

COUPLED RADIAL BASIS FUNCTIONS AND COVER SCHEME METHODS FOR CONSTRAINED OPTIMAL RENDEZVOUS AND PROXIMITY OPERATIONS

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In this work, a cover scheme method and Coupled Radial Basis Functions (CRBFs) are used to compute the optimal thrust and trajectory for a servicing satellite to meet the target spacecraft for in-space servicing via proximity operations. Local collocation is employed by dividing the problem domain into elements with overlapping covers to reinforce continuity. CRBFs are utilized as the basis functions for approximation. They offer flexibility in being insensitive to the shape parameter, not requiring a certain nodal distribution, and collocating at nodes where the solution is needed. This work presents the unconstrained equations for orbital relative motion, the Hill-Clohessy-Wiltshire (HCW) equations, to demonstrate the method against a standard solver. In addition, a constrained orbital relative motion problem is solved to account for an avoidance scenario, such as a neighboring spacecraft, debris, or no-fly zone.

INTRODUCTION

Optimal control is a branch of control system design that seeks to maximize (or minimize) performance criterion subject to a set of constraints.^{1,2} Optimal control problems can be either linear or nonlinear, and can be rather complex and require numerical methods to solve. Numerical methods transcribe optimal control problems into nonlinear programming problems or two-point boundary-value problems, which can then be solved using indirect or direct methods.^{2,3} With indirect methods, the optimal control problem is transcribed into a two-point boundary-value problem using the calculus of variations and Pontryagin's Maximum Principle. While indirect methods offer the advantage of high confidence in solution accuracy while satisfying first-order conditions for optimality, they tend to have a small radius of convergence and often require a priori knowledge of the system.²⁻⁴ Direct methods transcribe the optimal control problem into a nonlinear programming problem and avoid some of the requirements of indirect methods, such as adjoint and control equations as well as transversality conditions. However, they can have lacking or inaccurate costate information.³

To solve optimal control problem, collocation methods are used to approximate differential equations at collocation points across the problem domain. The two types of collocation methods are local collocation and global collocation. Local collocation uses a small number of collocation points in each subinterval and divides the problem's time domain into several subintervals, often connected

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using a continuity condition and the state.^{2,5} On the other hand, global collocation uses a global polynomial to collocate across the time interval. Pseudospectral methods, which are a class of direct transcription methods, are a popular type of global collocation used to solve optimal control problems. However, they are dependent on a specific distribution of nodes, making them inefficient with nonsmooth function interpolation.^{2,5,6}

In recent years, Radial Basis Functions (RBFs) have risen in popularity to solve optimal control problems.^{6,7} They offer advantages over traditional solving methods in that they can interpolate on an arbitrary nodal distribution, overcoming a major challenge associated with pseudospectral methods. However, RBFs rely on a shape parameter, which controls the flatness of the function. This can contribute to latency and increased error in approximation.⁸

A new class of RBFs called Coupled Radial Basis Functions (CRBFs) resolve this limitation of reliance on the shape parameter. CRBFs are a combination of a r^5 conical spline with an RBF, making them insensitive to the shape parameter while maintaining accuracy and robustness.⁹ CRBFs have been used in previous research to solve Zermelo's problem, the duffing oscillator problem, and the inverted cart-pole problem.^{10,11} This research builds off previous work where CRBFs were combined with a local cover scheme method to solve a simple pendulum problem and the unperturbed Lambert's problem.¹² The advantage of using a CRBF cover scheme method paired with local collocation is that it is more suitable for solving optimal control problems with longer integration times. This work demonstrates the method with two astrodynamics problems that require longer integration times, an unconstrained and a constrained orbital relative motion problem for rendezvous and proximity operations.

The next section describes the methodology behind the CRBF-cover scheme method, starting with how CRBFs are applied to optimal control problems and then describing how cover schemes are implemented. The following sections outline the unconstrained and constrained orbital relative motion problems, and apply the method to solving them. The results are analyzed and conclusions are drawn from the results.

CRBFS APPLIED TO OPTIMAL CONTROL PROBLEMS

The equation for a nonlinear dynamic system can be written as follows:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t), \quad t_0 \leq t \leq t_f \quad (1)$$

where the state $\mathbf{x}(t) \in \mathbf{R}^n$, the control input $\mathbf{u}(t) \in \mathbf{R}^m$, and the initial time t_0 are given.

When considering a continuous time problem with a fixed terminal time, the scalar performance index can be minimized by finding the control action $\mathbf{u}(t)$ that derives the state $\mathbf{x}(t)$:

$$J = \Phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} \mathcal{L}(\mathbf{x}(t), \mathbf{u}(t), t) dt \quad (2)$$

subject to the path and controls constraints:

$$\begin{aligned} \mathbf{P}(\mathbf{x}(t), t) &\leq \mathbf{0} \\ \mathbf{C}(\mathbf{u}(t), t) &\leq \mathbf{0} \end{aligned} \quad (3)$$

where $\Phi(\cdot)$ is the boundary terminal function and $\mathcal{L}(\cdot)$ is the Lagrange term.

Writing the Hamiltonian function:

$$\mathcal{H} = \mathcal{L}(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}^T \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \quad (4)$$

the augmented performance index can then be expressed as:

$$J = \Phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} (\mathcal{H} - \boldsymbol{\lambda}^T \dot{\mathbf{x}}) dt \quad (5)$$

Next, the necessary conditions for optimality can be obtained by applying the calculus of variations to the augmented cost function:^{1,7}

$$\begin{aligned} \dot{\mathbf{x}} &= \frac{\partial \mathcal{H}}{\partial \boldsymbol{\lambda}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, t) \\ -\dot{\boldsymbol{\lambda}} &= \frac{\partial \mathcal{H}}{\partial \mathbf{x}} = \frac{\partial \mathcal{L}}{\partial \mathbf{x}} + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right]^T \boldsymbol{\lambda} \\ \mathbf{0} &= \frac{\partial \mathcal{H}}{\partial \mathbf{u}} = \frac{\partial \mathcal{L}}{\partial \mathbf{u}} + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right]^T \boldsymbol{\lambda} \end{aligned} \quad (6)$$

where $\boldsymbol{\lambda}$ is the Lagrange multipliers vector.¹³ The complementary boundary conditions can be obtained from the optimal control problem's initial and final states.¹⁴

