} \frac{dt^{(m-1)}}{(m-1)!} \tag{27}$$

The algorithm for applying the Analytic Continuation method for propagating CR3BP states is presented in the Appendix.

NUMERICAL RESULTS

This section presents a comparison of the numerical results among ODE45, ODE87, and the Analytic Continuation method for simulating CR3BP. The study considers four distinct orbit types of the Earth-Moon system: Short-Period, Long-Period, Lyapunov, and Halo. The corresponding non-dimensional orbit periods are 6.5827, 24.0715, 2.6916, and 2.7430, respectively. The initial non-dimensional positions and velocities of these orbits are obtained from NASA Jet Propulsion Laboratory's Three-Body Periodic Orbit database²⁵ and are listed in Table 1.

Table 1: Initial Conditions and Orbit Period of the Simulated Orbits

Orbit Type	Position (Non-dim.)	Velocity (Non-dim.)
Short Period	$[0.48785, 0.86599, -3.6602 \times 10^{-24}]$	$[-6.2382 \times 10^{-5}, 2.5013 \times 10^{-5}, 2.3881 \times 10^{-24}]$
Long Period	$[0.48785, 0.67213, 1.7393 \times 10^{-35}]$	$[-0.24535, 0.13806, -3.0358 \times 10^{-35}]$
Lyapunov	$[0.83691, 2.65699 \times 10^{-29}, -6.5553 \times 10^{-32}]$	$[6.0314 \times 10^{-16}, 5.2232 \times 10^{-5}, 5.1825 \times 10^{-31}]$
Halo	$[0.82339, -1.9018 \times 10^{-28}, 9.8941 \times 10^{-4}]$	$[-2.3545 \times 10^{-15}, 0.12634, 2.2367 \times 10^{-16}]$

In order to compare the accuracy of the states using different integrators, the error in calculating Jacobi's constants is considered, as Jacobi's constant in CR3BP is analogous to the total energy of the unperturbed Two-body problem.⁵ The non-dimensional Jacobi's constant, C_J is defined as,

$$C_J = x^2(t) + y^2(t) + \frac{2(1 - \mu_{EM})}{d_1} + \frac{2\mu_{EM}}{d_2} - v^2(t) \tag{28}$$

where, $v = \sqrt{(x^{(1)}(t) + \mu_{EM})^2 + y^{(1)}(t)^2 + z^{(1)}(t)^2}$.

The error in Jacobi's constant is calculated by computing its value at every time step and subtracting its initial value, and taking the absolute value as follows:

$$C_{J_{Error}} = |C_J(t) - C_J(t_0)| \quad (29)$$

To numerically simulate the Short Period orbit around L4 using the Analytic Continuation method, 15 higher-order time derivatives of the position vector are calculated at each time step. A total of 50 such steps are used to propagate the trajectory for 10 orbital periods. Similarly, the Long Period orbit around L4, the Lyapunov Orbit around L1, and the Halo Orbit around L1 are each simulated for 10 orbital periods using 21, 19, and 19 higher-order time derivatives and 510, 700, and 616 time steps, respectively. MATLAB scripts used in computing arbitrarily higher-order time derivatives of the states are converted into .mex file format to improve computational efficiency. When simulating the same trajectories using MATLAB's ODE45 and ODE87 integrators, both the relative and absolute tolerances are set to 10^{-13} to ensure maximum possible accuracy of the results. Additionally, the initial and final times of the simulations are given to the integrators, allowing them to automatically determine optimal time steps for efficient trajectory computation.

The comparison among the errors in computing Jacobi's constant using ODE45, ODE87, and the Analytic Continuation method over 10 orbital periods for each orbit type, along with their corresponding computational times, obtained using MATLAB's tic and toc functions, is shown in Figure 1 - Figure 4. Additionally, a summary of the integration formula, time steps, and computational times is presented in Table 2 to compare the computational efficiency among ODE45, ODE87, and the Analytic Continuation method.

