

OPTIMIZED RADIAL BASIS FUNCTIONS WITH ADAPTIVE SAMPLING STRATEGY FOR ORBITAL UNCERTAINTY PROPAGATION

Pugazhenth Sivasankar ^{*}, Nassim Razaaly [†] and Tarek A. Elgohary [‡]

Orbital uncertainty propagation of Resident Space Objects (RSOs), a challenging area in astrodynamics, is central to applications such as Space Situational Awareness (SSA), mission design, and collision avoidance. In this context, the recently introduced uncertainty quantification method using Liouville's theorem and radial basis function approximation is advanced by the inclusion of optimization of approximation parameters and adaptive sampling to refine the probability density function (PDF) approximation at a future time of interest. Leveraging the mesh-free adaptability of radial basis functions, the curse of dimensionality is mitigated by optimizing the shape parameter associated with each dimension. The method outlined in this work is applied to propagate the uncertainty by approximating the future PDF of nonlinear dynamical systems, such as a Duffing oscillator and an Earth-bound planar orbit problem. Results show better computational efficiency and good agreement with Monte-Carlo simulations.

1 INTRODUCTION

The rapid expansion of space activities has led to an increasingly congested orbital environment, raising critical concerns about the need for improved Space Situational Awareness (SSA). Proposed by NASA scientist Donald J. Kessler in 1978, Kessler Syndrome describes a scenario where excessive space debris triggers a cascade of collisions, rendering certain orbits unusable.¹ With over 30,000 tracked debris objects and countless untrackable fragments, even small uncertainties in trajectory predictions can lead to catastrophic collisions in the long run. Additionally, the rise of CubeSats and large-scale constellations has dramatically increased the number of active satellites from about 3,000 in 2015 to over 8,000 in 2024. Many of these satellites operate in low Earth orbit (LEO), where close approaches are becoming increasingly frequent. In addition to these, the past few years have seen record-breaking launch rates, with 2025 marking over 1200 successful satellite launches between January and April. Recent debris-generating incidents, such as anti-satellite tests conducted in 2021 and accidental collisions, have added thousands of fragments to an already crowded space environment. Given the increase in space debris, more precise uncertainty propagation (UP) methods are required to mitigate collision risks, ensure satellite safety, and maintain the long-term sustainability of space operations.

^{*}Ph.D. Candidate, Mechanical and Aerospace Engineering, University of Central Florida, 12760 Pegasus Drive, Orlando, FL.

[†]Assistant Professor, Institute P', ISAE-ENSMA, Poitiers, France

[‡]Associate Professor, Mechanical and Aerospace Engineering, University of Central Florida, 12760 Pegasus Drive, Orlando, FL.

Accurate uncertainty quantification (UQ) of a resident space object's (RSO) state is critical in Space Situational Awareness (SSA), particularly for key tasks like conjunction analysis, collision probability estimation, and uncorrelated track association.^{2–9} Considering a 1-dimensional Gaussian PDF, for example, the region of sample space whose boundary lies at 2 Mahalanobis distances (i.e., 2σ) from the maximum likelihood point covers about 95% of all the probable outcomes. However, for a 3-dimensional Gaussian PDF, the same 2σ region covers only 75% of all probable outcomes, and for a 6-dimensional Gaussian PDF, the coverage reduces to approximately 33%. Conversely, for a Monte-Carlo (MC) simulation to approximate the Gaussian PDF bounded by the 2σ region, the number of samples required increases exponentially with the number of dimensions. This effect, known as the curse of dimensionality, becomes worse when the PDF becomes non-Gaussian. Traditional approaches to UQ often struggle with non-Gaussian probability density functions (PDFs), which can arise from nonlinear dynamics or extended propagation periods.¹⁰ For instance, propagating covariance using the state transition matrix has inherent limitations—it relies on linearized system dynamics and assumes a Gaussian PDF at the final state.^{3,11–13} To overcome these challenges, one approach involves incorporating higher-order derivatives through state transition tensor methods, enabling a more precise mapping of the final state PDF. However, this technique necessitates calculating (or approximating) high-order partial derivatives of the forcing function relative to the state-parameter variables, introducing a balance between computational expense and fidelity in results.^{14–17}

Among the most common uncertainty quantification techniques used in astrodynamics are Polynomial Chaos Expansions (PCE) and Gaussian Mixture Models (GMM), both of which have been actively employed for over two decades. PCE operates by constructing functional approximations to develop surrogate models that characterize how a system responds to different state-parameter inputs, making it particularly useful for astrodynamical applications involving nonlinear phenomena.¹⁸ A more recent advancement, Arbitrary Polynomial Chaos (APC), has demonstrated promise in orbital uncertainty propagation by combining the method of admissible regions with short-arc optical observations.¹⁹ This method is promising as it can utilize an arbitrarily shaped probability region at the initial time for uncertainty propagation. In contrast, GMM decomposes the initial uncertainty distribution into a weighted combination of multiple Gaussian probability density functions (PDFs). These individual PDFs are propagated forward in time and then recombined to form the posterior distribution, enabling a highly adaptable representation of uncertainty. A key strength of GMM lies in its capacity to integrate new observational data, allowing for progressive refinement of uncertainty estimates—a particularly valuable feature for orbit determination tasks where sequential tracking updates enhance prediction accuracy. However, a critical challenge in implementing GMM is selecting the appropriate number of Gaussian components. An insufficient number may result in underfitting, while an excessive number can lead to overfitting, both of which compromise the accuracy of the uncertainty representation.^{20,21}

In the domain of space object collision probability estimation, Probe et al. introduced an innovative technique employing Chebyshev polynomial approximations for PDF representation.²² This methodology has been successfully implemented across different orbital regimes, including the 6-dimensional PDF approximation for the GRACE satellite in Low Earth Orbit (LEO) as well as spacecrafts in Highly Elliptical Orbit (HEO) and Geostationary Earth Orbit (GEO). Validation studies using NASA JPL's navigation solutions and FireOPAL optical sensor data have confirmed the method's applicability.^{23,24} Nevertheless, the Chebyshev polynomial approach faces inherent constraints, particularly the requirement for Chebyshev grids that become progressively less effective

and less efficient as dimensionality increases in PDF approximation problems.

Hence, to effectively and efficiently quantify the uncertainty of space debris and space assets, a new method involving Liouville’s theorem and functional approximation using radial basis functions (RBFs) was introduced in previous works.^{25,26} Radial basis functions (RBFs) represent a family of real-valued functions whose values depend solely on the Euclidean distance between approximation nodes and center nodes. Of the various radial basis functions, multiquadric functions—originally introduced by Hardy—have proven especially useful in scientific computing.²⁷ RBFs gained prominence through Kansa’s work, which successfully applied RBFs to interpolate high-dimensional data and to solve partial differential equations.²⁸ The versatility of RBFs extends to optimal control applications, as demonstrated by Rad et al.²⁹ Recent advances include the development of Coupled Radial Basis Functions (CRBFs), whose shape parameter insensitivity has proven valuable for solving nonlinear optimal control problems such as Zermelo’s problem and inverted pendulum systems.³⁰

Zhang et al.³¹ further showcased the effectiveness of multiquadric RBFs in solving nonlinear integral equations across one and two dimensions. Several research papers discuss optimization techniques for RBFs, including shape parameter selection, center placement, and weight optimization. A few articles published more than a decade ago discuss the impact of shape parameters on RBF interpolation accuracy, introducing spatially variable shape parameters to mitigate the Runge phenomenon, and the implementation of the leave-one-out cross-validation (LOOCV) method to optimize the shape parameter in Gaussian and multiquadric RBFs.^{32,33} Another approach uses Bayesian methods to optimize RBF centers and weights, providing a probabilistic approach to model selection.³⁴ Multiquadric functions offer distinct advantages, including exceptional approximation accuracy with relatively few nodes. Their meshless nature provides unique flexibility, allowing for targeted node placement in regions requiring higher precision for probability density function approximation.

This work improves the author’s previously introduced uncertainty propagation technique of using Liouville’s theorem and functional approximation using RBFs, with a slight modification of the RBF formulation and the inclusion of the adaptive sampling strategy. The modified RBF formulation allows for the optimization of the shape parameters for each dimension as well as the power parameter. This results in an effective PDF approximation. This paper is organized as follows: Section 2 describes the methodology of using RBFs for the PDF approximation process, including the optimization strategy used, followed by the adaptive sampling process. Section 3.1 presents an example of using RBFs to approximate the PDF of a Duffing oscillator at a future time of interest. The results are compared with Monte-Carlo simulations in terms of the cumulative integral accuracy. Section 3.2 provides the numerical results of approximating the PDF of a planar orbit problem. It approximates the PDF of the satellite after propagating it to ten revolutions of its orbit period. Section 4 summarizes the results and presents a discussion on the accuracy metric and computational cost. Finally, Section 5 concludes the paper, and provides directions for future research.

2 METHODOLOGY

Structured in three parts, this section introduces the basics of the uncertainty propagation scheme in Section 2.1, followed by the nuances of optimizing the shape and power parameters in Section 2.2. Finally, Section 2.3 explains the adaptive sampling strategy for selective refinement of approximation domain. All these methods are implemented in the cases described in Section 3.

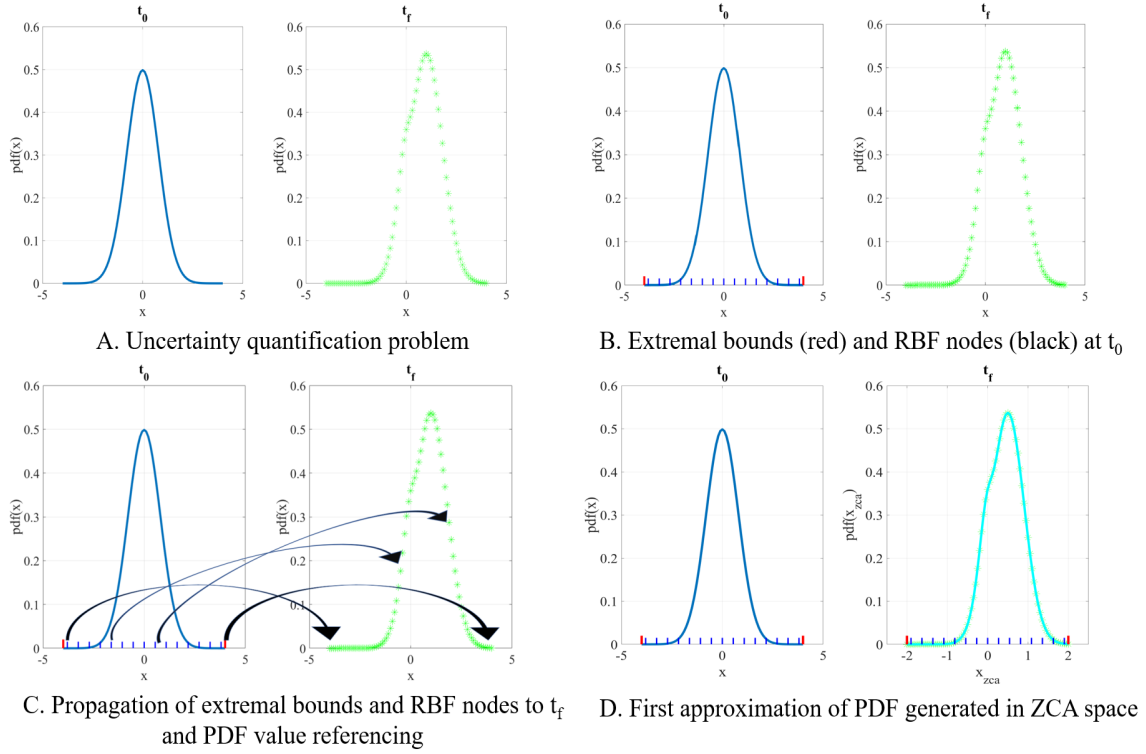


Figure 1. Steps to generate the first approximation of future PDF

2.1 Basics

The objective involves quantifying uncertainty by determining the probability density function (PDF) of a nonlinear dynamical system at a future time t_f , given knowledge of the initial PDF at time t_0 . Fig. 1A illustrates a simplified one-dimensional case, where the initial PDF appears as a solid black curve and the target (unknown) final PDF is depicted by a dashed green line. The PDF approximation process begins with defining the region of interest, the state-space domain where the posterior PDF approximation will be performed. This is achieved by identifying initial state-space points whose PDF values correspond to a user-specified boundary threshold. For common distributions like Gaussian or Cauchy, determining these extremal boundary points is relatively straightforward. A simple procedure to obtain these boundary points is provided in Section 3.1. Unless specified differently, all problems in this work assume an initial Gaussian distribution. The designated region, known as the Region of Interest (ROI), is then discretized using approximation nodes, as visualized in Fig. 1B. This study employs nodes generated from the Halton sequence, a low-discrepancy sequence based on coprime numbers.³⁵ These nodes serve as the foundation for the approximation scheme. The next step involves evaluating the initial PDF values at both the boundary points and the RBF approximation nodes at time t_0 using the known initial distribution.

Utilizing the ordinary differential equations that govern the dynamic system, these nodes and boundary points are propagated to t_f . According to Liouville's theorem, for a dynamic system under the influence of conservative forces, the PDF value of a point in state-space, defined by initial conditions and an apriori PDF, remains constant at all times as the point moves through a unique

trajectory in the state-space. This is mathematically represented by Liouville's equation as,³⁶

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \sum_{i=1}^m \left(\frac{\partial \rho}{\partial q_i} \dot{q}_i + \frac{\partial \rho}{\partial p_i} \dot{p}_i \right) = 0 \quad (1)$$

where, m is the number of random variables, and q, p represent the state variables. By a corollary of the theorem, the PDF values at the approximation nodes and the boundary points at t_0 can be transferred/referenced to the corresponding locations at t_f , as depicted in Fig. 1C. In the next step, shown in Fig. 1D, the boundary points are included as part of the approximation nodes, and together, they are transformed to the Zero-phase Component Analysis (ZCA) Whitening space, after which the corresponding referenced PDF values are used to create a functional approximation of the posterior PDF using RBFs,

$$f(x_{zca}) \cong \sum_{k,c=1}^n a_k \phi_k(r_k) \quad (2)$$

where, n is the number of RBF approximation nodes, $\phi_k(r_k)$ is the radial basis function whose value depends on the distance, r_k , between an approximation node x_k and a center node x_c in ZCA Whitening space such that $c \neq k$, f is the set of logarithm of the PDF values at the RBF approximation nodes, and a_k is the corresponding approximation coefficient. In this study, all the available approximation nodes are treated as centres, resulting in an $n \times n$ collocation matrix. For the purpose of numerical stability, all the cases considered in this paper involve the approximation of the logarithm of the PDF values instead of the actual PDF values. To convert to the actual PDF value, an exponential operation is performed on the logarithm of the PDF when necessary. Details about the ZCA transform and the formulation of r_k are provided in Section 2.2. The approximation coefficients are estimated by solving a system of linear equations. In matrix form, Eq. (2) can be represented by,

$$[\Phi][\mathbf{a}] = [\mathbf{F}] \quad (3)$$

and the coefficient vector is given by,

$$[\mathbf{a}] = [\Phi]^{-1}[\mathbf{F}] \quad (4)$$

which requires that the $n \times n$ collocation matrix of the system, $[\Phi]$ be invertible. To a certain degree, the number of RBF nodes and their locations determine the condition number of the collocation matrix. Because the value of the ϕ_k is a scalar that depends on the distance, r_k , the stochastic collocation matrix, $[\Phi]$ whose elements are ϕ_k , is agnostic to the number of dimensions in the problem. This property lends the RBF suitable for higher dimensional applications. Fig. 2 shows an example of a 2D functional approximation using RBFs.