Next, CRBFs are selected as trial functions. The following table displays four types of CRBFs along with their corresponding traditional RBF. The CRBFs have the addition of a conical spline r^5 , which scales the flatness by a value $c > 0$. This conical spline makes CRBFs insensitive to the shape parameter, while still maintaining the benefits of RBFs, such as robustness and accuracy of the solution.^{11,15}

Table 1: RBFs and CRBFs Definitions

Type	$\varphi(r, c)$ RBFs	$\varphi(r, c)$ CRBFs
Multiquadric	$\sqrt{1 + (cr)^2}$	$F + r^5$
Inverse multiquadric	$1/\sqrt{1 + (cr)^2}$	$1/F + r^5$
Gaussian	$e^{-(cr)^2}$	$e^{-h^2} + r^5$
Inverse Quadratic	$1/(1 + (cr)^2)$	$1/F^2 + r^5$

where $r = \|\mathbf{t}(i) - \mathbf{t}_i\|$, $h = \frac{r}{c}$ and $F = \sqrt{h^2 + 1}$.

After choosing a CRBF as a trial function, the states, control and costates can be approximated as:

$$\begin{aligned} \mathbf{x}(t) &\simeq \bar{\mathbf{x}}(r) = \sum_{i=1}^N \boldsymbol{\alpha}_i \varphi_i(r) \\ \mathbf{u}(t) &\simeq \bar{\mathbf{u}}(r) = \sum_{i=1}^N \boldsymbol{\beta}_i \varphi_i(r) \\ \boldsymbol{\lambda}(t) &\simeq \bar{\boldsymbol{\lambda}}(r) = \sum_{i=1}^N \boldsymbol{\gamma}_i \varphi_i(r) \end{aligned} \quad (7)$$

where α, β and γ are vectors of unknown coefficients and $i = 1, 2, \dots, N$ nodes. The notation $(\bar{x})$ is an approximated function of $x(t)$ using CRBFs. Similarly, $(\bar{u})$ is an approximated function of $u(t)$, and $(\bar{\lambda})$ is an approximated function of $\lambda(t)$.

Taking the time-derivatives of Equation (7):

$$\begin{aligned}\dot{\bar{x}}(r) &= \sum_{i=1}^N \alpha_i \dot{\phi}_i(r) \\ \dot{\bar{\lambda}}(r) &= \sum_{i=1}^N \gamma_i \dot{\phi}_i(r)\end{aligned}\tag{8}$$

Given the case of a univariate function with an independent time variable, the first time-derivative of the CRBF function is:

$$\frac{\partial \varphi}{\partial t} = \begin{cases} \frac{\partial \varphi(r)}{\partial r} \frac{\partial r}{\partial t}, & r \neq 0 \\ 0, & r = 0 \end{cases}\tag{9}$$

taking the second time-derivative of φ :

$$\frac{\partial^2 \varphi}{\partial t^2} = \begin{cases} \frac{\partial^2 \varphi(r)}{\partial r^2} \left(\frac{\partial r}{\partial t}\right)^2 + \frac{\partial \varphi(r)}{\partial r} \frac{\partial^2 r}{\partial t^2}, & r \neq 0 \\ \frac{1}{3}c^2, & r = 0 \end{cases}\tag{10}$$

Based on N , the number of collocation points, an approximation matrix φ is defined:

$$\varphi = \begin{bmatrix} \varphi(||t_1 - t_1||) & \varphi(||t_2 - t_1||) & \varphi(||t_3 - t_1||) & \dots & \varphi(||t_N - t_1||) \\ \varphi(||t_1 - t_2||) & \varphi(||t_2 - t_2||) & \varphi(||t_3 - t_2||) & \dots & \varphi(||t_N - t_2||) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \varphi(||t_1 - t_N||) & \varphi(||t_2 - t_N||) & \varphi(||t_3 - t_N||) & \dots & \varphi(||t_N - t_N||) \end{bmatrix}\tag{11}$$

Next, the necessary conditions can be written in a compact form. Recall Equation (7):

$$\alpha = \varphi^{-1} \bar{x}\tag{12}$$

taking the time-derivative:

$$\dot{\bar{x}} = \dot{\varphi} \alpha\tag{13}$$

Therefore:

$$\dot{\bar{x}} = \mathbf{D} \bar{x}\tag{14}$$

where $\mathbf{D} = \dot{\varphi} \varphi^{-1}$. In a similar manner, the costates can be expressed as $\dot{\bar{\lambda}} = \mathbf{D} \bar{\lambda}$. Substituting Equations (9) and (14) in Equation (6) leads to the necessary conditions of optimality:

$$\begin{aligned}\dot{x} &\simeq \dot{\bar{x}} = \mathbf{D} \bar{x} = f(\bar{x}, \bar{u}, r) \\ -\dot{\lambda} &\simeq -\dot{\bar{\lambda}} = \mathbf{D} \bar{\lambda} = \frac{\partial \mathcal{L}(\bar{x}, \bar{u}, r)}{\partial \bar{x}} + \frac{\partial f(\bar{x}, \bar{u}, r)}{\partial \bar{x}} \\ 0 &\simeq \frac{\partial \mathcal{L}(\bar{x}, \bar{u}, r)}{\partial \bar{u}} + \frac{\partial f(\bar{x}, \bar{u}, r)}{\partial \bar{u}}\end{aligned}\tag{15}$$

From here, a standard solver can be used to directly solve Equations (15), for the states, costates, and controls.