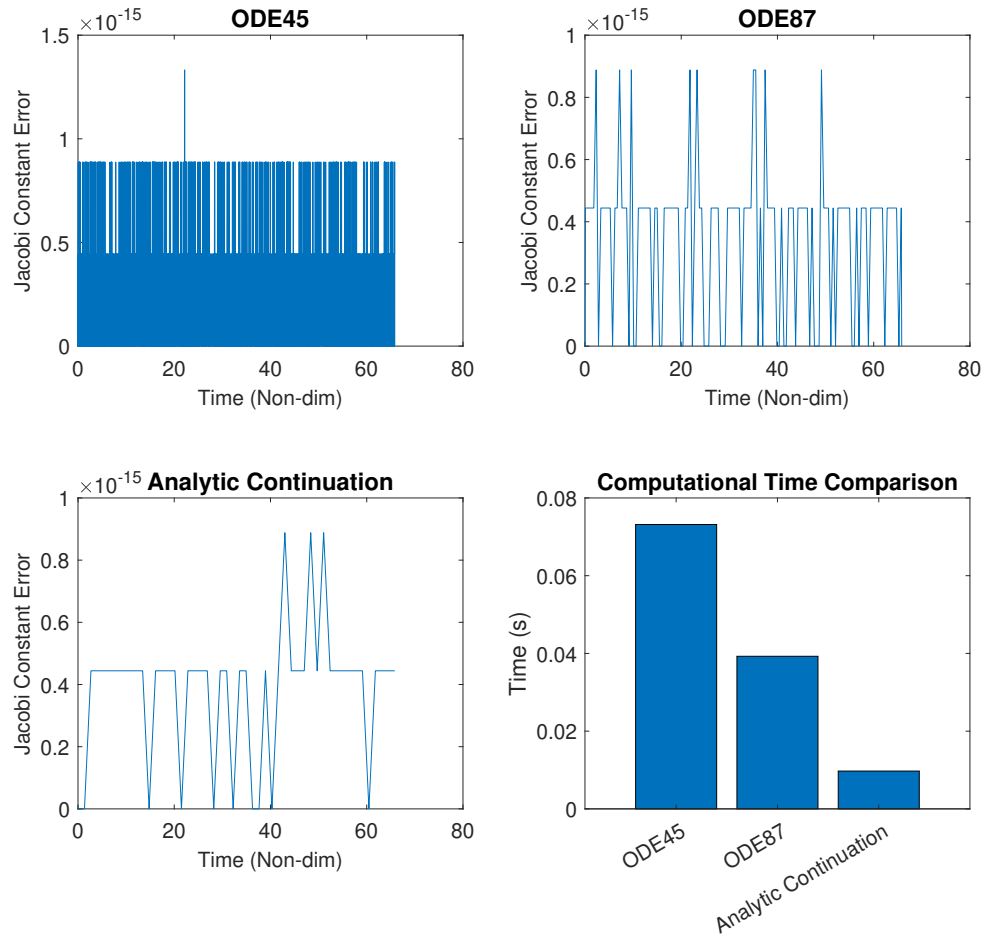


Figure 1: Jacobi's Constant Error and Computational Time for a Short Period Orbit around L_4