2.2 Optimization of shape and power parameters

The first approximation of the posterior PDF, depicted in Fig. 1D, takes place in the ZCA Whitening space. The RBF approximation nodes are transformed to the ZCA Whitening space, hereafter known as the ZCA space, to level the range of random variables for smoother computation of the

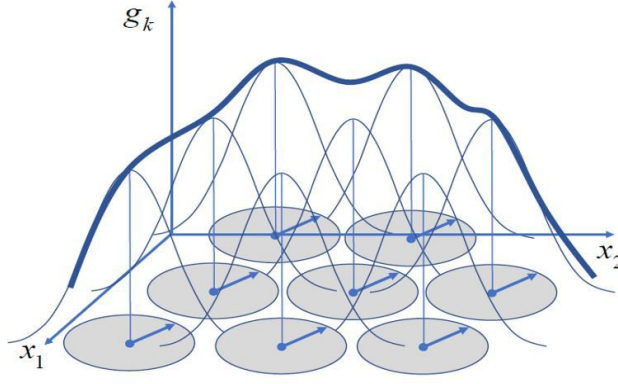


Figure 2. Bivariate functional approximation using RBFs [Source: H. Harry Asada, *Machine learning & Nonlinear System Modeling*, Department of Mech. Engg, MIT]

distance metric. This is an invertible linear transformation given by the following equations,

$$\begin{aligned}
 x_{\text{cen}} &= x - E[x] \\
 \Sigma &= \frac{1}{n} x_{\text{cen}}^T x_{\text{cen}} \\
 \Sigma &= U \Lambda U^T \\
 W_{\text{zca}} &= [U \Lambda^{-1/2} U^T]^T \\
 x_{\text{zca}} &= x_{\text{cen}} W_{\text{zca}}
 \end{aligned} \tag{5}$$

where x is the set of RBF approximation nodes in inertial space and x_{zca} is the same set of RBF approximation nodes in ZCA space. The impact of this transformation can be seen in plots of RBF nodes in Fig. 3 and Fig. 4. In Fig. 3, the random variables X_1, X_2, X_3, X_4 correspond to X, Y, V_x, V_y coordinates of a planar orbit in inertial space after 10 revolutions. It is clear that the position and velocity are three orders of magnitude apart, numerically, in the inertial space. However, after the ZCA transformation, Fig. 4 shows the new set of random variables which have the same order of magnitude between -3 and 4. With all the random variables (i.e. the different dimensions) in a similar range, approximation using RBFs becomes easier. The radial basis kernel mentioned in Eq. (2) is formulated between two approximation nodes, x_k and x_c , as follows,

$$\begin{aligned}
 \phi(r_k) &= (1 + r_k^2)^q \\
 r_k^2 &= \sum_{i=1}^d ((x_{k,i} - x_{c,i}) l_i)^2 \\
 l_i &= 10^{\epsilon_i}
 \end{aligned} \tag{6}$$

where d is the number of dimensions, ϵ_i is the shape parameter per dimension and q is the power parameter. This formulation is applied to all the approximation nodes, resulting in the $n \times n$ collocation matrix, $[\Phi]$, which is then plugged into Eq. (4) to yield the approximation coefficient vector, $[\mathbf{a}]$. Though the quality of the first approximation of the posterior PDF, shown in Fig. 1D, can be quite accurate at the approximation nodes, it is only useful as long as the approximant can predict (i.e. interpolate) the PDF values with a reasonable degree of accuracy at locations other than the approximation nodes, but still within the ROI. The approximation quality is evaluated by the following procedure:

- Create a separate set of points spanning the entire ROI of the dynamic system at t_0 . It is desirable that the size of this test set be equal to or greater than the size of the approximation nodes (i.e. train nodes).
- Obtain the PDF values at these test set points using the given initial distribution.
- Use Liouville's theorem and the system dynamics to propagate these test set points and their PDF values from t_0 to t_f . For a system with conservative dynamics, these PDF values remain unchanged and are considered as the true PDF values at the test set locations, $[\mathbf{F}_{\text{true}}]_{\text{test}}$.
- Use the approximation coefficient vector $[\mathbf{a}]$ to interpolate/predict the PDF values at the test set locations. In matrix form, the interpolation/prediction on a test set of size m is written as,

$$[\mathbf{F}_{\text{pred}}]_{\text{test}} = [\mathbf{\Gamma}][\mathbf{a}] \quad (7)$$

where $[\mathbf{\Gamma}]$ is an $m \times n$ interpolation matrix whose elements are given by $\phi_k(r_k)$ with r_k as defined in Eq. (6) with $k = 1, 2, \dots, m$ and $c = 1, 2, \dots, n$. Here, the RBF kernel is formulated between each approximation node and each test set point, resulting in the $m \times n$ interpolation matrix.

- Evaluate the approximation quality by computing the Q^2 norm for the test set as follows,

$$Q^2_{\text{test}} = 1 - \frac{\sum_{i=1}^m (f_{\text{pred},i} - f_{\text{true},i})^2}{\sum_{i=1}^m (f_{\text{true},i} - \bar{f}_{\text{true}})^2} \quad (8)$$

where $\bar{f}_{\text{true}}$ is the mean of the true PDF values at the test set locations.

For a good-quality PDF prediction, the Q^2_{test} should approach the value 1, indicating the ability of the RBF approximant to generalize. In general, depending on the dimensionality and the nonlinearity of the system dynamics involved, the first RBF approximation does not produce an approximant with Q^2_{test} value approaching one. A major advantage of the formulation shown in Eq. (6) is that it allows for the improvement of approximation quality through optimization of the power parameter q and the dimension-wise shape parameters, ϵ_i , while retaining the same set of training nodes. The optimization of RBF approximation is accomplished by the following procedure:

- The objective function to minimize is the Mean Squared Error (MSE) between the predicted PDF and the true PDF at the test point locations.
- For a given set of training nodes, the MSE is minimized by altering the RBF approximation by varying the shape parameters, ϵ_i , and the power parameter, q involved.
- *fmincon* optimizer from MATLAB is used with the sequential quadratic programming algorithm for faster convergence. The maximum iterations is set to 100 with an optimality tolerance and step tolerance of 10^{-4} . Parallel computing with multiple CPU cores is utilized for accelerated computation.
- In general, the power parameter, q , is allowed to vary between 0.5 and 2, whereas the shape parameters, ϵ_i , are allowed to vary between -3 and 3.
- As this is a nonlinear optimization problem, 10 different initial starting points (i.e. restarts) are used, resulting in 10 different minimum values for the MSE. The RBF parameter set corresponding to the minimum of these 10 minimum MSEs is taken as the global minimum solution. A sample RBF optimization is shown in Fig. 5.

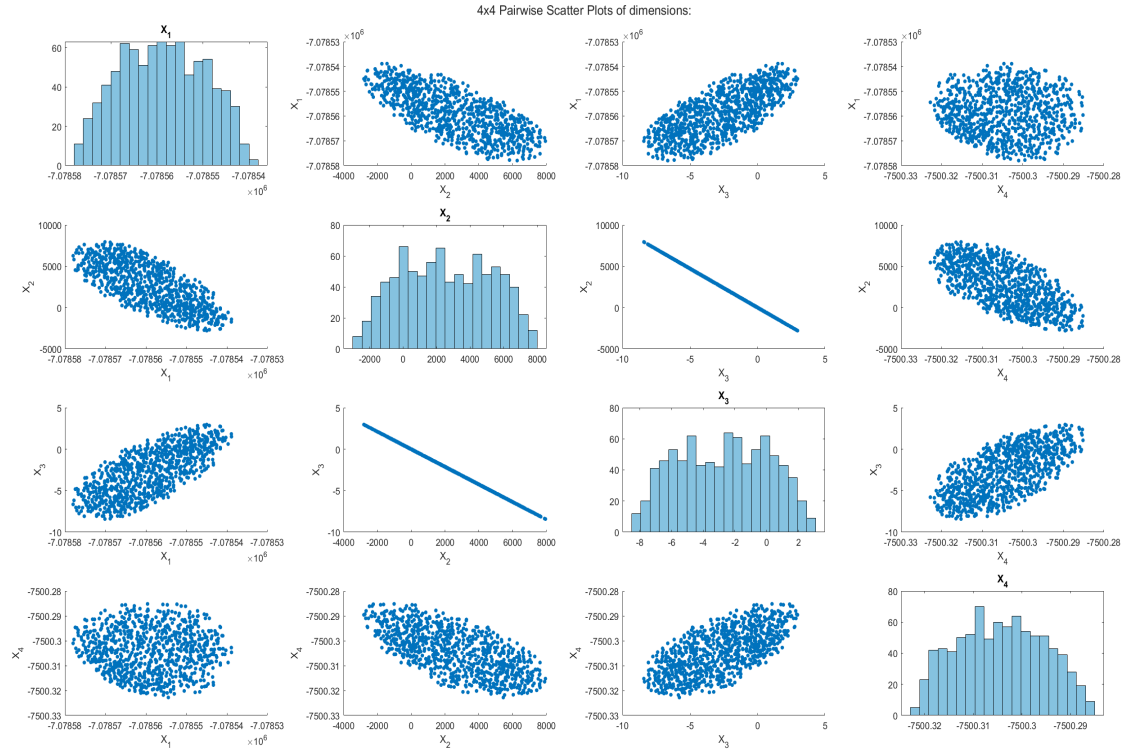


Figure 3. Planar orbit example: RBF nodes in inertial space after 10 orbit revolutions

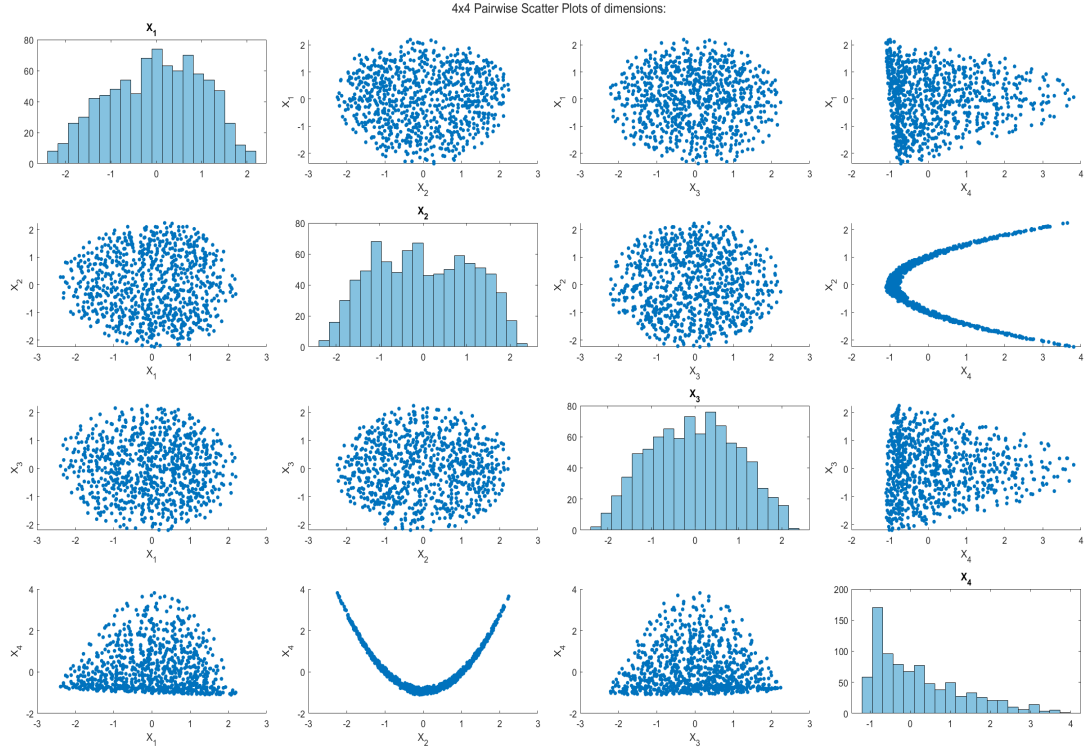


Figure 4. Planar orbit example: RBF nodes in ZCA space after 10 orbit revolutions

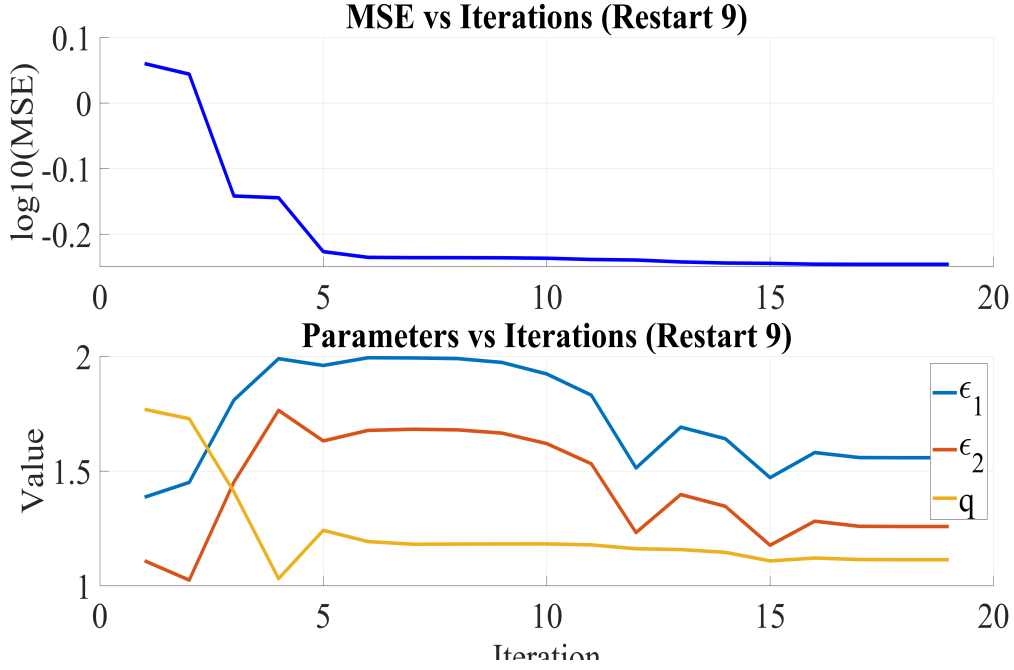


Figure 5. Sample RBF optimization in a 2D problem: variation of the MSE and the RBF parameters (2 shape parameters and 1 power parameter)