For the constrained optimal control problem, the Hamiltonian can be augmented as follows:

$$\mathcal{H} = \mathcal{L}(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\lambda}^T \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) + \boldsymbol{\mu}^T \mathbf{S}^{(q)} \quad (16)$$

where the inequality path constraint is expressed as:

$$\mathbf{S}(\mathbf{x}(t), t) \leq \mathbf{0} \quad (17)$$

In this case, S is a scalar and q represents the total time derivative of S . When setting up the constrained Hamiltonian, the time derivative of Equation (17) is taken and $f(\mathbf{x}, \mathbf{u}, t)$ is substituted for $\dot{\mathbf{x}}$ until the control action $\mathbf{u}$ appears. Thus, to control the inequality constraint S , the q -th time derivative is used in Equation (16) and the tangency conditions must be met:¹⁴

$$N(\mathbf{x}, t) \triangleq \begin{Bmatrix} S(\mathbf{x}, t) \\ S^1(\mathbf{x}, t) \\ \dots \\ S^{(q-1)}(\mathbf{x}, t) \end{Bmatrix} = 0$$

When the constraint is active:

$$\mu = \begin{cases} 0 & \text{if } S < 0 \\ \geq 0 & \text{if } S = 0 \end{cases} \quad (18)$$

Therefore, the Hamiltonian in Equation (16) is only active when the solution encounters the path constraint. Otherwise, the solution is governed by the Hamiltonian in Equation (4).

To summarize, solving optimal control problems using CRBFs as trial functions involves expressing the system dynamics and constraints as in Equations (1) and (3), and writing the cost function as shown in Equation (2). The necessary conditions for optimality are written as a TPBVP which is then cast into a system of nonlinear algebraic equations (NAEs). A standard solver can then be used to solve the system of NAEs for the states, costates, control, and boundary conditions.

COVER SCHEME METHODS FOR SOLVING OPTIMAL CONTROL PROBLEMS

In this research, a cover scheme method is proposed where the optimal control problem domain is divided into M segments called "elements" that are locally collocated¹⁶ and have overlapping regions called "covers". The union of all elements captures the whole problem domain Ω , and the cover intersections can include multiple collocation nodes, N , to enforce continuity.

$$\Omega = \bigcup_{i=1}^M \Omega_i \quad (19)$$

where $i = 1, 2, \dots, M$ elements.

To reduce approximation error, the number of nodes, covers, and elements can be varied.

We are proposing dividing the problem domain into elements with overlapping covers to better suit problems that require longer integration times. As the problem's time domain increases, the number of nodes also increases and the CRBF matrix can become ill-conditioned. Only global collocation was used when applying the CRBF method in previous work.^{10, 11, 17} CRBFs are chosen for this cover scheme implementation because they do not rely on a certain nodal distribution like other methods, such as pseudospectral methods, that are found in the literature. Therefore, the number and distribution of nodes can be varied around areas of interest, such as when a constraint is met or active.

Figure 1 shows how a problem's domain can be divided into different elements with overlapping covers. In this Figure, the black box is an approximation point in the intersection of Elements 1 and 2. The approximation of x at the black box approximation point in terms of the number of nodes contained between the covers can be expressed as:

$$\bar{x} = \sum_{i=1}^N \alpha_i \varphi_i(r) \quad (20)$$

where $N = 14$ nodes (between Elements 1 and 2).

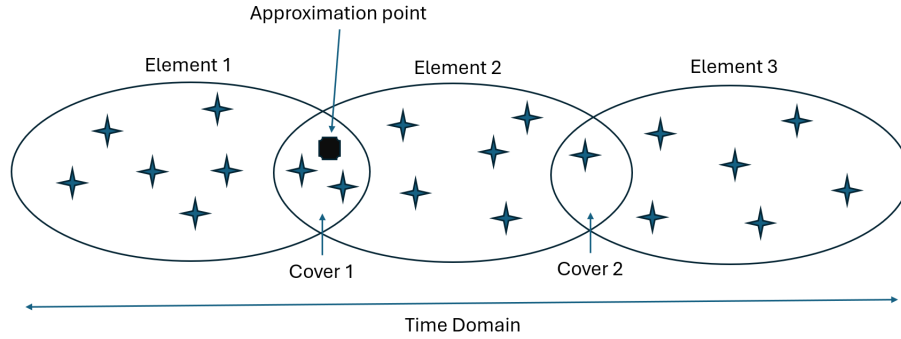


Figure 1: Cover Scheme Method - Example

HILL-CLOHESSY-WILTSHIRE RENDEZVOUS AND PROXIMITY OPERATIONS PROBLEM

The relative motion in orbit between two bodies in space can be described using nonlinear equations. Consider the first body to be the "target" with a circular orbit, and the second body to be "chasing" the target body in either an elliptical or circular orbit. This second body is often a maneuvering spacecraft that can use low-thrust to rendezvous with the target body.¹⁸

This concept was first described by George Hill in the 1870s, using the moon as the target body.¹⁹ His work was built upon by Clohessy and Wiltshire in the 1960s using a first-order approximation to model the chasing motion of the maneuvering spacecraft.²⁰ The geometry for the orbits of the target and maneuvering bodies is shown in Figure 2, where $\vec{r}_1$ is the position vector for the target body, and $\vec{r}_2$ is the position vector for the maneuvering body. $\vec{r}$ is the position of the bodies relative to each other.

The Hill-Clohessy-Wiltshire (HCW) Equations serve as an excellent model for the relative motion

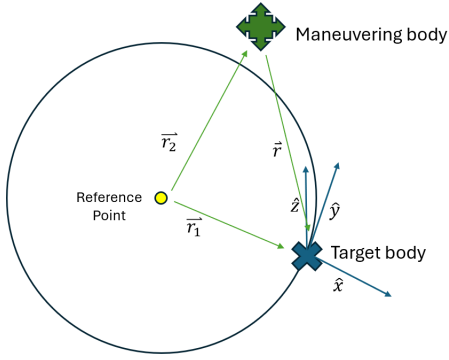


Figure 2: Relative Motion in Orbit Geometry

of two spacecraft due to its linearity.¹⁸ The HCW Equations can be described by the following system of second order equations:

$$\begin{aligned}\ddot{x} &= 3n^2x + 2n\dot{y} \\ \ddot{y} &= -2n\dot{x} \\ \ddot{z} &= -n^2z\end{aligned}\tag{21}$$

where $n = \sqrt{\frac{\mu}{a^3}}$, μ is the gravitational parameter, and a is the radius of circular orbit for the target body.