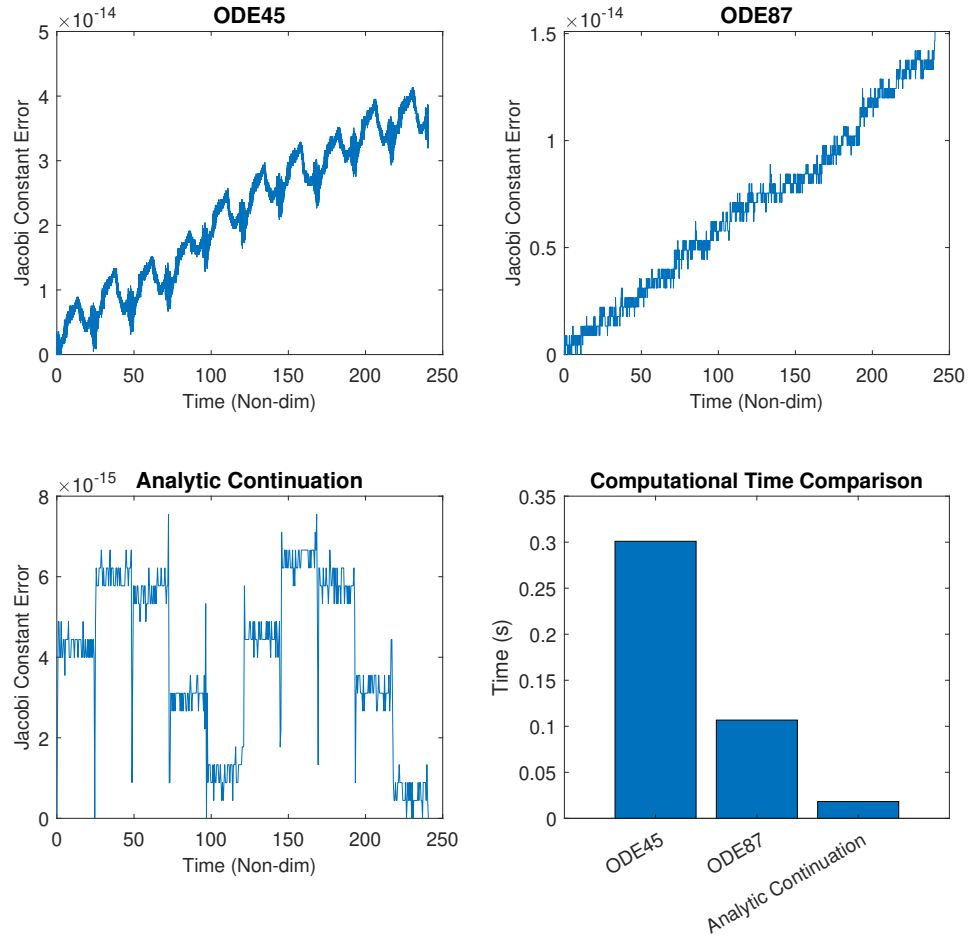


Figure 2: Jacobi's Constant Error and Computational Time for a Long Period Orbit around L_4