2.3 Adaptive sampling for selective refinement

The objective of the RBF optimization procedure discussed in the previous section is to improve the approximation quality, indicated by Q_{test}^2 . However, there is a limit to which the optimization of shape and power parameters can improve Q_{test}^2 so that it gets closer to one. Beyond this limit, any improvement in Q_{test}^2 is possible if the Design of Experiment (i.e. training nodes) is enhanced in such a way that the approximation domain is *selectively refined*. This is done through adaptive sampling by the following procedure:

1. The first step is to create an initial pool of possible nodes in the approximation domain in the ZCA space. This is done by drawing samples from the uniform distribution with the extremities of the existing training nodes as bounds. This is shown in the top left plot of Fig. 6. Usually, the size of this initial pool is kept quite large to densely fill the approximation domain.
2. From the initial pool, only those points within a certain threshold distance of the DOE are retrieved. This threshold distance is a function of the dimensionality of the problem. The retrieved points are termed as L1 candidates and are shown in the top right plot of Fig. 6.
3. Now, the optimized RBF model is used to predict the PDF values at these L1 locations, and only those L1 candidates for which the PDF value lies above the ROI boundary PDF value are considered to be within the ROI. From these internal L1 candidates, a pre-determined number of candidates that are *farthest* from the DOE are chosen as the best candidates. This pre-determined number is also known as the back propagation budget.
4. The best candidate nodes are transformed to inertial space, and are back propagated using the

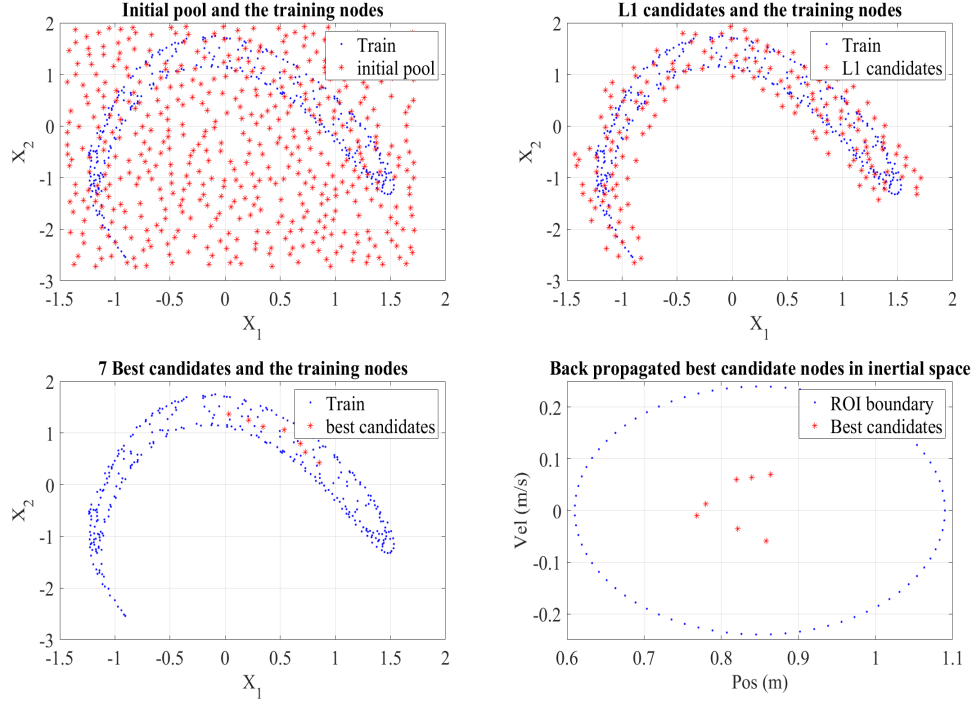


Figure 6. Adaptive sampling for selective refinement in a 2D PDF approximation problem.

system dynamics to t_0 when their actual PDF values can be obtained using the known initial distribution. By Liouville's theorem, the PDF values associated with the best candidate nodes can be referenced to t_f

5. These best candidate nodes and their PDF values are now included in the DOE, and the RBF approximation is done with the updated DOE and the optimized RBF parameters resulting in an improved Q_{test}^2 .
6. Steps 1 to 5 are repeated until the desired value of Q_{test}^2 is achieved.
7. The RBF approximant from Step 6 is put through another shape and power parameter optimization process to further improve Q_{test}^2 .

Three different stages of the adaptive sampling process are illustrated in Fig. 7. It is called adaptive sampling because the new nodes are picked based on the existing DOE (i.e. training nodes). By modifying the boundary PDF value in Step 3, the new nodes can be concentrated in a specific region. Besides Q_{test}^2 metric, another way of ascertaining the PDF approximation quality is to compute the cumulative integral/probability using the PDF approximant. In any statistical system, the total probability integral or the cumulative integral of the PDF with respect to all random variables at any given time is always equal to one. In a dynamical system with two random variables x_1 and

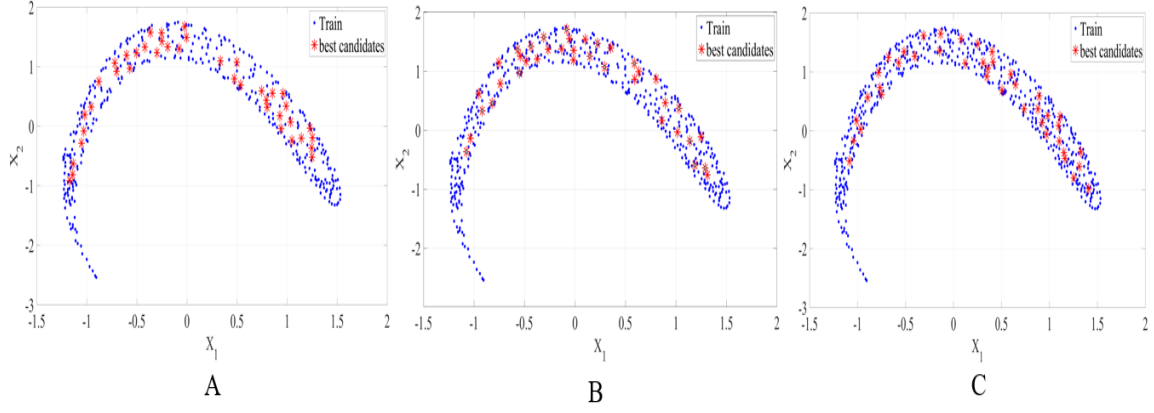


Figure 7. Adaptive sampling in 2D: initial refinement (A), intermediate refinement (B), and final refinement (C)(Current DOE is shown in black and new nodes are shown in red).

x_2 , this cumulative integral of the PDF at t_0 and t_f can be written as,

$$\begin{aligned}
 I_0 &= \int_{x_{10}} \int_{x_{20}} f(x_{10}, x_{20}) dx_{20} dx_{10} \\
 I_f &= \int_{x_{1f}} \int_{x_{2f}} f(x_{1f}, x_{2f}) dx_{2f} dx_{1f} \\
 I_0 &= I_f = 1
 \end{aligned} \tag{9}$$

where x_{10}, x_{20} are the random variables at t_0 and x_{1f}, x_{2f} are the same random variables at t_f . For an n-D system, a tensorial grid of sufficient density can be created using the extremities of the DOE (i.e. training nodes). Using the optimized RBF from the final step of the adaptive sampling process, the PDF values at the tensorial grid can be predicted. This can then be used with a quadrature integration scheme to obtain the marginal PDFs, and eventually the cumulative integral. The complete process of uncertainty propagation using RBF is shown in Fig. 8.

3 NUMERICAL EXPERIMENTS

In this section, the proposed uncertainty propagation scheme is applied to two different nonlinear dynamical systems: i) an undamped Duffing oscillator, and ii) a planar orbit propagated up to 10 revolutions. Wherever possible, the outcomes are validated against Monte-Carlo simulations. Unless stated otherwise, all the simulations in this work are performed in a computer with 13th Generation Intel(R) Core(TM) i9-13900H, 2.600 Ghz, 14 Core(s), 20 Logical Processors.

3.1 Characterizing the PDF of an undamped Duffing oscillator

A Duffing oscillator is a nonlinear oscillator whose dynamics is given by the general equation,

$$\ddot{x} + \delta \dot{x} + \alpha x + \beta x^3 = \gamma \cos \omega t \tag{10}$$

where δ is the damping coefficient, α is the linear stiffness coefficient, β is the nonlinear stiffness coefficient, γ and ω are the amplitude and the angular frequency of the periodic driving force, respectively. In this work, the unforced Duffing oscillator ($\gamma = 0$) with a softening spring ($\beta < 0$)

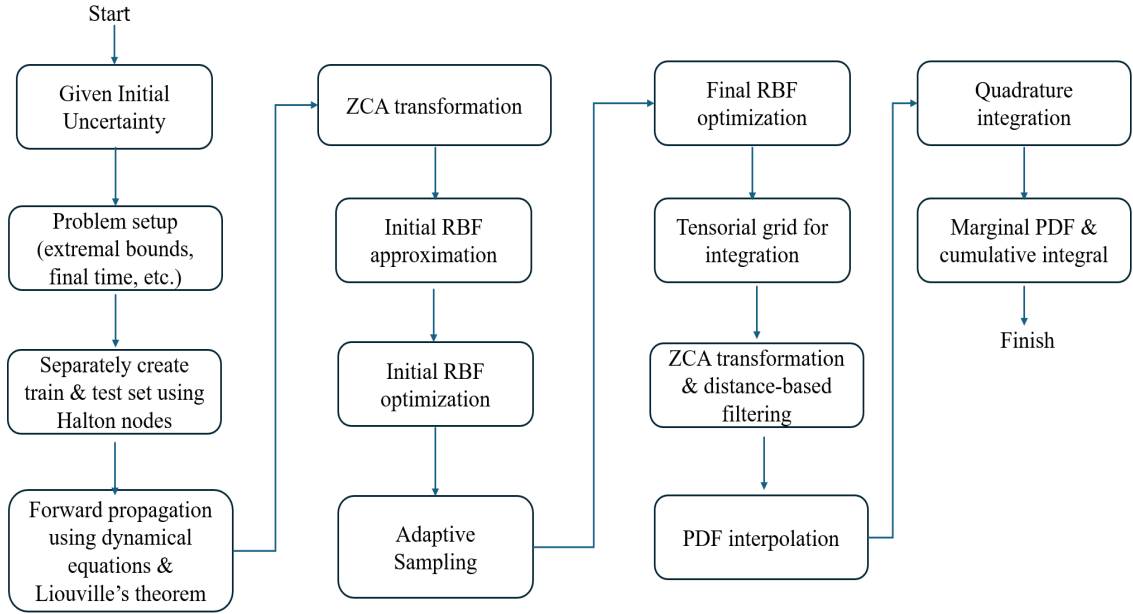


Figure 8. Uncertainty Propagation Scheme using RBFs

is studied. For this problem, the value of the damping coefficient is set to zero, the mass is set to unity and k is used as a common stiffness coefficient in the linear and nonlinear stiffness term, Eq. 11. Hence, it becomes a nonlinear conservative system in which the PDF depends on two random variables: position and velocity ($[x \dot{x}]^T$).

$$\ddot{x} + (1 + k^2)x - 2k^2x^3 = 0 \quad (11)$$

The simulation parameters of this case are shown in Tables 1 and 2. The ROI is set as 8σ from the nominal state space point at t_0 . For this 2-dimensional system, 80 boundary points marking the 8σ initial extremal bound are created using the statistical parameters shown in the Table 2 and by the following steps:

- A set of 80 randomly spaced or equispaced points is generated on a unit circle. The number of these points marking the boundary should be increased with dimensionality for better coverage. Creating equispaced points on the surface of the unit hyper-sphere in higher dimensions is expensive as it requires solving the Thompson optimization problem.
- The standard deviations provided in Table 2 are used to form the initial diagonal covariance matrix of the system.
- The eigenvalues and the eigenvectors of the covariance matrix are obtained.
- The points created on the unit circle are scaled to lie on the 8σ boundary using the eigenvalues, and oriented using the eigenvectors of the covariance matrix.

To allow for some margin of error, a buffer boundary is created with 80 points at 9σ from the nominal state space point at t_0 . Next, approximation nodes are created up to the buffer boundary

using the Halton sequence. The initial number of these approximation nodes, n_{RBF} , is typically chosen based on the dimensionality of the problem, Table 3. The approximation nodes and the boundary points are propagated for seven seconds using Eq. (11) and the *ode45* integration routine in MATLAB. Fig. 9 shows the transformation of a circular 8σ Gaussian extremal bound at t_0 to an irregularly shaped extremal bound at t_f . As described in Section 2.2, the approximation nodes are transformed to the ZCA space, shown in Fig. 10, to aid the approximation process, and the minimum distance tolerance of 0.01 is used to remove the approximation nodes that lie too close to each other to ensure numerical stability of the RBF collocation matrix. Similarly, 460 test set points are created, spanning from the nominal state space point up to the buffer boundary at t_0 , and propagated to t_f , shown in Fig. 11, along with their PDF values. The concentration of these test points within the approximation domain is dictated by the degree of refinement required by the different probability density value regions within the approximation domain, and can be explored further in future works.

Table 1. 2D conservative Duffing oscillator: simulation parameters for UP with optimized RBF

Parameter	Value
t_0 (s)	0
t_f (s)	7.00
Generalized stiffness, k (N/m)	0.5
Mahalanobis distance from nominal at t_0	8
RBF kernel type	$(1 + r_k^2)^q$

Table 2. 2D conservative Duffing oscillator: random variables at initial time (t_0) for UP with optimized RBF

Statistical parameter	Position (m)	Velocity (m/s)
μ	0.85	0.00
σ	0.03	0.03

Using the RBF formulation described in Eq. (6), the shape and power parameters, the initial set of training nodes (i.e. the RBF approximation nodes or the Design of Experiment (DOE)) presented in Table 3 and their referenced PDF values, an initial RBF approximation of the PDF is created at t_f . The generated approximation coefficients, along with the shape and power parameters, are used to predict the PDF values at the test set locations as described in Eq. (7). The quality of prediction is evaluated using the Q_{test}^2 metric, as shown in Fig. 12. In the figure, the prediction represented by the blue dots doesn't align very well with the $y = x$ line in this initial approximation, and this is quantified by $Q_{test}^2 = -0.085883$. When the predicted value of the PDF matches the true PDF value perfectly, the blue dots will align with the $y = x$ line, which corresponds to $Q_{test}^2 = 1$. In the next stage, the initial RBF approximation is improved by following the optimization procedure described in Section 2.2. This optimizes the shape and power parameters without modifying the set of training nodes (i.e. the DOE), Fig. 13, and this results in a big improvement of Q_{test}^2 from -0.085883 to 0.972413, shown in Fig. 14.