The Unconstrained Case

To solve the unconstrained HCW equations for a rendezvous and proximity operations problem, the equations of motion are cast into a system of first-order differential equations where the thrust is applied in the x, y, and z directions such that $\vec{u} = \langle u_x, u_y, u_z \rangle$:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= 3n^2x_1 + 2ny_2 + u_x \\ \dot{y}_1 &= y_2 \\ \dot{y}_2 &= -2nx_2 + u_y \\ \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -n^2z_1 + u_z\end{aligned}\tag{22}$$

Subject to the performance index:

$$J = \frac{1}{2} \int_{t_0}^{t_f} \vec{x}^T Q \vec{x} + \vec{u}^T R \vec{u} dt\tag{23}$$

where Q and R are 3x3 weighting factor matrices, assumed to be unity.

From here, the Hamiltonian can be derived:

$$\begin{aligned}\mathcal{H} = & \frac{1}{2}(\vec{x}^T Q \vec{x} + \vec{u}^T R \vec{u}) + \lambda_1 x_2 + \lambda_2(3n^2 x_1 + 2ny_2 + u_x) + \\ & \lambda_3 y_2 - \lambda_4(2nx_2 - u_y) + \lambda_5 z_2 - \lambda_6(n^2 z_1 - u_z)\end{aligned}\quad (24)$$

Utilizing Equation (6) for the necessary condition for optimality (for stationarity), the control action $\vec{u} = \langle u_x, u_y, u_z \rangle$ can be found:

$$\begin{aligned}u_x &= -\lambda_2 \\ u_y &= -\lambda_4 \\ u_z &= -\lambda_6\end{aligned}\quad (25)$$

By substituting Equation (25) into Equation (24), the Hamiltonian can be expressed as:

$$\begin{aligned}\mathcal{H} = & \frac{1}{2}\vec{x}^T Q \vec{x} + \lambda_1 x_2 + \lambda_2(3n^2 x_1 + 2ny_2 - \frac{1}{2}\lambda_2) + \\ & \lambda_3 y_2 - \lambda_4(2nx_2 + \frac{1}{2}\lambda_4) + \lambda_5 z_2 - \lambda_6(n^2 z_1 + \frac{1}{2}\lambda_6)\end{aligned}\quad (26)$$

In addition, the necessary conditions for optimality can be written as:

$$\begin{aligned}\dot{\lambda}_1 &= -\frac{\partial \mathcal{H}}{\partial x_1} = -q_{11}x_1 - 3n^2\lambda_2 \\ \dot{\lambda}_2 &= -\frac{\partial \mathcal{H}}{\partial x_2} = -q_{22}x_2 - \lambda_1 + 2n\lambda_4 \\ \dot{\lambda}_3 &= -\frac{\partial \mathcal{H}}{\partial y_1} = -q_{33}y_1 \\ \dot{\lambda}_4 &= -\frac{\partial \mathcal{H}}{\partial y_2} = -q_{44}y_2 - 2n\lambda_2 - \lambda_3 \\ \dot{\lambda}_5 &= -\frac{\partial \mathcal{H}}{\partial z_1} = -q_{55}z_1 + n^2\lambda_6 \\ \dot{\lambda}_6 &= -\frac{\partial \mathcal{H}}{\partial z_2} = -q_{66}z_2 - \lambda_5\end{aligned}\quad (27)$$

From here, Equations (7), (9) and (10) can be used to write the set of differential equations as nonlinear algebraic equations in a residual vector form $\mathbf{R}$. Using, Equations (7), (9) and (10):

$$\begin{aligned}
{}^1R_1^1 &= {}^1x_1^1 - x_{1_0} \\
{}^kR_1^i &= D^k x_1^i - {}^kx_2^i \\
{}^{q+1}R_1^1 &= {}^qR_1^N \\
{}^M R_1^N &= {}^M x_1^N - x_{2_f} \\
{}^1R_2^1 &= {}^1x_2^1 - x_{2_0} \\
{}^kR_2^i &= D^k x_2^i - (3n^2 {}^kx_1^i + 2n {}^ky_2^i - \frac{{}^k\lambda_2^i}{2}) \\
{}^{q+1}R_2^1 &= {}^qR_2^N \\
{}^M R_2^N &= {}^M x_2^N - x_{2_f} \\
{}^1R_3^1 &= {}^1y_1^1 - y_{1_0} \\
{}^kR_3^i &= D^k y_1^i - {}^ky_2^i \\
{}^{q+1}R_3^1 &= {}^qR_3^N \\
{}^M R_3^N &= {}^M y_1^N - y_{1_f} \\
{}^1R_4^1 &= {}^1y_2^1 - y_{2_0} \\
{}^kR_4^i &= D^k y_2^i + (2n {}^kx_2^i + \frac{{}^k\lambda_4^i}{2}) \\
{}^{q+1}R_4^1 &= {}^qR_4^N \\
{}^M R_4^N &= {}^M y_2^N - y_{2_f} \\
{}^1R_5^1 &= {}^1z_1^1 - z_{1_0} \\
{}^kR_5^i &= D^k z_1^i - {}^kz_2^i \\
{}^{q+1}R_5^1 &= {}^qR_5^N \\
{}^M R_5^N &= {}^M z_1^N - z_{1_f} \\
{}^1R_6^1 &= {}^1z_2^1 - z_{2_0} \\
{}^kR_6^i &= D^k z_2^i + (n^2 {}^kz_1^i + \frac{{}^k\lambda_6^i}{2}) \\
{}^{q+1}R_6^1 &= {}^qR_6^N \\
{}^M R_6^N &= {}^M z_2^N - z_{2_f}
\end{aligned} \tag{28}$$