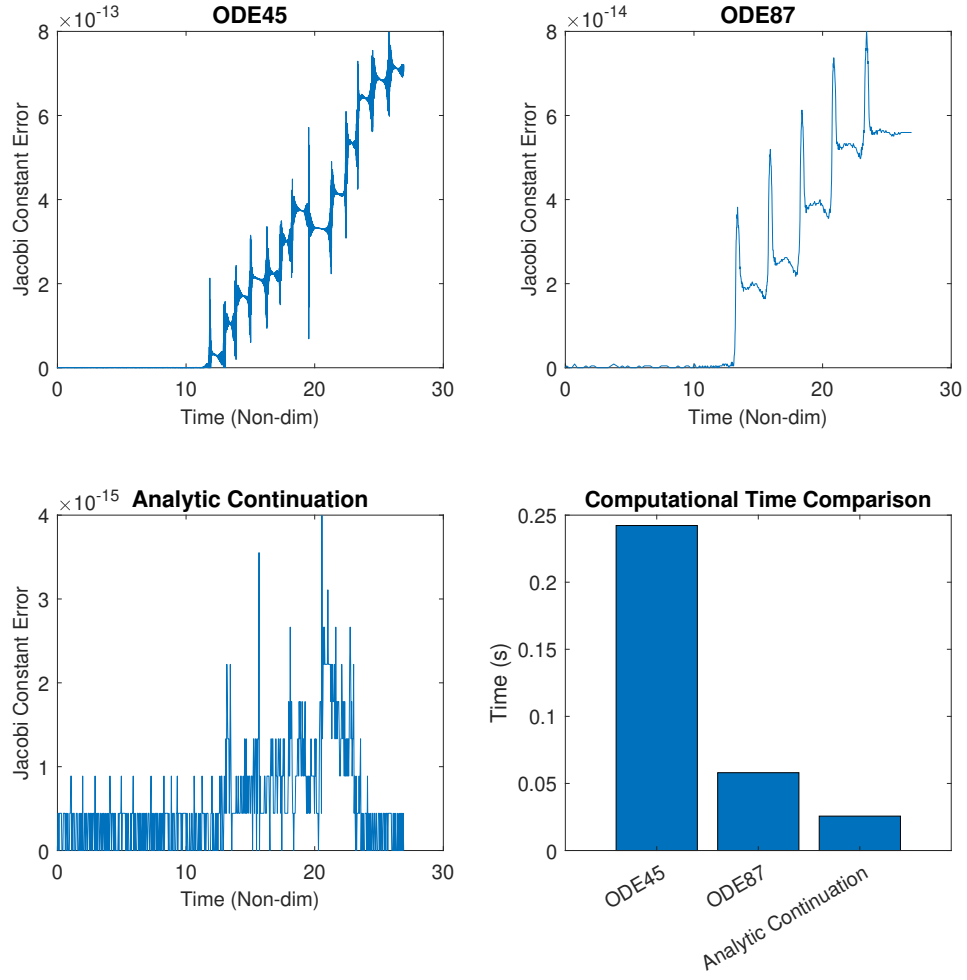


Figure 3: Jacobi's Constant Error and Computational Time for a Lyapunov Orbit around $L1$

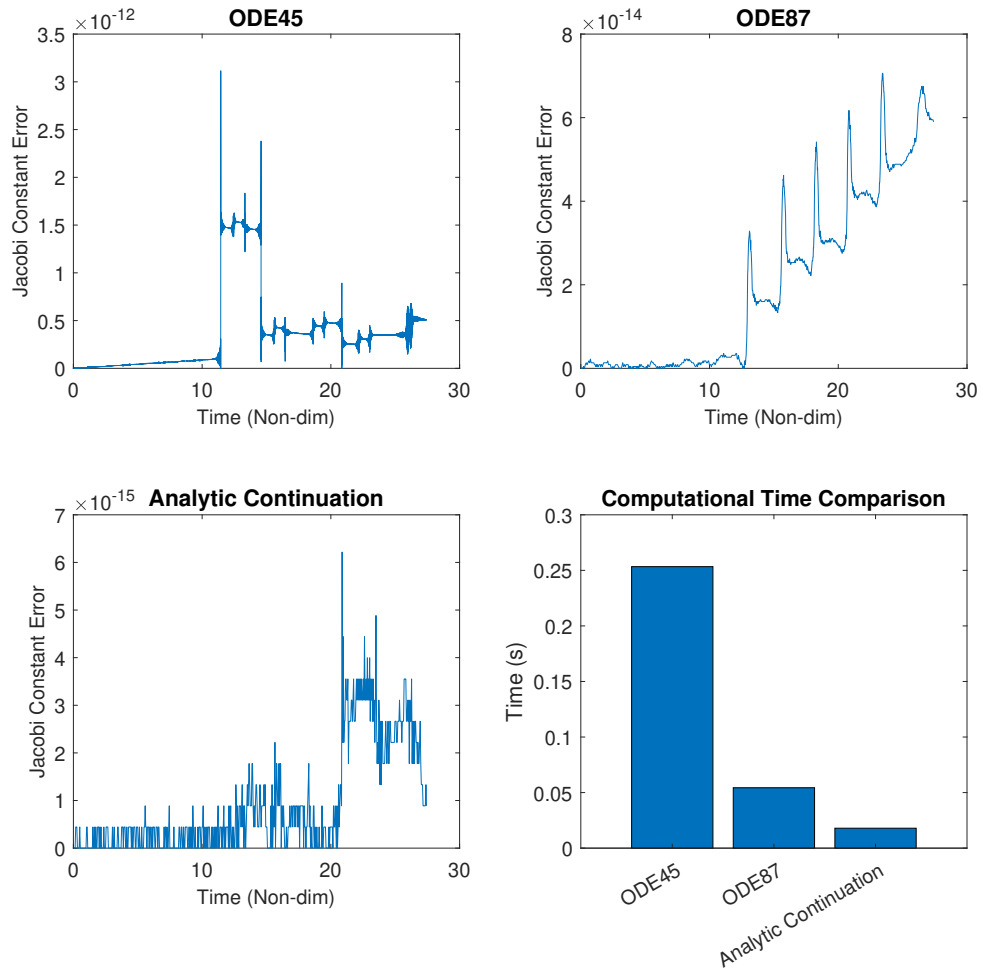


Figure 4: Jacobi's Constant Error and Computational Time for a Halo Orbit around L_1

