

CISLUNAR ASTRODYNAMICS LEVERAGING GENERALIZED EQUINOCTIAL ORBITAL ELEMENTS

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The cislunar dynamical environment is strongly influenced by the gravity of both the Earth and the Moon. The resulting nonlinearities and complex dynamics pose challenges for spacecraft navigation and Space Domain Awareness operations in this regime, where the knowledge of current and future spacecraft states is essential. As such, the Generalized Equinoctial Orbital Elements (GEqOEs) are investigated as an alternative orbital parameterization that are embedded with perturbations arising from conservative forces. Cislunar propagation leveraging lunar gravity-perturbed GEqOEs is assessed against various orbits obtained via the Earth-Moon Circular Restricted Three-Body Problem. For each orbit, the GEqOE solution demonstrates agreement with conventional Cartesian propagation.

INTRODUCTION

The cislunar dynamical environment is characterized by highly nonlinear and chaotic behavior. In addition to the dominant gravitational influence of the Earth, the third-body effects imparted by the gravity of the Moon significantly perturb the trajectory of any spacecraft operating in this regime. The complexities of this space are especially pronounced for spacecraft navigation and cislunar Space Domain Awareness applications, where the accurate knowledge of current and future spacecraft states is sought. While the osculating values of Keplerian orbital elements may be determined at discrete time steps for cislunar trajectories, relevant perturbing accelerations must be incorporated into the equations of motion that govern the Keplerian elements, an approach that is not commonly employed. As such, element sets that are directly embedded with relevant orbital perturbations have been explored. Offshoots of Equinoctial Orbital Elements (EqOEs), such as the J_2 EqOEs and the oblate spheroidal EqOEs provide options for element sets that accommodate J_2 effects near the vicinity of the Earth.^{1,2} For modeling cislunar astrodynamics, however, there is a need for an orbital element set that encapsulates the lunar gravity.

This work considers another offshoot of EqOEs, proposed by Baù et al. and termed the Generalized Equinoctial Orbital Elements (GEqOEs), for cislunar astrodynamics with the addition of lunar gravity effects.³ The GEqOE set is characterized by the total energy of the system and, as such, may be utilized to directly embed any perturbations arising from conservative forces directly into the orbital elements. For near-Earth orbits, Hernando-Ayuso et al. and McGee leverage GEqOEs for accurate state and uncertainty propagation under the addition of J_2 and third-body gravity effects.^{4,5} In the current work, the GEqOE set is employed for propagating cislunar orbits that are significantly impacted by lunar gravity. Inherently non-Keplerian orbits that are conventionally associated with three-body dynamical models, such as orbits from the Earth-Moon Circular Restricted Three-Body

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Problem (CR3BP), are examined and propagated using the GEqOEs. The propagation performance of the GEqOEs is evaluated for various orbits that span the cislunar domain.

GENERALIZED ORBITAL MOTION

Consider a spacecraft of mass m orbiting the Earth or the Moon. In the associated body-centered inertial frame, the perturbed two-body motion of the spacecraft is represented as

$$\ddot{\mathbf{r}} + \frac{\mu_{CB}\mathbf{r}}{r^3} = \mathbf{a}_p(\mathbf{r}, \dot{\mathbf{r}}, t), \quad (1)$$

where $\mathbf{r}$ represents the inertial position vector of the spacecraft, $\dot{\mathbf{r}}$ is the velocity, and $\ddot{\mathbf{r}}$ is the spacecraft acceleration. The vector $\mathbf{a}_p$ represents any perturbing accelerations acting on the spacecraft. The variables r and t denote the magnitude of the spacecraft position and time respectively. The parameter μ_{CB} represents the gravitational parameter for the central body (either the Earth or the Moon), where the mass of the spacecraft is assumed to be infinitesimal relative to the mass of the central body.