Additionally,

$$\begin{aligned}
{}^k\mathbf{R}_7^j &= D^k \lambda_1^j - 3n^2 {}^k\lambda_2^j \\
{}^{q+1}\mathbf{R}_7^1 &= {}^q\mathbf{R}_7^N \\
{}^k\mathbf{R}_8^j &= D^k \lambda_2^j - ({}^k\lambda_1^j - 2n {}^k\lambda_4^j) \\
{}^{q+1}\mathbf{R}_8^1 &= {}^q\mathbf{R}_8^N \\
{}^k\mathbf{R}_9^j &= D^k \lambda_3^j \\
{}^{q+1}\mathbf{R}_9^1 &= {}^q\mathbf{R}_9^N \\
{}^k\mathbf{R}_{10}^j &= D^k \lambda_4^j - (2n {}^k\lambda_2^j + {}^k\lambda_3^j) \\
{}^{q+1}\mathbf{R}_{10}^1 &= {}^q\mathbf{R}_{10}^N \\
{}^k\mathbf{R}_{11}^j &= D^k \lambda_5^j + n^2 {}^k\lambda_6^j \\
{}^{q+1}\mathbf{R}_{11}^1 &= {}^q\mathbf{R}_{11}^N \\
{}^k\mathbf{R}_{12}^j &= D^k \lambda_6^j - {}^k\lambda_5^j \\
{}^{q+1}\mathbf{R}_{12}^1 &= {}^q\mathbf{R}_{12}^N
\end{aligned} \tag{29}$$

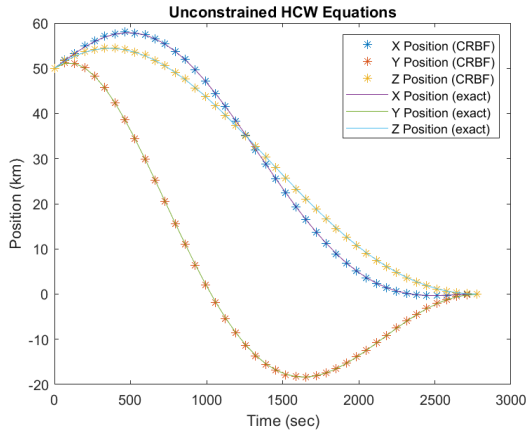
where the node indices are $i = j = 2, 3, \dots, N$, and the element index is $k = 1, 2, \dots, M$. N and M are the total number of nodes and covers, respectively. In addition, $q = 1, 2, \dots, M - 1$, is the index for enforcing the continuity conditions between elements.

The initial parameters used to solve the problem are shown in Table 2, with the boundary conditions coming from the literature to conduct rendezvous and docking with the International Space Station (ISS).¹⁸

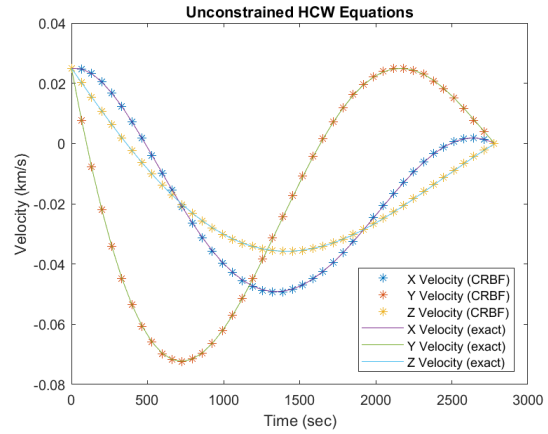
Results To drive the system to perform proximity operations at a final state of $x_f = y_f = z_f = 0$, the fixed final state problem is considered. The optimal trajectories for the position and velocities computed using the CRBF-cover scheme method are displayed in Figures 3a and 3b. The trajectories appear continuous and closely match the exact solution propagated using MATLAB's *ode45*. The 3-dimensional optimal trajectory is displayed in Figure 4, with the starting point represented by the red dot, and the ending point represented by the green dot. The optimal trajectories terminate at $x_f = y_f = z_f = 0$, which meets criteria for proximity operations. The absolute error for position displayed in Figure 5a is less than 0.2 km for all trajectories, and the absolute error for velocity displayed in Figure 5b is on the magnitude of 10^{-4} km/s . The small absolute error in both cases shows the CRBF-cover scheme method closely approximates the exact solution.

Table 2: HCW Equations Initial Parameters

Initial Conditions	Value
N (nodes)	15 per element
M (covers)	3
ϵ (shape parameter)	10
x_0	50 km
$\dot{x}_0$	0.025 km/s
x_f	0 km
$\dot{x}_f$	0 km/s
y_0	50 km
$\dot{y}_0$	0.025 km/s
y_f	0 km
$\dot{y}_f$	0 km/s
z_0	50 km
$\dot{z}_0$	0.025 km/s
z_f	0 km
$\dot{z}_f$	0 km/s
t_f	$\frac{\pi}{n_o}$ sec
n_o (orbital frequency)	1.1313658×10^{-3}
n (mean motion)	$\sqrt{\frac{\mu}{a^3}}$
μ (gravitational Earth parameter)	$398600.4 \frac{km^3}{s^2}$
a (orbit radius)	6778.14 km
ISS altitude (above Earth)	400 km
$\varphi(r, c)$ (CRBF)	Multiquadric ($F + r^5$)



(a) Optimal Position Values



(b) Optimal Velocity Values

Figure 3: Unconstrained HCW Equations - Optimal Position Values (a) and Optimal Velocity Values (b)