Table 2: Summary of Computational Efficiency Comparison

Orbit Type	Integrator	Formula	Time Steps	Computational Time (s)
Short Period	ODE45	4th and 5th Order Runge-Kutta	4273	0.073093
	ODE87	8th and 7th Order Runge-Kutta	138	0.038632
	Analytic Continuation	15 derivatives in Taylor series	50	0.009646
Long Period	ODE45	4th and 5th Order Runge-Kutta	76077	0.300832
	ODE87	8th and 7th Order Runge-Kutta	1718	0.106583
	Analytic Continuation	21 derivatives in Taylor series	510	0.018163
Lyapunov	ODE45	4th and 5th Order Runge-Kutta	48317	0.242157
	ODE87	8th and 7th Order Runge-Kutta	551	0.057898
	Analytic Continuation	19 derivatives in Taylor series	700	0.025540
Halo	ODE45	4th and 5th Order Runge-Kutta	61729	0.253251
	ODE87	8th and 7th Order Runge-Kutta	716	0.054216
	Analytic Continuation	19 derivatives in Taylor series	616	0.017830

DISCUSSION

Figure 1 demonstrates that the Analytic Continuation method, as well as ODE45 and ODE87 solvers, can retain machine precision accuracy level in Jacobi’s constant computation in Short Period orbit for over 10 orbital periods. However, as shown in Figure 2 - Figure 4, the accuracy of ODE45 and ODE87 significantly degrades in the Long Period, Lyapunov, and Halo orbits due to the accumulation of round-off errors. In contrast, the Analytic Continuation method consistently preserves machine-precision accuracy across all four test cases, due to its ability to compute arbitrarily high-order time derivatives of the states of CR3BP using recursive formulas and utilize them through Taylor series expansions.

Furthermore, the Analytic Continuation method demonstrates superior computational efficiency in all simulations. As reported in Table 2, it is approximately 4, 5.86, 2.25, and 3 times faster than ODE87 and 7.5, 16.5, 9.5, and 14.2 times faster than ODE45 for the Short Period, Long Period, Lyapunov, and Halo orbits, respectively, when simulating for over 10 orbital periods.

CONCLUSION

The algorithm presented in this paper, using the Analytic Continuation method, computes arbitrarily higher-order time derivatives of the CR3BP states through recursive formulations. To compare the accuracy of the proposed method with MATLAB’s ODE45 and ODE87 integrators, the conservation of Jacobi’s constant is monitored, while computational efficiency is evaluated by recording execution times using MATLAB’s tic and toc functions. In the Earth-Moon three-body system, four types of orbits are tested: the Short Period and Long Period orbits around L_4 , and the Lyapunov and Halo orbits around L_1 . The numerical results indicate that the proposed method consistently achieves near machine-precision accuracy across all test cases, outperforming MATLAB’s ODE45 and ODE87 integrators. In terms of computational efficiency, the algorithm shows a significant advantage, performing at least twice as fast as ODE87 and over seven times faster than ODE45 when simulating the orbits for ten orbital periods.

In the current implementation of the algorithm, both the number of time derivatives in every step and the total number of steps to simulate each orbit type are manually tuned to achieve the desired accuracy of the results. A fixed time step is used in the simulation for each orbit, although

the value of the time step varies between different orbit types. Future developments of the method will focus on incorporating adaptive time steps and expansion orders to enhance computational efficiency while meeting any specified accuracy requirements. Additionally, the algorithm will be applied to study non-periodic and chaotic trajectories within the Earth-Moon system, modeled using the CR3BP. Further expansion will also include the computation of the State Transition Matrix (STM) and Higher-Order State Transition Tensors for analyzing cis-lunar trajectories.

APPENDIX: ALGORITHM

Algorithm 1 Analytic Continuation method for propagating CR3BP orbits

Initialize $\mathbf{r}(t) = [x(t_0), y(t_0), z(t_0)]$, $\mathbf{r}^{(1)}(t) = [x^{(1)}(t_0), y^{(1)}(t_0), z^{(1)}(t_0)]$, final time, T_f , select number of time derivatives of the Taylor's series, N_{der} , select number of time steps of the integration, N_{step} , and calculate the time step of the integration, dT .