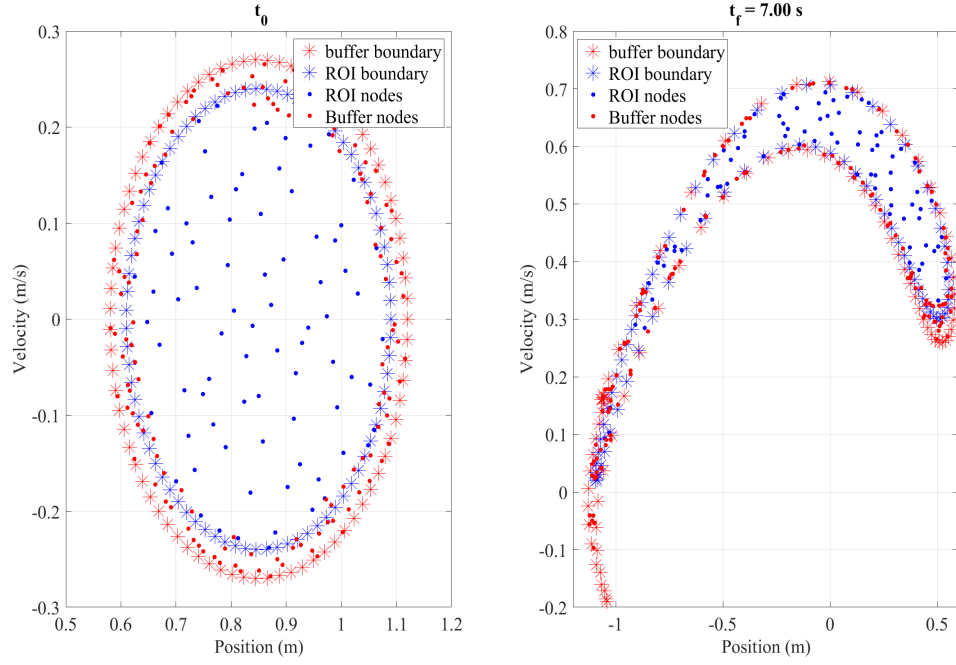


Figure 9. 2D conservative Duffing oscillator: Uncertainty bounds filled with RBF nodes at t_0 and $t_f = 7s$. Points marking the boundary of the Region of Interest (ROI) and buffer region are also shown.

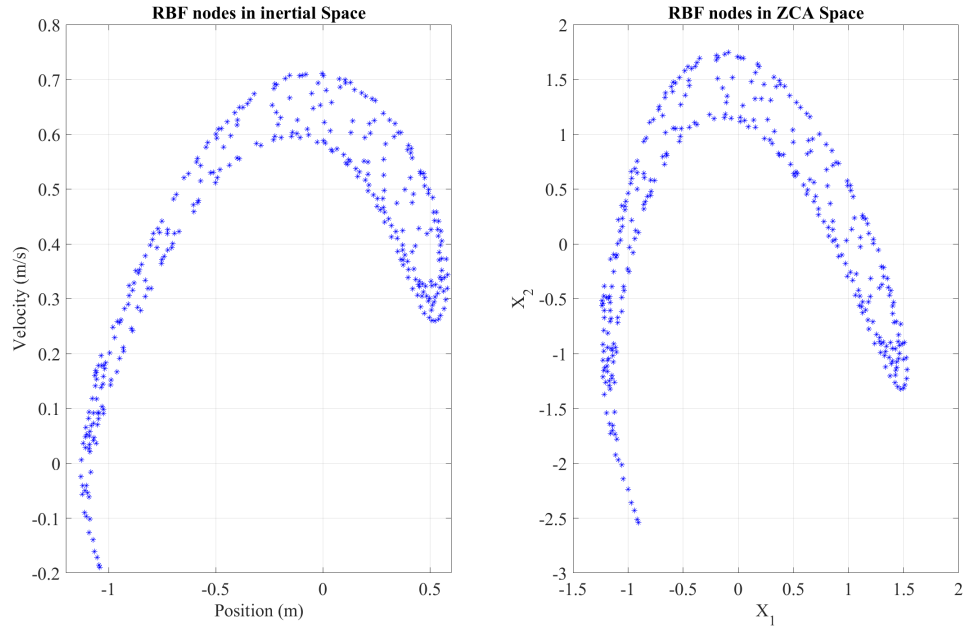


Figure 10. 2D conservative Duffing oscillator ($t_f = 7s$): transforming the training nodes from inertial to ZCA space.

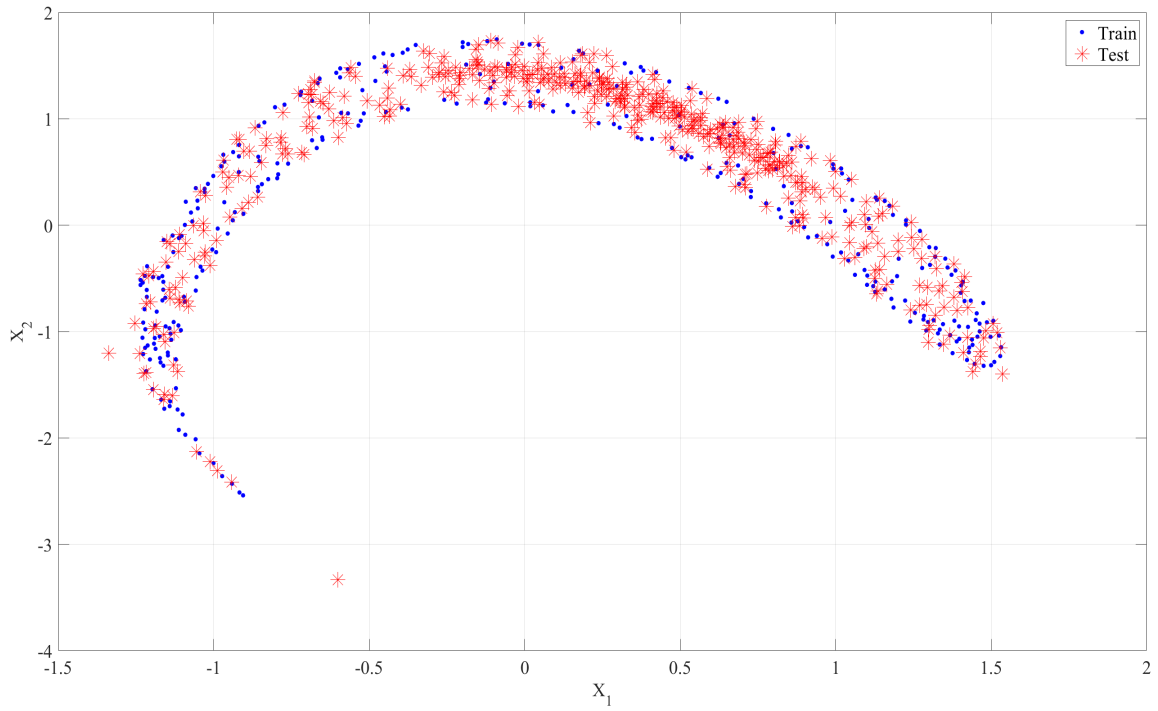


Figure 11. 2D conservative Duffing oscillator ($t_f = 7s$) in ZCA space: Train set (RBF nodes) after implementing minimum distance filtering and the test set.

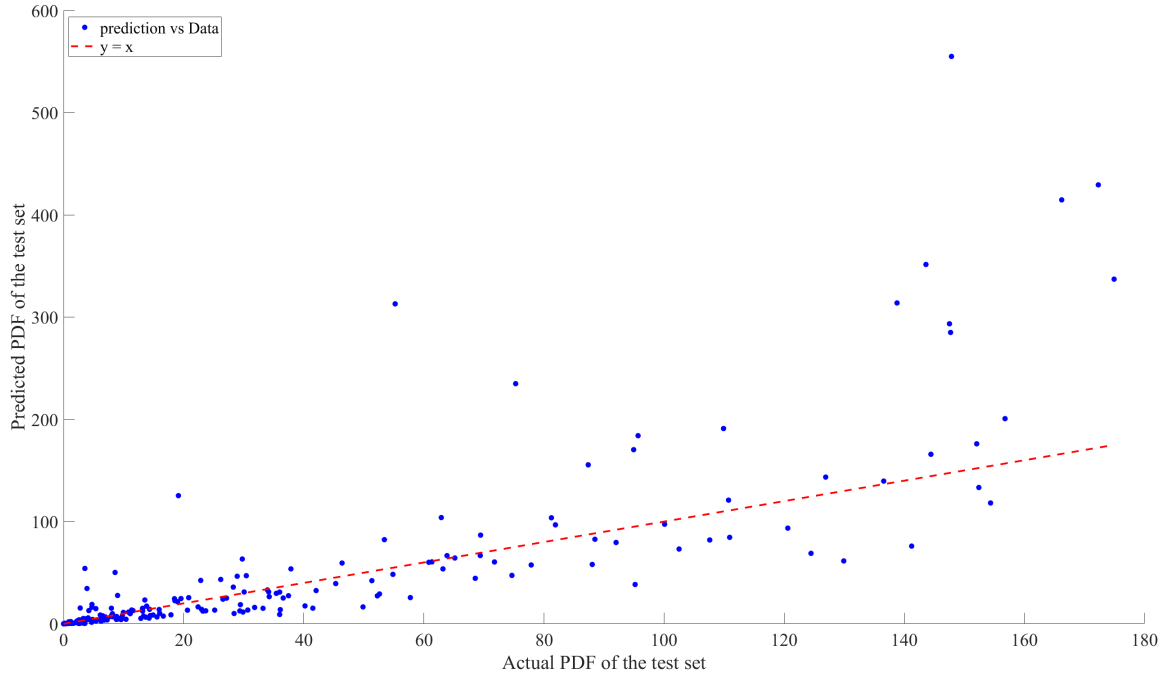


Figure 12. Duffing oscillator ($t_f = 7s$) in ZCA space: predicted PDF (vs) actual PDF at the test set locations after initial RBF approximation ($Q_{test}^2 = -0.085883$).

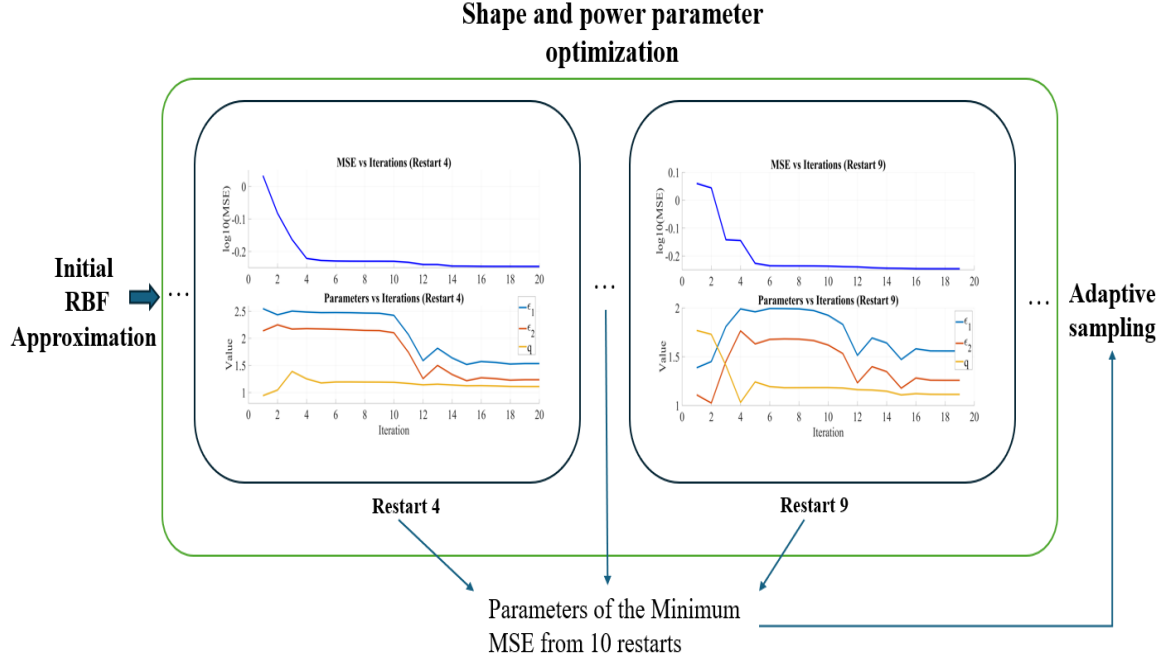


Figure 13. 2D conservative Duffing oscillator ($t_f = 7s$) in ZCA space: initial optimization of the two shape parameters (ϵ_1, ϵ_2) and one power parameter, q , using the Sequential Quadratic Programming algorithm (SQP) resulting in the least Mean Square Error (MSE) of 10 different restarts.

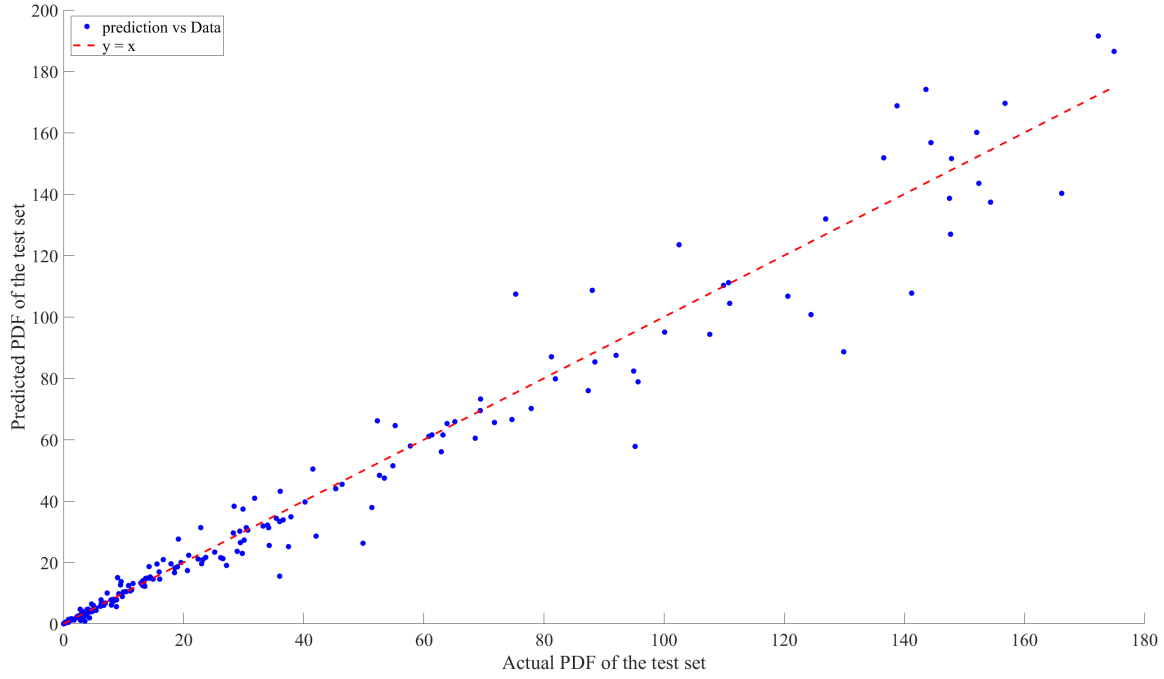


Figure 14. 2D conservative Duffing oscillator ($t_f = 7s$) in ZCA space: predicted PDF (vs) actual PDF at the test set locations after the initial parameters optimization ($Q_{test}^2 = 0.972413$).