The origin of the inertial reference frame, denoted O , is centered on the celestial body and is defined as

$$\Sigma = \{O; \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}. \quad (2)$$

Here, $\mathbf{e}_x = [1, 0, 0]^T$, $\mathbf{e}_y = [0, 1, 0]^T$, and $\mathbf{e}_z = [0, 0, 1]^T$. Additionally, the orbital reference frame, denoted Σ_{or} , is defined by the orthonormal basis,

$$\Sigma_{or} = \{O; \mathbf{e}_r, \mathbf{e}_f, \mathbf{e}_h\}, \quad (3)$$

where $\mathbf{e}_r$ points along the spacecraft position vector, $\mathbf{e}_h$ is directed along the angular momentum vector, and $\mathbf{e}_f$ completes the dextral orthonormal triad

$$\mathbf{e}_r = \frac{\mathbf{r}}{r}, \quad \mathbf{e}_f = \mathbf{e}_h \times \mathbf{e}_r, \quad \mathbf{e}_h = \frac{\mathbf{r} \times \dot{\mathbf{r}}}{|\mathbf{r} \times \dot{\mathbf{r}}|} = \frac{\mathbf{h}}{h}. \quad (4)$$

The perturbing accelerations in Equation (1) arise from a total perturbing force, denoted $\mathbf{F}$, that can be written as a sum of forces that are derived from a potential energy, such as the gravitational acceleration from additional celestial bodies, and forces that are not derived from a potential, such as atmospheric drag and solar radiation pressure. Mathematically, the total perturbing force is then represented as

$$\mathbf{F} = \mathbf{P} - \nabla U, \quad (5)$$

where $\mathbf{P}$ represents the perturbing forces that do not arise from a potential, and $-\nabla U$ models the contribution of forces derived from a potential energy, U . The total orbital energy of the spacecraft, $\mathcal{E}$, is a sum of its Keplerian energy, $\mathcal{E}_K$, and this potential energy, such that

$$\mathcal{E} = \mathcal{E}_K + U. \quad (6)$$

Expressing the spacecraft velocity vector, $\dot{\mathbf{r}}$, in the orbital frame as

$$\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + \frac{h}{r}\mathbf{e}_h, \quad (7)$$

the total orbital energy may be represented as

$$\mathcal{E} = \frac{\dot{r}^2}{2} + \frac{h^2}{2r^2} - \frac{\mu_{CB}}{r} + U, \quad (8)$$

where $\dot{r}$ denotes the radial component of the velocity, and h is the magnitude of the angular momentum of the spacecraft. Then, the effective potential energy, denoted U_{eff} , is defined as

$$U_{\text{eff}} = \frac{h^2}{2r^2} + U. \quad (9)$$

Thus, the total energy of the spacecraft is

$$\mathcal{E} = \frac{\dot{r}^2}{2} - \frac{\mu_{CB}}{r} + U_{\text{eff}}. \quad (10)$$

Using the effective potential energy, the *generalized* angular momentum of the spacecraft, denoted $\tilde{h}$, is introduced,

$$\tilde{h} = \sqrt{2r^2 U_{\text{eff}}} \quad (11)$$

which defines the *generalized* velocity vector

$$\tilde{\mathbf{v}} = \dot{r}\mathbf{e}_r + \frac{\tilde{h}}{r}\mathbf{e}_h. \quad (12)$$

Parameters denoted by tildes represent generalized quantities that are defined by the total orbital energy, $\mathcal{E}$, and are, thus, embedded with perturbations acting on the spacecraft. Together, the vectors $(\mathbf{r}, \tilde{\mathbf{v}})$ characterize a non-osculating ellipse with one focus located at the center of the Earth and perturbed by additional forces. Although the preceding formulation is provided in the context of orbits centered on the Earth or the Moon, the equations are applicable to orbits centered on other celestial bodies as well.