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for  $k = 0 \rightarrow N_{step}$  do
     $\mathbf{X}(t) = \mathbf{r}(t) + \begin{bmatrix} \mu_{EM} \\ 0 \\ 0 \end{bmatrix}$  and  $\mathbf{Y}(t) = \mathbf{r}(t) + \begin{bmatrix} \mu_{EM} - 1 \\ 0 \\ 0 \end{bmatrix}$ 
     $\mathbf{X}^{(1)}(t) = \mathbf{r}^{(1)}(t)$  and  $\mathbf{Y}^{(1)}(t) = \mathbf{r}^{(1)}(t)$ ;
     $f_1(t) = \mathbf{X}(t) \cdot \mathbf{X}(t)$ ;  $f_2(t) = \mathbf{Y}(t) \cdot \mathbf{Y}(t)$ ;
     $f_1^{(1)}(t) = 2\mathbf{X}(t) \cdot \mathbf{X}^{(1)}(t)$ ;  $f_2^{(1)}(t) = 2\mathbf{Y}(t) \cdot \mathbf{Y}^{(1)}(t)$ ;

     $g_{p1}(t) = f_1(t)^{-\frac{p}{2}}$ ;  $g_{p2}(t) = f_2(t)^{-\frac{p}{2}}$ ;
     $g_{p1}^{(1)}(t) = -\frac{p}{2} f_1^{(1)}(t) \frac{g_{p1}(t)}{f_1(t)}$ ;  $g_{p2}^{(1)}(t) = -\frac{p}{2} f_2^{(1)}(t) \frac{g_{p2}(t)}{f_2(t)}$ ;

    Get  $x^{(2)}(t), y^{(2)}(t), z^{(2)}(t)$ ; //  $\triangleright$  Eq. (15), (16), (17)

     $\mathbf{r}^{(2)}(t) = [x^{(2)}(t), y^{(2)}(t), z^{(2)}(t)]$ ;
     $\mathbf{X}^{(2)}(t) = \mathbf{r}^{(2)}(t)$  and  $\mathbf{Y}^{(2)}(t) = \mathbf{r}^{(2)}(t)$ ;

     $f_1^{(2)}(t) = 2\mathbf{X}^{(1)}(t) \cdot \mathbf{X}^{(1)}(t) + 2\mathbf{X}(t) \cdot \mathbf{X}^{(2)}(t)$ ;
     $f_2^{(2)}(t) = 2\mathbf{Y}^{(1)}(t) \cdot \mathbf{Y}^{(1)}(t) + 2\mathbf{Y}(t) \cdot \mathbf{Y}^{(2)}(t)$ ;

     $g_{p1}^{(2)}(t) = -\frac{1}{f_1(t)} \left( \frac{p}{2} f_1^{(1)}(t) g_{p1}^{(1)}(t) + \frac{p}{2} f_1^{(2)}(t) g_{p1}(t) + f_1^{(1)}(t) g_{p1}^{(1)}(t) \right)$ ;
     $g_{p2}^{(2)}(t) = -\frac{1}{f_2(t)} \left( \frac{p}{2} f_2^{(1)}(t) g_{p2}^{(1)}(t) + \frac{p}{2} f_2^{(2)}(t) g_{p2}(t) + f_2^{(1)}(t) g_{p2}^{(1)}(t) \right)$ ;
    where  $p$  is 3.

    for  $n = 1 \rightarrow (N_{der} - 2)$  do
        Get  $x^{(n+2)}(t), y^{(n+2)}(t), z^{(n+2)}(t)$ ; //  $\triangleright$  Eq. (23), (24), (25)

         $\mathbf{r}^{(n+2)}(t) = [x^{(n+2)}(t), y^{(n+2)}(t), z^{(n+2)}(t)]$ ;
         $\mathbf{X}^{(n+2)}(t) = \mathbf{r}^{(n+2)}(t)$ ;
         $\mathbf{Y}^{(n+2)}(t) = \mathbf{r}^{(n+2)}(t)$ ;

        Get  $f_1^{(n)}(t), f_2^{(n)}(t), g_{p1}^{(n)}(t)$ , and  $g_{p2}^{(n)}(t)$ ; //  $\triangleright$  Eq. (19), (20), (21), (22)
    end

    Get  $\mathbf{r}(t + dT), \mathbf{r}^{(1)}(t + dT)$ ; //  $\triangleright$  Eq. (26), (27)
end

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