The adaptive sampling procedure described in Section 2.3 is used to improve Q_{test}^2 . For this case, 7 iterations are conducted with each iteration having a maximum back propagation budget of 50 nodes while maintaining the same minimum distance tolerance in ZCA space of 0.01, as shown in Fig. 15. After the seventh iteration, the current DOE and the RBF parameters are subject to a final optimization process, shown in Fig. 16, which results in a very good quality PDF prediction, as shown in Fig. 17. As the number of iterations in the adaptive sampling increases, the Q_{test}^2 gets closer to one, as shown in Fig. 18. This also increases the back propagation load as a larger number of best candidates need to be back-propagated to obtain their true PDF values. Hence, a trade-off is required here, between the number of iterations and the desired PDF approximation quality indicated by Q_{test}^2 .

Table 3. 2D conservative Duffing oscillator: overview of optimization and adaptive sampling at $t_f = 7s$

Parameters of interest	Initial RBF Approximation	Final RBF Approximation
n_{RBF}	324	560
Minimum distance tolerance in ZCA space	0.01	0.01
Shape parameter 1, ϵ_1	0.8	0.5119
Shape parameter 2, ϵ_2	0.8	0.0986
Power parameter, q	0.5	-1.1597
Q_{test}^2	-0.0859	0.9970

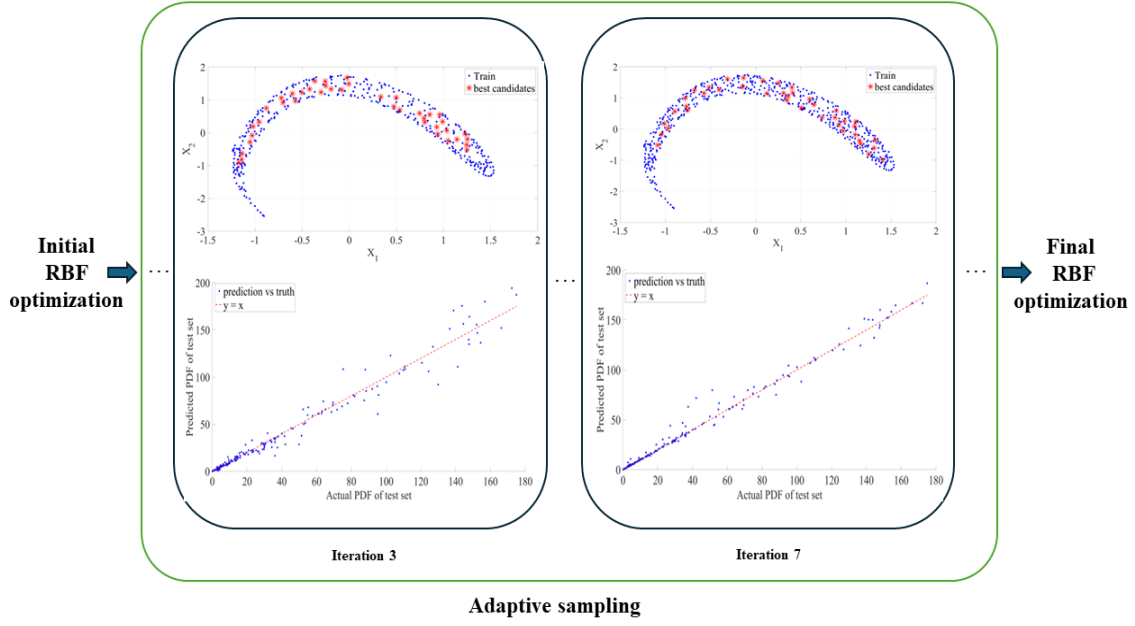


Figure 15. 2D conservative Duffing oscillator ($t_f = 7s$) in ZCA space: adaptive sampling.

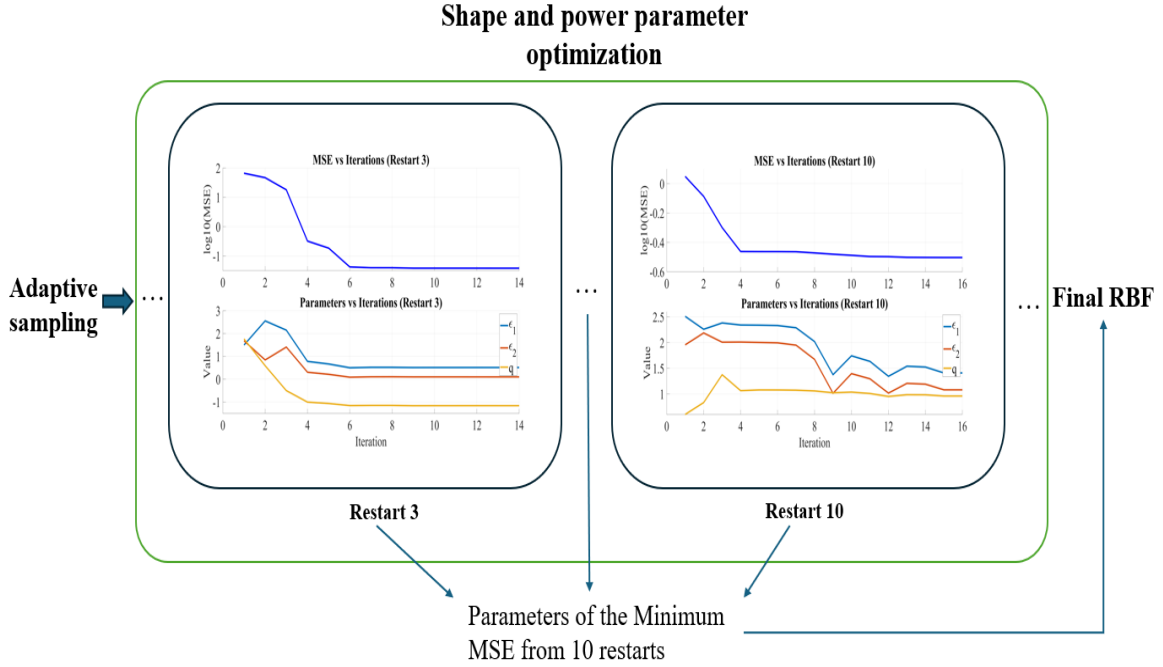


Figure 16. 2D conservative Duffing oscillator ($t_f = 7s$) in ZCA space: final optimization of the two shape parameters (ϵ_1 , ϵ_2) and one power parameter, q , using the Sequential Quadratic Programming algorithm (SQP) resulting in the least Mean Square Error (MSE) of 10 different restarts.

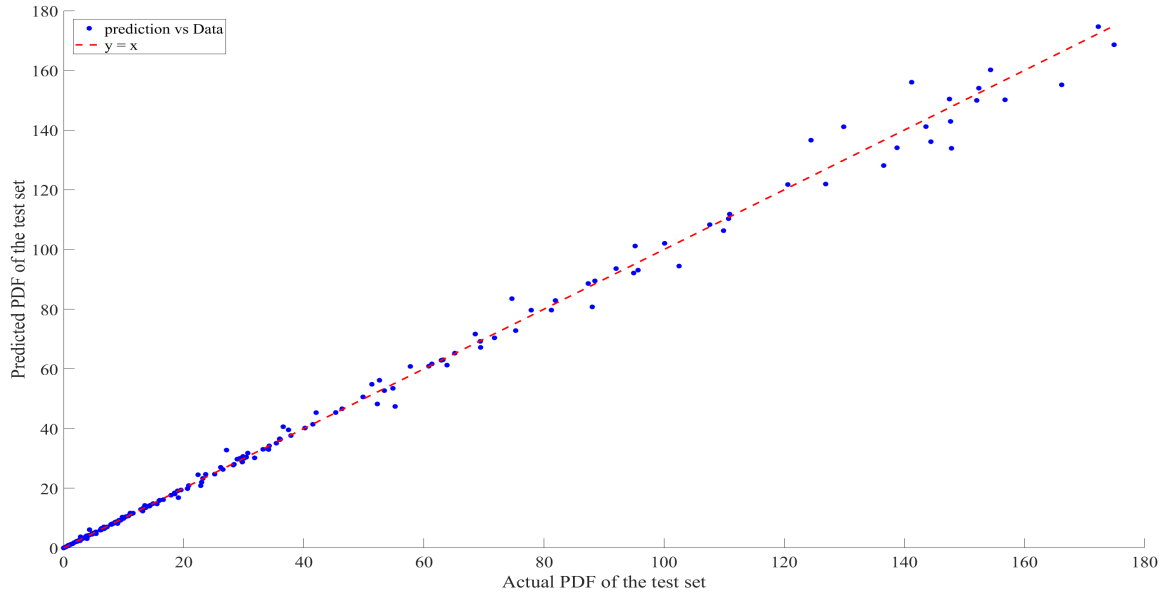


Figure 17. 2D conservative Duffing oscillator ($t_f = 7s$) in ZCA space: predicted PDF (vs) actual PDF at the test set locations after the final optimization ($Q_{test}^2 = 0.996999$).

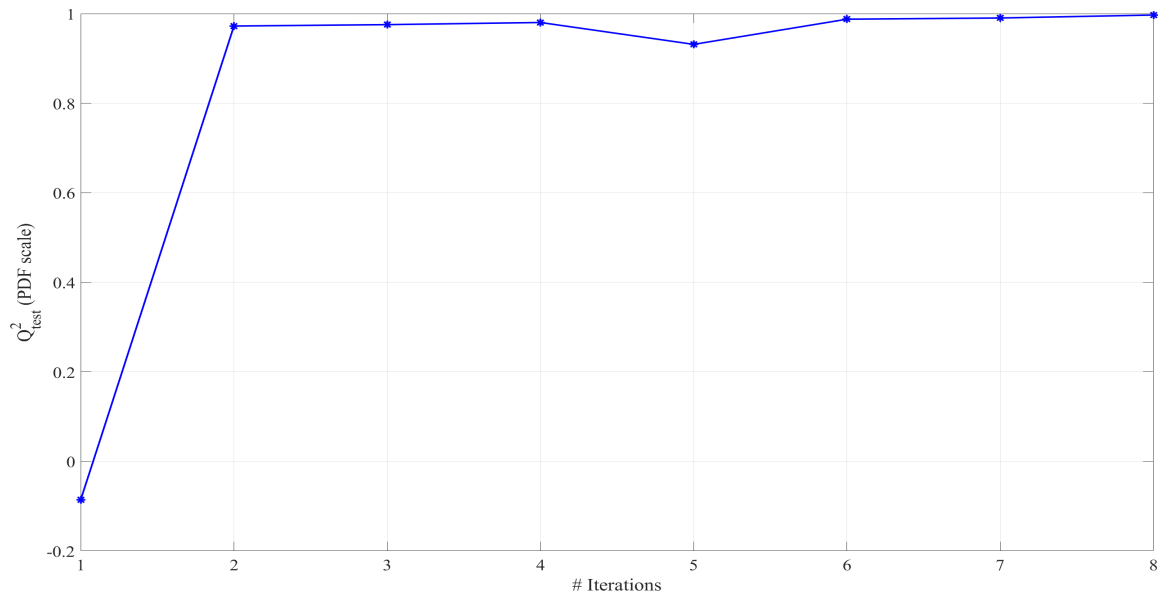


Figure 18. 2D conservative Duffing oscillator ($t_f = 7s$) in ZCA space: variation of the Q_{test}^2 during the optimization and adaptive sampling stages.

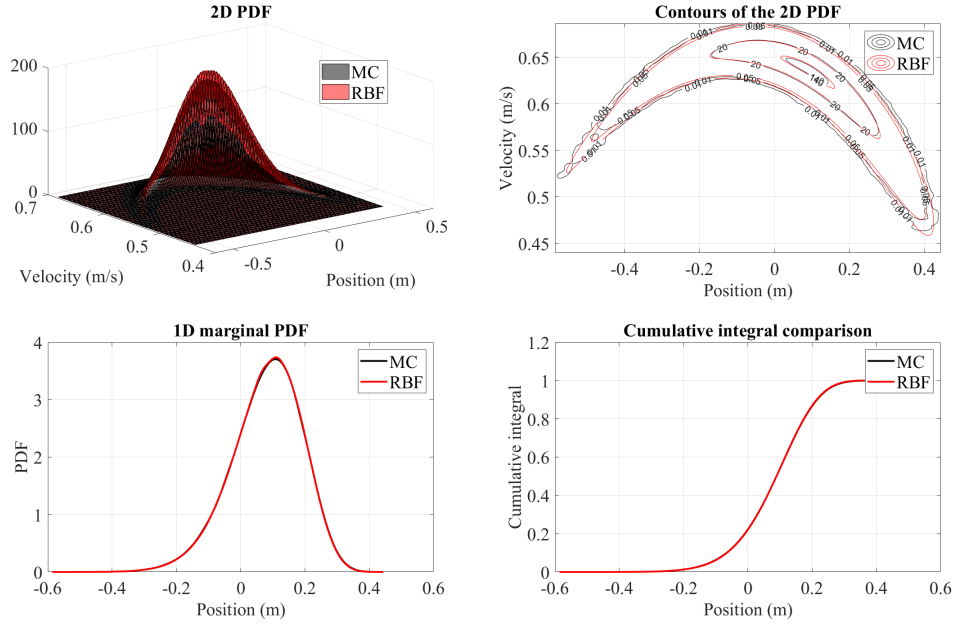


Figure 19. 2D conservative Duffing oscillator: interpolated PDF, 1D marginal PDF, and cumulative integral compared with a Monte-Carlo simulation involving 1 million samples.

To obtain the 1-D marginal PDF and the cumulative integral curve, a tensor grid of 120×120 resolution is created using the extremities of the final DOE in inertial space at t_f for PDF interpolation. This grid is transformed to ZCA space, and the good-quality final RBF approximant, with Q_{test}^2 approaching 1, is now used to predict the PDF values at the interpolating grid points within the ROI. For those grid points outside the ROI, the PDF value is assumed to be zero. The predicted PDF values are inverted from the logarithmic scale, and quadrature integration with the trapezoidal rule is implemented to yield the cumulative integral curve, as shown in Fig. 19. Comparison of the outcomes of the final RBF approximant with a one-million-sample Monte-Carlo (MC) simulation shows a good match in terms of the contours of the 2-D PDFs, the 1-D marginal PDF, and the cumulative integral. Table 3 shows that the final approximant uses about 560 RBF nodes to achieve the same accuracy as the one-million sample MC simulation.