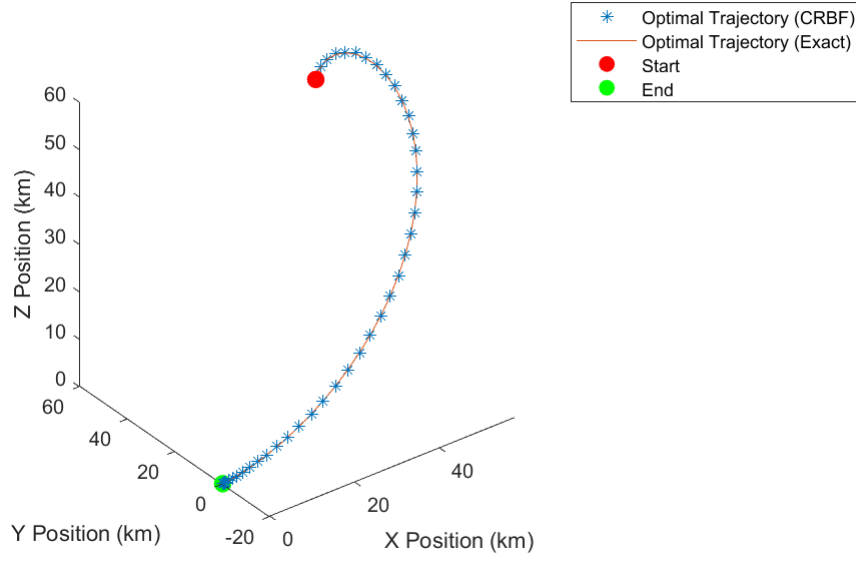
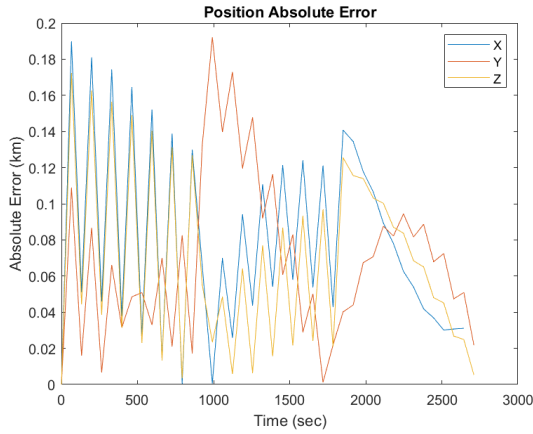
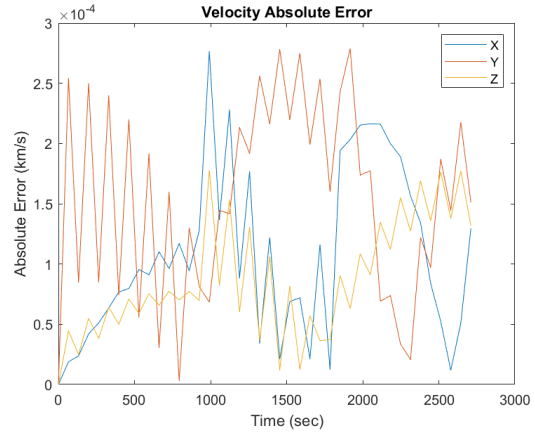


Figure 4: Optimal Trajectory



(a) Absolute Error for Optimal Position Values



(b) Absolute Error for Optimal Velocity Values

Figure 5: Unconstrained HCW Equations - Absolute Error for Optimal Position Values (a) and Absolute Error for Optimal Velocity Values (b)

The Constrained Case: Spherical No-Fly-Zone

Next, an avoidance scenario is considered by solving the HCW Equations with a nonlinear no-fly-zone constraint to simulate an obstacle, such as an asteroid or space debris.

The no-fly-zone for this problem is modeled as a sphere, however, the method can be extended to any shaped no-fly-zone and any number of no-fly-zones. Consider the equation for a spherical

no-fly-zone constraint:

$$R_j^2 = \frac{1}{2}(x(t) - x_{cj})^2 - (y(t) - y_{cj})^2 - (z(t) - z_{cj})^2 \quad (30)$$

where $j = 1, 2, \dots$ for the number of no-fly-zones, and x_{cj}, y_{cj}, z_{cj} is the center of the j -th no-fly-zone.

Taking the time-derivative of the constraint and substituting $f(x, u, t)$ for $\dot{x}$ until the control effort explicitly appears:

$$\begin{aligned} \mathbf{S}(\mathbf{x}(t), t) &= \frac{1}{2}[R_j^2 - (x(t) - x_{cj})^2 - (y(t) - y_{cj})^2 - (z(t) - z_{cj})^2] \\ \mathbf{S}^{(1)}(\mathbf{x}(t), t) &= -(x(t) - x_{cj})^2 \dot{x}_1 - (y(t) - y_{cj})^2 \dot{y}_1 - (z(t) - z_{cj})^2 \dot{z}_1 \\ \mathbf{S}^{(2)}(\mathbf{x}(t), t) &= -\dot{x}_1^2 + (x_{cj} - x(t))\dot{x}_2 - \dot{y}_1^2 + (y_{cj} - y(t))\dot{y}_2 - \dot{z}_1^2 + (z_{cj} - z(t))\dot{z}_2 \end{aligned} \quad (31)$$

Using the HCW equations in Equation (22), the Hamiltonian can be augmented with the q -th time derivative of the constraint in Equation (30) and substituted in the dynamics as follows:

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}(\vec{x}^T Q \vec{x} + \vec{u}^T R \vec{u}) + \lambda_1 x_2 + \lambda_2(3n^2 x_1 + 2ny_2 + u_x) + \\ &\quad \lambda_3 y_2 - \lambda_4(2nx_2 - u_y) + \lambda_5 z_2 - \lambda_6(n^2 z_1 - u_z) + \\ &\quad \mu_1(3x_{cj}n^2 - 3n^2 x_1 - u_x) + \mu_2(2nx_2 - u_y) + \mu_3(-z_{cj}n^2 + 2n^2 z_1 - u_z) \end{aligned} \quad (32)$$

On the constraint, the control u is assumed to be zero. Thus, the u_x, u_y , and u_z terms in Equation (32) drop out. Utilizing Equation (6) for the necessary condition for optimality (for stationarity), μ^T can be found:

$$\begin{aligned} \mu_1 &= \lambda_2 \\ \mu_2 &= \lambda_4 \\ \mu_3 &= \lambda_6 \end{aligned} \quad (33)$$