Assuming that the total energy is negative, the generalized analogs for the semi-major axis, eccentricity, and the semi-latus rectum are evaluated as

$$\tilde{a} = -\frac{\mu_{CB}}{2\mathcal{E}} \quad (13)$$

$$\tilde{e} = \frac{\sqrt{\mu_{CB}^2 + 2\mathcal{E}\tilde{h}^2}}{\mu_{CB}} \quad (14)$$

$$\tilde{p} = \tilde{a}(1 - \tilde{e}^2). \quad (15)$$

The generalized eccentricity vector, $\tilde{\mathbf{e}}$, orients the ellipse in the orbital plane and is given as

$$\mu_{CB}\tilde{\mathbf{e}} = \tilde{\mathbf{v}} \times (\mathbf{r} \times \tilde{\mathbf{v}}) - \mu_{CB}\mathbf{e}_r. \quad (16)$$

The generalized true anomaly, $\tilde{\theta}$, defines the angle between the $\tilde{\mathbf{e}}$ and $\mathbf{r}$ vectors and satisfies the relations

$$\tilde{e} \cos \tilde{\theta} = \frac{\tilde{h}^2}{\mu_{CB}r} - 1 \quad \text{and} \quad \tilde{e} \sin \tilde{\theta} = \frac{\tilde{h}\dot{r}}{\mu_{CB}}. \quad (17)$$

Other angular relationships that parameterize the non-osculating ellipse include the classical true longitude

$$L = \omega + \Omega + \theta , \quad (18)$$

where θ is the classical true anomaly, and ω and Ω denote the argument of periapsis and the right ascension of ascending node, respectively. The angle Ψ denotes the generalized longitude of periapsis and is determined as

$$\Psi = L - \tilde{\theta} = \omega + \Omega + \theta - \tilde{\theta} . \quad (19)$$

In the absence of orbital perturbations ($U = 0$), the generalized and classical true anomalies are equal, and $\Psi = \omega + \Omega$.

Generalized Equinoctial Orbital Elements

Using the generalized quantities that characterize the orbital motion and the angular relationships, the Generalized Equinoctial Orbital Elements (GEqOEs) are defined as

$$\{\tilde{n}, p_1, p_2, q_1, q_2, \tilde{\lambda}\} . \quad (20)$$

The first element denotes the generalized mean motion and is expressed as

$$\tilde{n} = \frac{1}{\mu_{CB}}(-2\mathcal{E})^{3/2} . \quad (21)$$

The second and third elements parameterize the eccentricity vector and are defined

$$p_1 = \tilde{e} \sin \Psi \quad (22)$$

$$p_2 = \tilde{e} \cos \Psi . \quad (23)$$

The elements q_1 and q_2 orient the equinoctial reference frame relative to the inertial reference frame. These elements are a function of the classical inclination, i , and the classical right ascension of the ascending node, Ω , expressed as

$$q_1 = \tan \frac{i}{2} \sin \Omega \quad (24)$$

$$q_2 = \tan \frac{i}{2} \cos \Omega . \quad (25)$$

Finally, $\tilde{\lambda}$ represents the generalized mean longitude and is a function of the generalized mean anomaly, $\tilde{M}$, and the generalized longitude of periapsis, Ψ :

$$\tilde{\lambda} = \tilde{M} + \Psi , \quad (26)$$

where $\tilde{M}$ is determined by solving Kepler's equation for the generalized quantities

$$\tilde{M} = \tilde{E} - \tilde{e} \sin \tilde{E} . \quad (27)$$

Define the generalized eccentric longitude as

$$\tilde{F} = \tilde{E} + \Psi . \quad (28)$$

Substituting Equations (27) and (28) into Equation (26) yields

$$\begin{aligned}\tilde{\lambda} &= \tilde{E} - \tilde{e} \sin(\tilde{F} - \Psi) + (\tilde{F} - \tilde{E}) \\ &= \tilde{F} - \tilde{e}(\sin \tilde{F} \cos \Psi - \cos \tilde{F} \sin \Psi) \\ &= \tilde{F} + p_1 \cos \tilde{F} - p_2 \sin \tilde{F},\end{aligned}\tag{29}$$

where the definitions of the elements p_1 and p_2 via Equations (22) and (23) provide the relevant substitutions.