3.2 Characterising the PDF of a satellite in a planar orbit

Characterizing the PDF of a planar orbit problem serves as an excellent application of the RBF approximation method to higher dimensions. For this problem, assume a satellite is in a near-circular orbit around Earth, subject to Newtonian gravity potential without perturbations. Under such conditions, the angular momentum of the orbit is conserved, which results in the orbit confined to a single plane in space. For this work, a planar orbit in the Low Earth Orbit (LEO) region is considered. The state vector and its time derivative for the planar orbit problem can be written as,

$$\begin{aligned} X &= [x \ y \ \dot{x} \ \dot{y}]^T \\ \dot{X} &= [\dot{x} \ \dot{y} \ \ddot{x} \ \ddot{y}]^T \end{aligned} \tag{12}$$

with its dynamics given by,

$$\begin{aligned}\ddot{x} &= -\frac{\mu_e}{r^3}x \\ \ddot{y} &= -\frac{\mu_e}{r^3}y \\ r &= \sqrt{x^2 + y^2}\end{aligned}\tag{13}$$

The probability density function (PDF) of the satellite in the planar orbit is a function of 4 random variables $[x \ y \ \dot{x} \ \dot{y}]^T$ at any given time. Tables 4 through 6 present the simulation parameters of this case. As this is a 4-dimensional system, the 6σ boundary from the initial nominal state is marked with 500 points that are created by following the procedure mentioned in Section 3.1 and the statistical parameters shown in Table 6. Compared to defining the boundary of a 2-dimensional PDF, it requires more points to define the boundary of a 4-dimensional PDF, and hence the 500 boundary points. A buffer boundary with 500 points is created at 7σ distance from the initial nominal state. Next, the ROI and the buffer region, which is the space beyond the ROI and within the buffer boundary, are populated with 850 RBF approximation nodes created using the Halton sequence. This initial number of approximation nodes, n_{RBF} , generally depends on the dimensionality of the problem. A closer look at Tables 3 and 7 shows the difference between the number of initial RBF approximation nodes required to fill the approximation domain of a 2-dimensional and a 4-dimensional system. The top two plots of Fig. 21 partially depict this scenario of the 4-dimensional system. It is important to note that a complete depiction of a 4-D point cloud using two-dimensional plots is quite challenging because of our inability to perceive 4-dimensional entities.

Table 4. Planar orbit problem: orbital elements of the LEO for UP with optimized RBFs

Orbit elements	Value	Unit
Semi-major axis (a)	7071.5×10^3	m
Apogee altitude (h_a)	700.0000×10^3	m
Eccentricity (e)	0.0010	(none)
Inclination (i)	0.0000	deg
Right Ascension of the Ascending Node (Ω)	(not applicable)	deg
Argument of perigee (ω)	(not applicable)	deg
Period of the orbit (T)	5.9180×10^3	s
Propagation time (t_f)	$10 * T$	s
Mahalanobis distance from nominal at t_0	6	(no unit)
RBF kernel type	$(1 + r_k^2)^q$	(no unit)

Next, the RBF approximation nodes, the boundary points, and the nominal state are propagated to 10 revolutions of the orbit using the *ode45* integration routine in MATLAB. The initial nominal state in Cartesian space is obtained using the orbital elements shown in Table 4. To ensure accuracy in numerical integration, the stepsize is kept as one second with the relative and absolute tolerances of the integration routine at 10^{-12} . Fig. 20 shows the nominal trajectory, the boundary points at t_0 (marked in green) and the boundary points at $t_f = 10 * T$ (marked in red). A close look at the zoomed-in versions shows how the initial position uncertainty in the order of a few kilometers expands to the final position uncertainty in the order of a few thousand kilometers in both the X and Y positions after 10 revolutions. Fig. 21 shows the 4-D point cloud of ROI nodes (i.e. the

Table 5. Other relevant parameters used in the planar orbit problem for UP with optimized RBF

Parameters	Value	Unit
Earth gravitational parameter (μ_e)	3.9860×10^{14}	m^3/s^2
Radius of the Earth (r_e)	6378.5600×10^3	m
Apogee radius (r_a)	7078.560×10^3	m
Apogee velocity (v_a)	7.5003×10^3	m/s

Table 6. Planar orbit problem: random variables at initial time (t_0) for UP with optimized RBFs.

State variable	Mean value (μ)	Standard deviation (σ)
x (m)	$-r_a$	30.00
y (m)	0	30.00
$\dot{x}$ (m/s)	0	0.03
$\dot{y}$ (m/s)	$-v_a$	0.03

Table 7. Planar orbit problem: overview of optimization and adaptive sampling at $t_f = 10 * T$.

Parameters of interest	Initial RBF Approximation	Final RBF Approximation
n_RBF	850	951
Minimum distance tolerance in ZCA space	0.07	0.07
Shape parameter 1, ϵ_1	0.8	1.4051
Shape parameter 2, ϵ_2	0.8	3.8074
Shape parameter 3, ϵ_3	0.8	2.1155
Shape parameter 4, ϵ_4	0.8	2.6730
Power parameter, q	0.5	3.4292
Q_{test}^2	-0.1294	0.9997

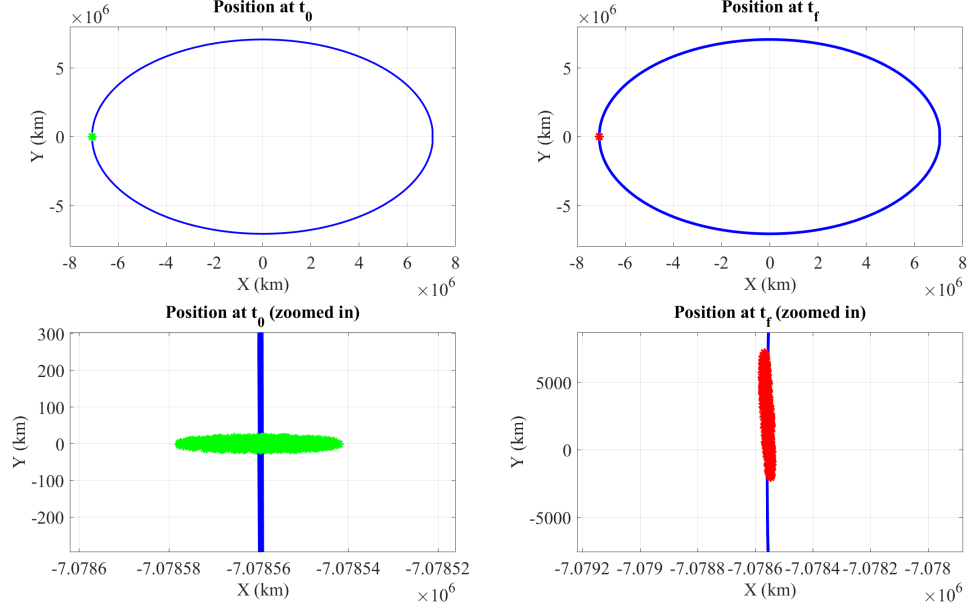


Figure 20. Planar orbit: Nominal trajectory (blue) and the 6σ boundary points (green at t_0 and red at t_f) propagated to $t_f = 59180.00$ s ($= 10.00 \cdot T$).

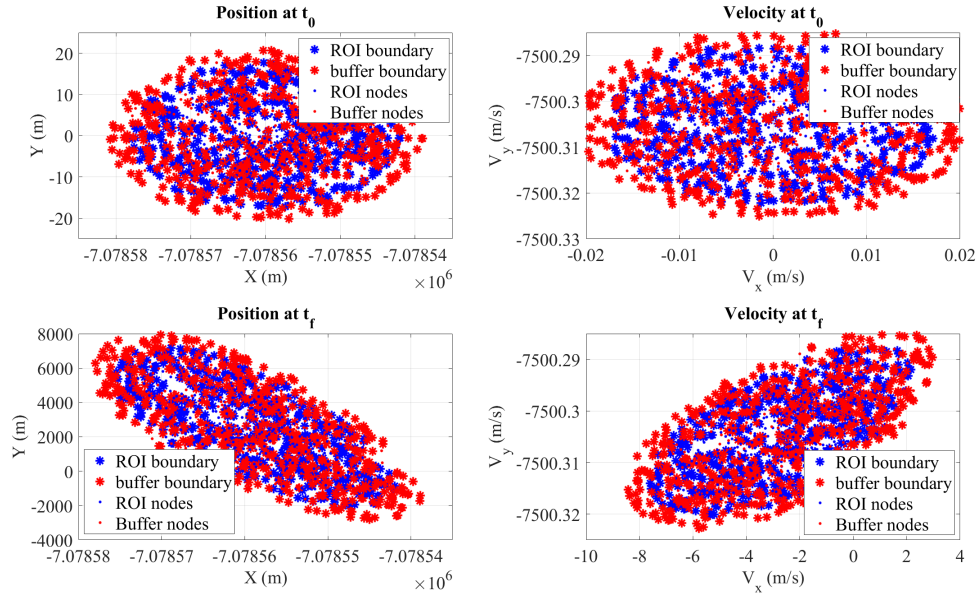


Figure 21. Planar orbit: Uncertainty bounds filled with RBF nodes at t_0 and $t_f = 10 \cdot T$. Points marking the boundary of the Region of Interest (ROI) and buffer region are also shown.

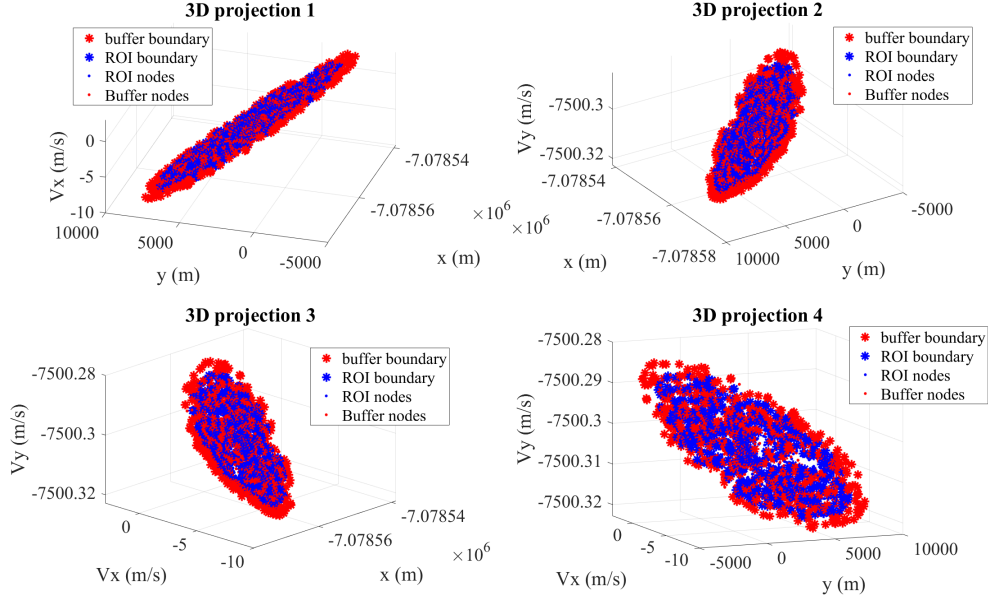


Figure 22. Planar orbit ($t_f = 10 * T$): the 4-dimensional RBF nodes and boundary points are shown in four different 3-D projections.

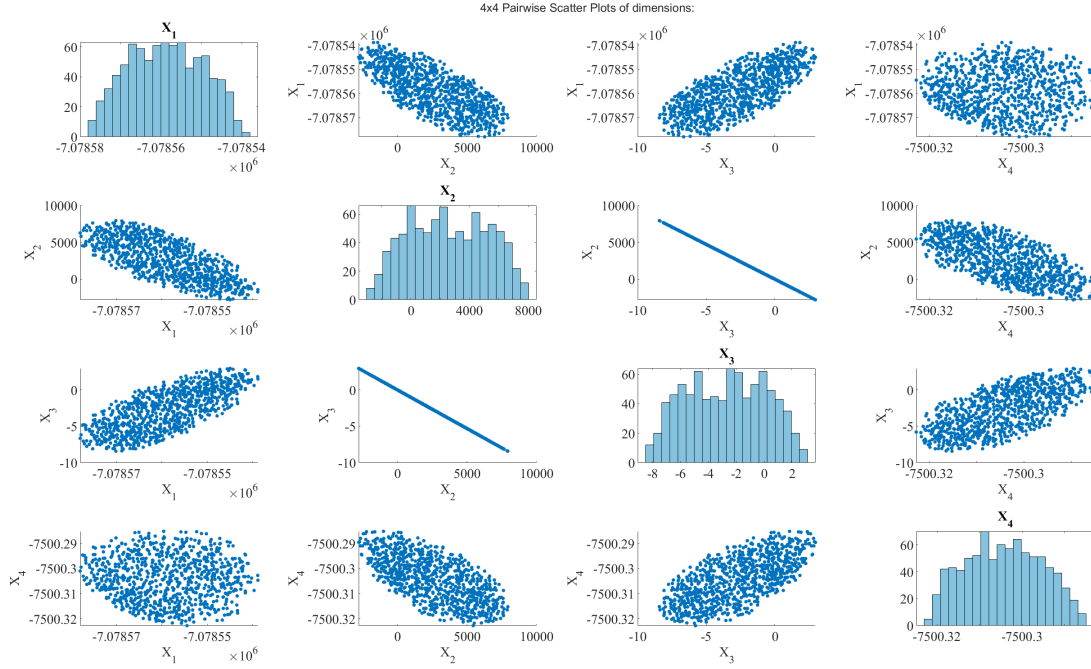


Figure 23. Planar orbit ($t_f = 10 * T$): each dimension of the 4-dimensional RBF nodes in inertial space is plotted against every other dimension in 2D scatter plots ($[X_1 \ X_2 \ X_3 \ X_4] = [X \ Y \ \dot{X} \ \dot{Y}]$).