By substituting Equation (33) into Equation (24), the Hamiltonian can be expressed as:

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}(\vec{x}^T Q \vec{x}) + \lambda_1 x_2 + \lambda_2(3x_{cj}n^2 - 3n^2 x_1) + \\ &\quad \lambda_3 y_2 + \lambda_5 z_2 + \lambda_6(-z_{cj}n^2 + n^2 z_1) \end{aligned} \quad (34)$$

In addition, the necessary conditions for optimality can be shown as:

$$\begin{aligned}
\dot{\lambda}_1 &= -\frac{\partial \mathcal{H}}{\partial x_1} = -q_{11}x_1 + 3n^2\lambda_2 \\
\dot{\lambda}_2 &= -\frac{\partial \mathcal{H}}{\partial x_2} = -q_{22}x_2 - \lambda_1 \\
\dot{\lambda}_3 &= -\frac{\partial \mathcal{H}}{\partial y_1} = -q_{33}y_1 \\
\dot{\lambda}_4 &= -\frac{\partial \mathcal{H}}{\partial y_2} = -q_{44}y_2 - \lambda_3 \\
\dot{\lambda}_5 &= -\frac{\partial \mathcal{H}}{\partial z_1} = -q_{55}z_1 - n^2\lambda_6 \\
\dot{\lambda}_6 &= -\frac{\partial \mathcal{H}}{\partial z_2} = -q_{66}z_2 - \lambda_5
\end{aligned} \tag{35}$$

From here, the residual vector of nonlinear algebraic equations can be written using Equations (7), (9) and (10):

$$\begin{aligned}
{}^1R_1^1 &= {}^1x_1^1 - x_{1_0} \\
{}^kR_1^i &= D^k x_1^i - {}^k x_2^i \\
{}^{q+1}R_1^1 &= {}^qR_1^N \\
{}^M R_1^N &= {}^M x_1^N - x_{2_f} \\
{}^1R_2^1 &= {}^1x_2^1 - x_{2_0} \\
{}^kR_2^i &= D^k x_2^i - (3n^2 - 3n^2 {}^k x_1^i) \\
{}^{q+1}R_2^1 &= {}^qR_2^N \\
{}^M R_2^N &= {}^M x_2^N - x_{2_f} \\
{}^1R_3^1 &= {}^1y_1^1 - y_{1_0} \\
{}^kR_3^i &= D^k y_1^i - {}^k y_2^i \\
{}^{q+1}R_3^1 &= {}^qR_3^N \\
{}^M R_3^N &= {}^M y_1^N - y_{1_f} \\
{}^1R_4^1 &= {}^1y_2^1 - y_{2_0} \\
{}^kR_4^i &= D^k y_2^i \\
{}^{q+1}R_4^1 &= {}^qR_4^N \\
{}^M R_4^N &= {}^M y_2^N - y_{2_f} \\
{}^1R_5^1 &= {}^1z_1^1 - z_{1_0} \\
{}^kR_5^i &= D^k z_1^i - {}^k z_2^i \\
{}^{q+1}R_5^1 &= {}^qR_5^N \\
{}^M R_5^N &= {}^M z_1^N - z_{1_f} \\
{}^1R_6^1 &= {}^1z_2^1 - z_{2_0} \\
{}^kR_6^i &= D^k z_2^i + (-z_{cj}n^2 + n^2 {}^k z_1^i) \\
{}^{q+1}R_6^1 &= {}^qR_6^N \\
{}^M R_6^N &= {}^M z_2^N - z_{2_f}
\end{aligned} \tag{36}$$

Additionally,

$$\begin{aligned}
{}^k\mathbf{R}_7^j &= D {}^k\lambda_1^j + 3n^2 {}^k\lambda_2^j \\
{}^{q+1}\mathbf{R}_7^1 &= {}^q\mathbf{R}_7^N \\
{}^k\mathbf{R}_8^j &= D {}^k\lambda_2^j - {}^k\lambda_1^j \\
{}^{q+1}\mathbf{R}_8^1 &= {}^q\mathbf{R}_8^N \\
{}^k\mathbf{R}_9^j &= D {}^k\lambda_3^j \\
{}^{q+1}\mathbf{R}_9^1 &= {}^q\mathbf{R}_9^N \\
{}^k\mathbf{R}_{10}^j &= D {}^k\lambda_4^j - {}^k\lambda_3^j \\
{}^{q+1}\mathbf{R}_{10}^1 &= {}^q\mathbf{R}_{10}^N \\
{}^k\mathbf{R}_{11}^j &= D {}^k\lambda_5^j - n^2 {}^k\lambda_6^j \\
{}^{q+1}\mathbf{R}_{11}^1 &= {}^q\mathbf{R}_{11}^N \\
{}^k\mathbf{R}_{12}^j &= D {}^k\lambda_6^j - {}^k\lambda_5^j \\
{}^{q+1}\mathbf{R}_{12}^1 &= {}^q\mathbf{R}_{12}^N
\end{aligned} \tag{37}$$

The same initial parameters in Table 2 that were used to solve the unconstrained problem are used in this case to drive the system to a final state of $x_f = y_f = z_f = 0$. Table 3 displays the parameters used to model the no-fly-zone.

Table 3: HCW Equations No-Fly-Zone Parameters

Parameter	Value
Sphere center	$[33.44, -13.88, 33.81] \text{ km}$
Sphere radius	3.73 km

Results Figure 6 shows the optimal trajectory when a no-fly-zone is encountered. The constrained optimal trajectory is represented by blue dots, and the unconstrained trajectory is represented by red dots for comparison. From this Figure, we can see the unconstrained trajectory going straight through the no-fly-zone, whereas the constrained optimal trajectory avoids it. To display a better view of the trajectory missing the no-fly-zone, the Y-position view is shown in Figure 7a and the Z-position view is shown in Figure 7b. These different views provide a better perspective for comparing the unconstrained and constrained trajectories. From both Figures, we can conclude that the constrained optimal trajectory successfully avoids the no-fly-zone.