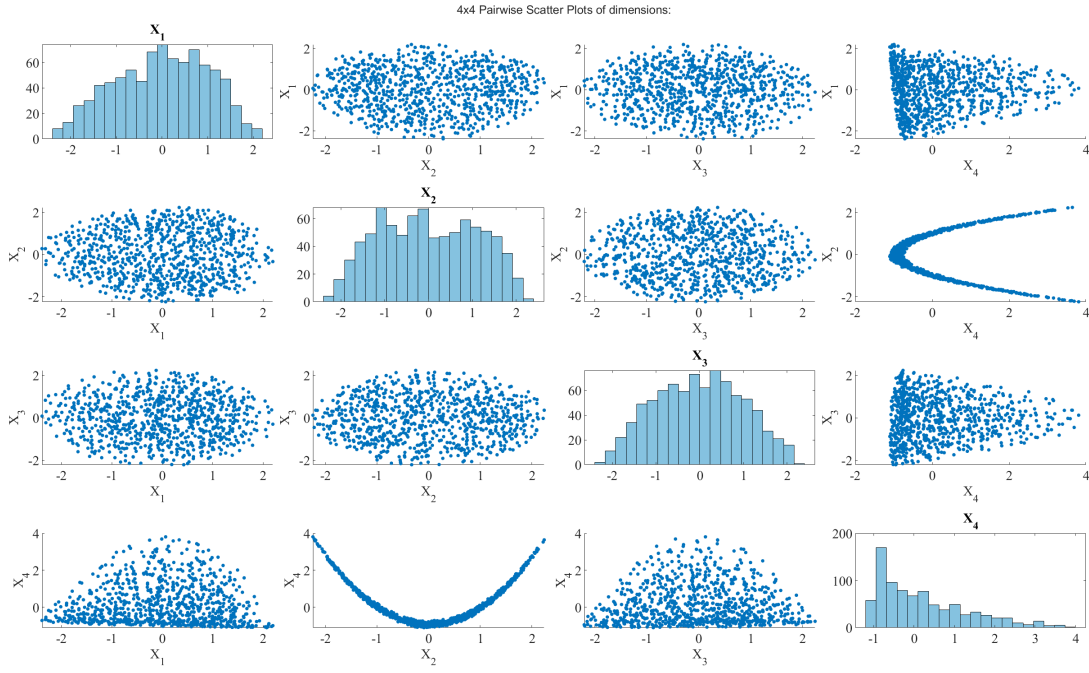


Figure 24. Planar orbit ($t_f = 10 * T$): each dimension of the 4-dimensional RBF nodes in ZCA space is plotted against every other dimension in 2D scatter plots.

approximation nodes) and the boundary points at t_f in the bottom two plots. Not only has the final uncertainty expanded in both position and velocity dimensions, but it is no longer Gaussian after the satellite is propagated for 10 revolutions. This is illustrated in Fig. 22, which shows the four different 3-D projections of the 4-D point cloud at t_f . The point cloud is spread out in thin leaf-like structures that are bent and have different spatial orientations depending on the projection. The objective here is to approximate the multivariate final PDF, which is a function of this complex 4-D point cloud. To accomplish this, the approximation nodes are transformed from the inertial space to ZCA space using Eq. (5), and the minimum distance tolerance of 0.07 is used to remove the approximation nodes that lie too close to each other to ensure numerical stability of the collocation process during the approximation process. The advantage of the ZCA transformation can be seen in the pairwise scatter plots of the dimensions of the point cloud shown in Fig. 24 when compared to a similar plot in inertial space, Fig. 23. A key difference is the range of the random variables. In inertial space, there is a numerical difference of three orders of magnitude between the position ($[X \ Y]$) and velocity ($[\dot{X} \ \dot{Y}]$) random variables. However, in the ZCA space, all four random variables ($[X_1 \ X_2 \ X_3 \ X_4]$) vary between -2 and 4, which aids the approximation process. A crucial point to note is that the transformation from inertial to ZCA space is bijective, but the random variables in ZCA space don't have the physical dimensions of position and velocity. For the shape and power parameter optimization, a test set comprised of state-space points and their associated PDF values is required. Similar to the process of creating, propagating, and transforming the approximation nodes (i.e., the training nodes or the Design of Experiment (DOE)), a separate test set of 1600 points is created, propagated, and transformed to the ZCA space, shown in Fig. 25, along with their PDF values.

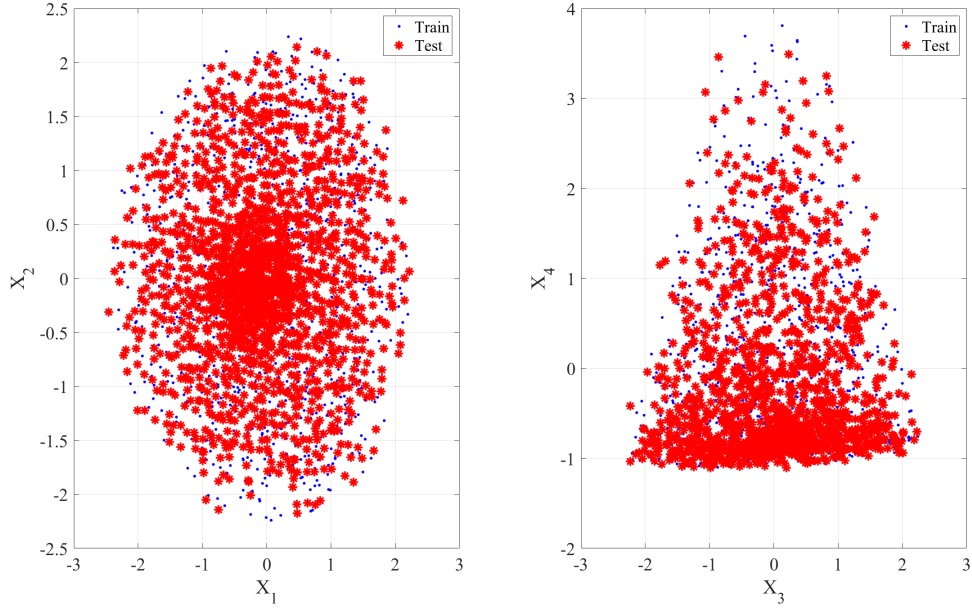


Figure 25. Planar orbit ($t_f = 10 * T$) in ZCA space: train set (RBF nodes) after implementing minimum distance filtering and the test set.

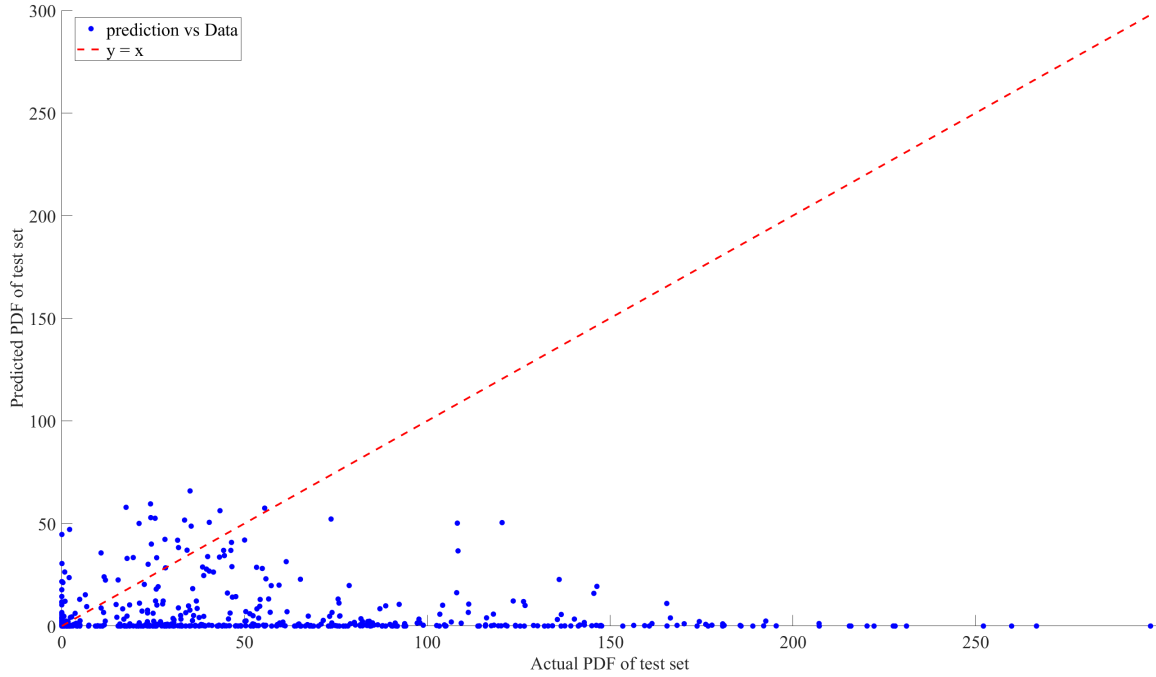


Figure 26. Planar orbit ($t_f = 10 * T$) in ZCA space: predicted PDF (vs) actual PDF at the test set locations after initial RBF approximation ($Q_{test}^2 = -0.129401$).

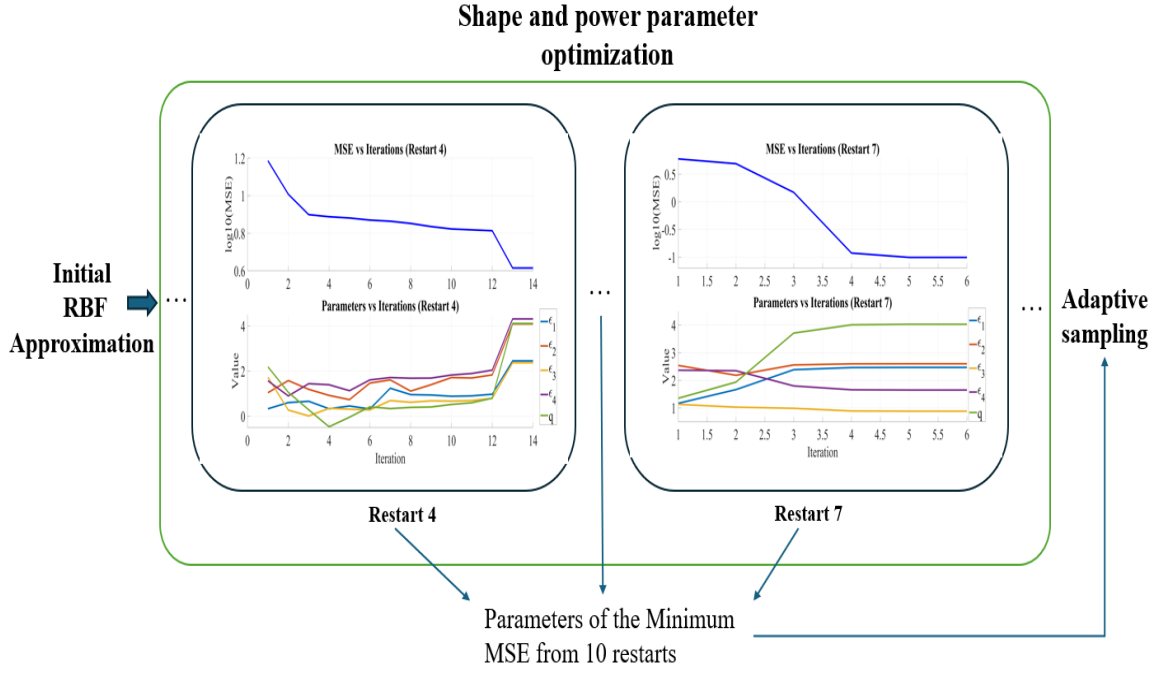


Figure 27. Planar orbit ($t_f = 10 * T$) in ZCA space: initial optimization of the four shape parameters ($\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$) and one power parameter, q , using the Sequential Quadratic Programming algorithm (SQP) resulting in the least Mean Square Error (MSE) of 10 different restarts.

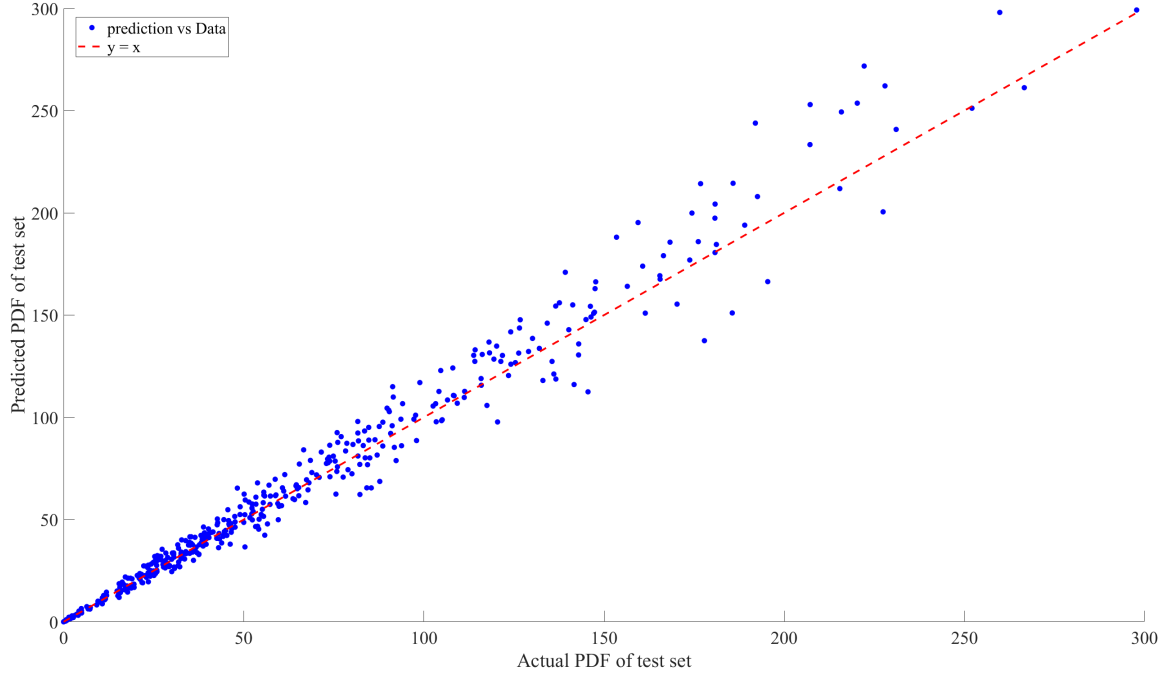


Figure 28. Planar orbit ($t_f = 10 * T$) in ZCA space: predicted PDF (vs) actual PDF at the test set locations after the initial parameter optimization ($Q_{test}^2 = 0.982903$).

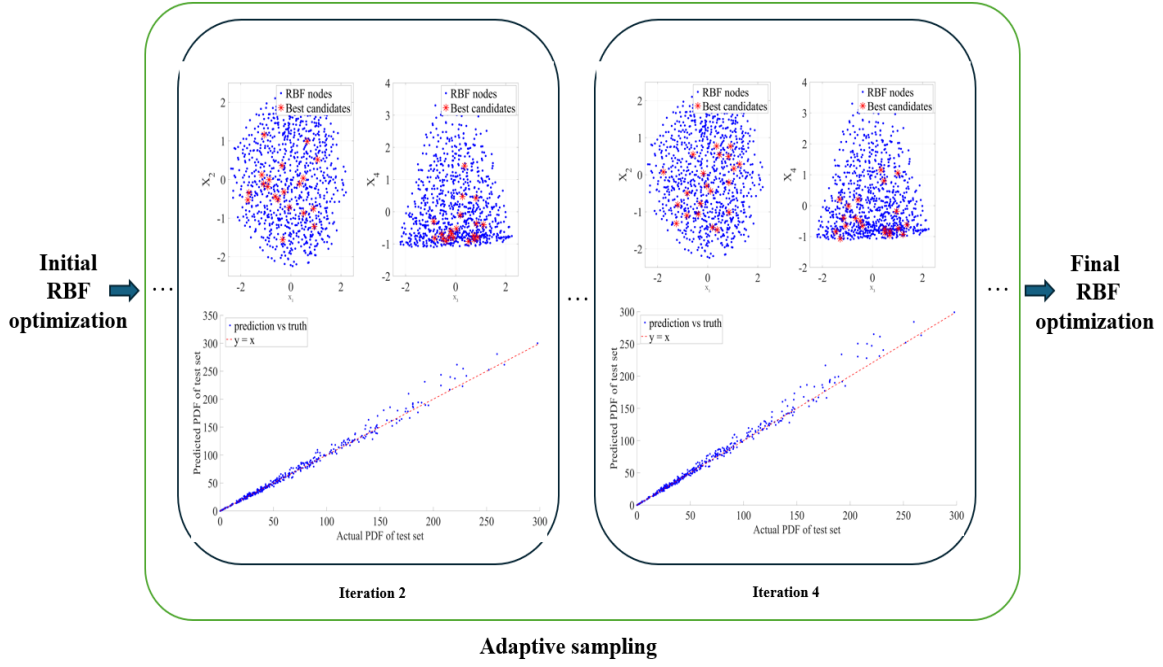


Figure 29. Planar orbit ($t_f = 10 * T$) in ZCA space: adaptive sampling.

The shape and power parameters, the initial set of training nodes presented in Table 7 and their referenced PDF values are used to create an initial RBF approximation of the PDF at t_f as described in Eq. (3) and Eq. (6). This initial RBF approximant is generally of poor quality. At the test set locations, this initial RBF approximant is used to predict the PDF values as described in Eq. (7). The Q_{test}^2 metric and the associated prediction are shown in Fig. 26. With a poor PDF prediction quality indicated by $Q_{test}^2 = -0.129401$, the plot also shows that the initial RBF approximant performs poorly in the regions with high PDF values. In the next stage, the initial RBF approximant is improved by optimizing the four shape parameters ($\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$) and the power parameter, q , according to the procedure mentioned in Section 2.2. As the optimization process using the Sequential Quadratic Programming (SQP) algorithm modifies the shape and power parameters without modifying the set of training nodes (i.e., the approximation nodes or the DOE), shown in Fig. 27, it results in a considerable improvement of Q_{test}^2 from -0.129401 to 0.982903, Fig. 28.

Further improvement in Q_{test}^2 is made possible by implementing the adaptive sampling procedure described in Section 2.3, which enhances the DOE. The adaptive sampling is repeated for 5 iterations, with each iteration having a maximum back propagation budget of 50 nodes while maintaining the same minimum distance tolerance in ZCA space of 0.07. The Region of Interest (ROI) gradually becomes dense with more and more best candidates that become part of the approximation nodes, Fig. 29. The locations of the best candidate nodes are based on the locations of the DOE at the current iteration. The PDF prediction quality metric, Q_{test}^2 , also increases. After the fifth iteration, the current DOE and the RBF parameters are subject to a final optimization process, as shown in Fig. 30, resulting in a very good quality PDF prediction, as shown in Fig. 31. The history of the variation of Q_{test}^2 from the initial RBF approximation to the final RBF approximation is shown in Fig. 32. As mentioned in Section 3.1, a trade-off is required between the number of iterations in the

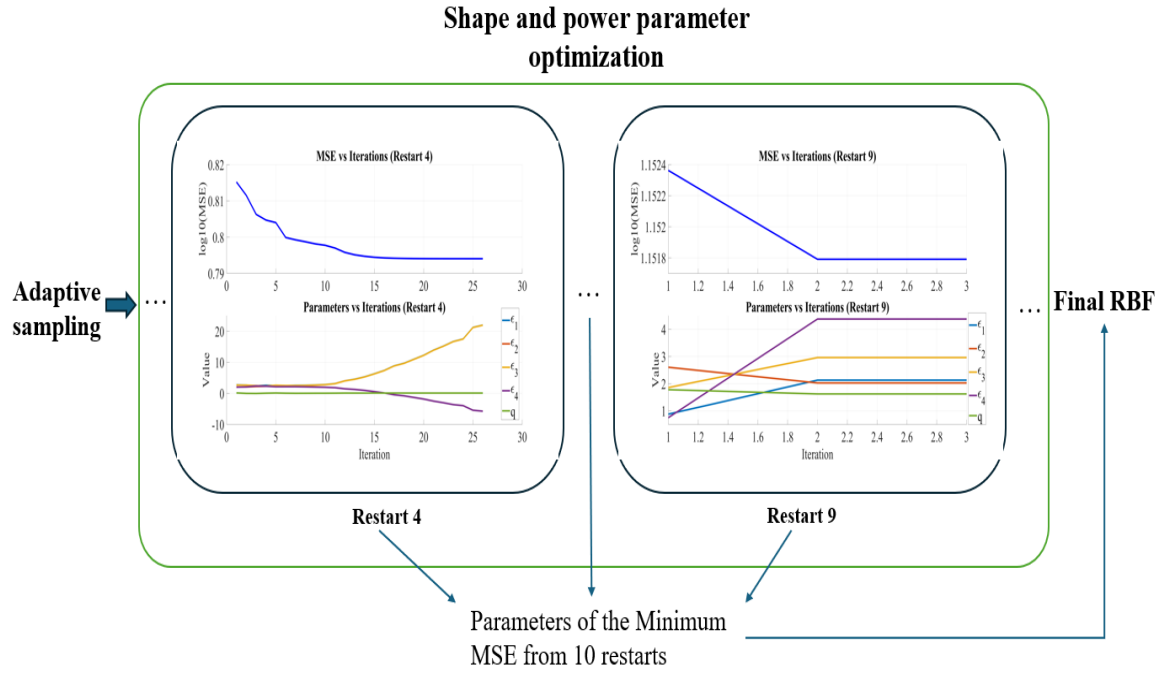


Figure 30. Planar orbit ($t_f = 10 * T$) in ZCA space: final optimization of the four shape parameters ($\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$) and one power parameter, q , using the Sequential Quadratic Programming algorithm (SQP) resulting in the least Mean Square Error (MSE) of 10 different restarts.

adaptive sampling process and the computational power required for back propagation, especially when long periods such as 10 revolutions are involved.

To obtain the lower-dimensional marginal PDFs and the cumulative integral curve, a 4-D tensor grid of $120 \times 120 \times 120 \times 120$ resolution was created using the extremities of the final DOE (i.e., the final set of RBF approximation nodes) in inertial space for PDF interpolation. When this grid was transformed to the ZCA space, it was found that hardly 50 grid points were actually inside the ROI due to the intricate and slender shape of the 4-D approximation point cloud. Since this was not sufficient, the task of computing the cumulative integral has been reserved for future work.

4 DISCUSSION

The proposed method with optimized RBFs proves to be a valuable tool to propagate uncertainty in systems like the conservative Duffing oscillator and the planar orbit. Both systems showed considerable improvement in Q_{test}^2 metric when the process of shape and power parameter optimization was implemented along with the adaptive sampling procedure. The objective of effective long-term uncertainty propagation with fewer samples is accomplished in the planar orbit problem. With almost 950 approximation nodes/points, the UP with optimized RBFs achieves a good quality PDF prediction, as evidenced by $Q_{test}^2 = 0.9997$. Between Fig. 26 and Fig. 31, the PDF prediction in the low probability region improves considerably, indicated by the close alignment with the line $y = x$. To improve the PDF approximation quality in a certain subspace of the ROI, the adaptive sampling procedure can be tuned to focus on this subspace, thereby guiding the best candidate locations to be within this subspace. The computational efficiency of UP with optimized RBFs is

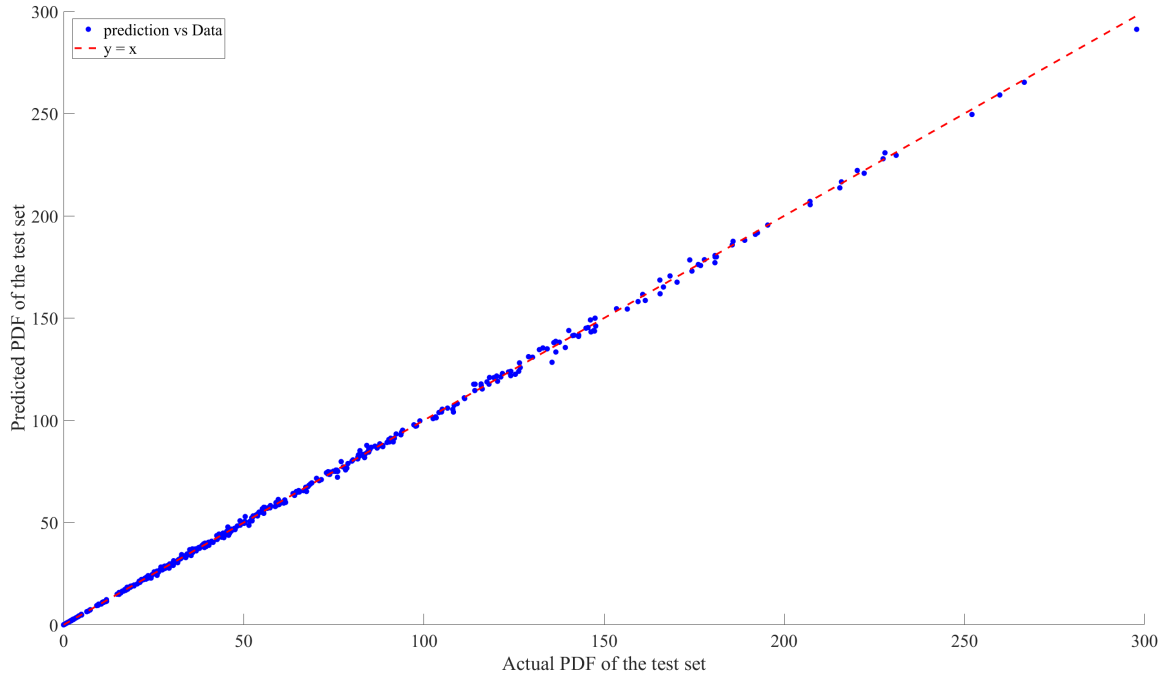


Figure 31. Planar orbit ($t_f = 10 * T$) in ZCA space: predicted PDF (vs) actual PDF at the test set locations after the final optimization ($Q_{test}^2 = 0.999749$).

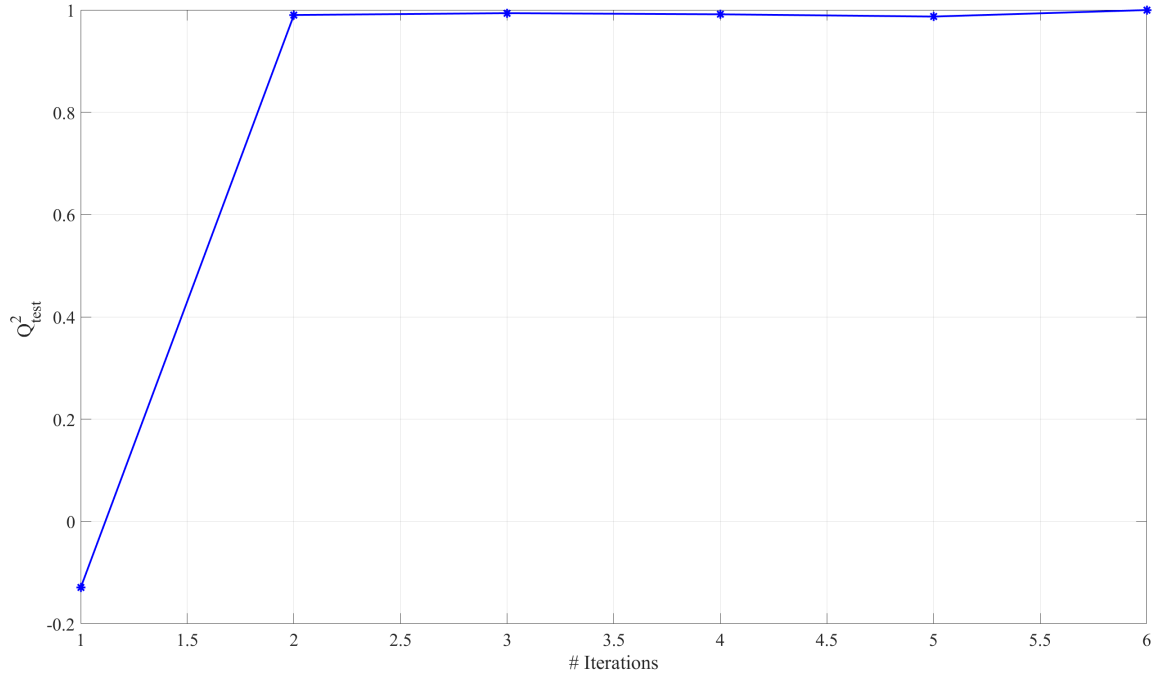


Figure 32. Planar orbit ($t_f = 10 * T$) in ZCA space: variation of the Q_{test}^2 during the optimization and adaptive sampling stages.

assessed in terms of the number of approximation nodes rather than the computation time, as it can vary depending on the hardware architecture and the period of uncertainty propagation. The overall computational efficiency can be increased by improving the optimization process to quickly narrow down the search space of parameters in the subsequent iterations, or by using a different optimization algorithm altogether.

5 CONCLUSION

The strategy of using optimized radial basis functions with adaptive sampling is combined with Liouville's theorem to propagate uncertainty in nonlinear dynamical systems. The basic methodology of RBF approximation is explained using a 1-dimensional PDF approximation example, followed by the modified RBF formulation that lends itself to shape parameter optimization per dimension as well as the power parameter. The adaptive sampling strategy that uses the existing set of approximation nodes to selectively refine the approximation domain is then enunciated. The proposed method is applied to propagate the state uncertainty in two nonlinear dynamical systems: the Duffing oscillator and the planar orbit problem. The cumulative integral and the Q_{test}^2 score are used as metrics to test the quality of the PDF approximation. In both systems, the proposed method achieves good-quality PDF approximations, indicated by the Q_{test}^2 score, which approaches 1, by optimally filling the approximation domain with RBF nodes. Future works can explore the possibility of using different RBF kernels for UP with optimized RBFs, as well as the use of Monte-Carlo integration techniques to compute the cumulative integral at the final time. Of more interest to the astrodynamics community is the probability of collision estimation, which can be done by sampling from the optimized RBF-based PDF approximant. Extending the UP with optimized RBFs for higher-dimensional and more complex problems, such as the uncertainty propagation of the state of a satellite in a full 3D orbit with large eccentricity, requires the method to incorporate a mixture of experts. This will involve several local RBF approximants combined to provide a good quality global approximation of the PDF.

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