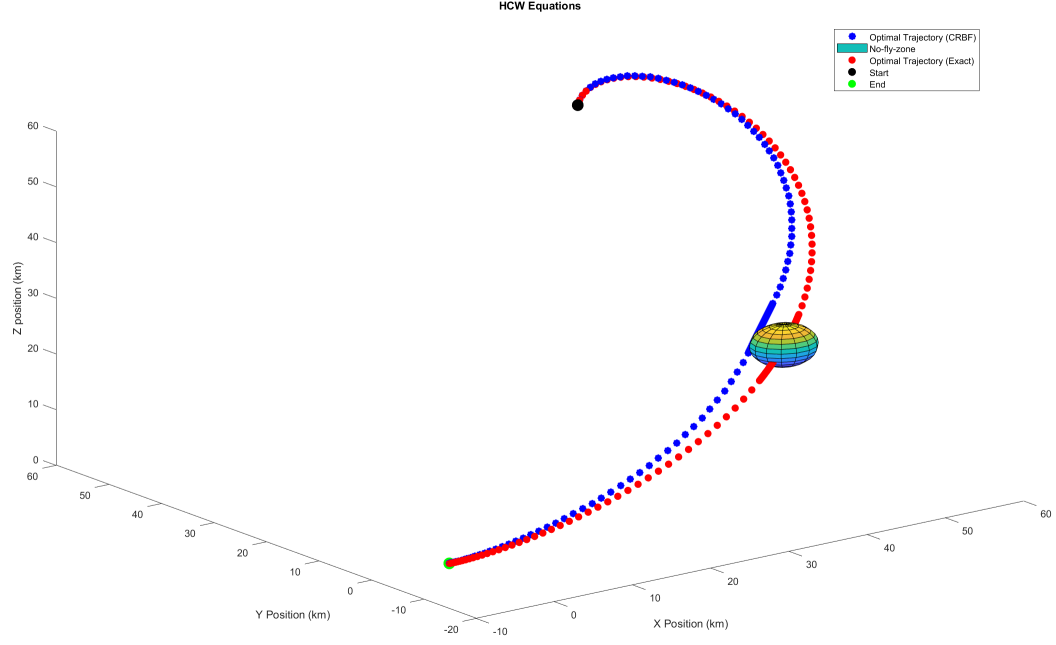
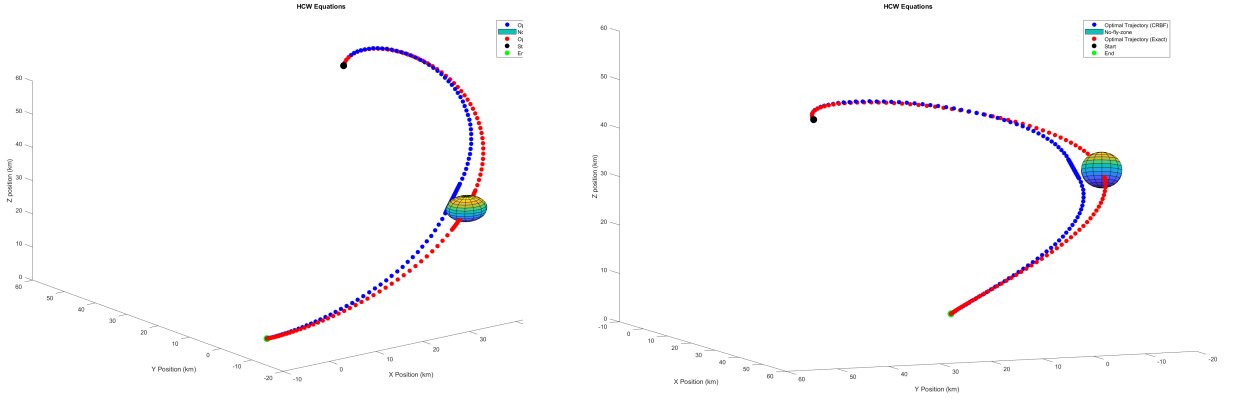


Figure 6: Optimal Trajectories for HCW Equations with No-Fly-Zone



(a) Optimal Trajectories - Y-Position View

(b) Optimal Trajectories - Z-Position View

Figure 7: Constrained HCW Equations - Optimal Trajectory Views for Y-Position (a) and Z-Position (b)

CONCLUSION

In this work, we demonstrated a Coupled Radial Basis Function and cover scheme method for solving optimal control problems for rendezvous and proximity operations. The method was ap-

plied to solving the unconstrained Hill-Clohessy-Wiltshire (HCW) equations. It was then extended to the constrained case with a spherical no-fly-zone. The HCW equations were cast into a two-point boundary value problem (TPBVP) and the necessary conditions of optimality were derived. Coupled Radial Basis Functions (CRBFs) were used as the approximating function over a problem domain divided into segments called "elements" with overlapping sections called "covers". Each element was locally collocated and the overlapping covers ensured continuity.

The CRBF-cover scheme method provides the advantage of less sensitivity to the shape parameter versus traditional Radial Basis Functions for approximation while preventing the CRBF matrix from becoming ill-conditioned with longer integration times. When solving the HCW equations, the CRBF and cover scheme method demonstrated accuracy and robustness when compared to the exact solution, with the absolute error for all trajectories being less than 0.2 km .

Overall, solving the unconstrained HCW equations for rendezvous and proximity operations serves as an excellent example for validating the method, while the constrained problem provided an excellent real-world application of the method. By applying a no-fly-zone constraint to the model, the optimal trajectory successfully avoided the no-fly-zone compared to the unconstrained solution. The CRBF-cover scheme method offers several advantages over the current methods in the literature. These include being able to solve optimal control problems with longer integration time problems, not requiring *a priori* information, and not relying on certain nodal distributions.

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