Transformation from Cartesian Coordinates to the GEqOE Set

Let $\mathbf{x}_C = [\mathbf{r} \ \dot{\mathbf{r}}]^T$ represent the Cartesian state vector comprised of the position and velocity of the spacecraft. The transformation from Cartesian into the GEqOE coordinates is summarized below:

- To obtain the first element, $\tilde{n}$, the total energy of the system is found via Equation (6) and substituted directly into Equation (21).
- The fourth and fifth generalized elements, q_1 and q_2 , are determined as

$$q_1 = \frac{\mathbf{e}_h \cdot \mathbf{e}_x}{1 + \mathbf{e}_h \cdot \mathbf{e}_z} \quad \text{and} \quad q_2 = -\frac{\mathbf{e}_h \cdot \mathbf{e}_y}{1 + \mathbf{e}_h \cdot \mathbf{e}_z}.\tag{30}$$

- The second and third elements, p_1 and p_2 , are determined by first obtaining the unit vectors that define the equinoctial reference frame, $\Sigma_{eq} = \{O; \mathbf{e}_X, \mathbf{e}_Y, \mathbf{e}_Z\}$:

$$\begin{aligned}\mathbf{e}_X &= \frac{1}{1 + q_1^2 + q_2^2} [1 - q_1^2 + q_2^2, \ 2q_1q_2, \ -2q_1]^T \\ \mathbf{e}_Y &= \frac{1}{1 + q_1^2 + q_2^2} [2q_1q_2, \ 1 + q_1^2 - q_2^2, \ 2q_2]^T \\ \mathbf{e}_Z &= \frac{1}{1 + q_1^2 + q_2^2} [2q_1, \ -2q_2, \ 1 - q_1^2 - q_2^2]^T.\end{aligned}\tag{31}$$

The generalized eccentricity vector is then computed via Equation (16), supplying p_1 and p_2 as

$$p_1 = \tilde{\mathbf{e}} \cdot \mathbf{e}_Y \quad \text{and} \quad p_2 = \tilde{\mathbf{e}} \cdot \mathbf{e}_X.\tag{32}$$

- To obtain the final element, $\tilde{\lambda}$, the position vector in the equinoctial reference frame is found

$$X = \mathbf{r} \cdot \mathbf{e}_X \quad \text{and} \quad Y = \mathbf{r} \cdot \mathbf{e}_Y.\tag{33}$$

Next, the intermediary variables α and β are defined

$$\alpha = \frac{1}{1 + \beta} \quad \text{and} \quad \beta = \sqrt{1 - p_1^2 - p_2^2}\tag{34}$$

that, along with the generalized semi-major axis obtained via Equation (13), yield functions of the generalized eccentric longitude

$$\begin{aligned}\cos \tilde{F} &= p_2 + \frac{1}{\tilde{a}\beta} [(1 - \alpha p_2^2)X - \alpha p_1 p_2 Y] \\ \sin \tilde{F} &= p_1 + \frac{1}{\tilde{a}\beta} [(1 - \alpha p_1^2)Y - \alpha p_1 p_2 X].\end{aligned}\tag{35}$$

Using these values, the generalized mean longitude is finally obtained as

$$\tilde{\lambda} = \text{atan2}(\sin \tilde{F}, \cos \tilde{F}) + \frac{1}{\tilde{a}\beta}(Xp_1 - Yp_2) . \quad (36)$$

This transformation from Cartesian coordinates to the GEqOE set is utilized to initialize spacecraft motion, enabling direct propagation in GEqOE coordinates.

Transformation from the GEqOE Set to Cartesian Coordinates

Let $\mathbf{x}_G = [\tilde{n} \ p_1 \ p_2 \ q_1 \ q_2 \ \tilde{\lambda}]^T$ represent the Generalized Equinoctial Orbital Elements at a given time. The transformation from GEqOE coordinates into Cartesian coordinates is detailed below:

- Using p_1 , p_2 , and $\tilde{\lambda}$, Equation (29) is solved to determine the generalized eccentric longitude.
- Next, the quantities α and β are determined using p_1 and p_2 , as defined in Equation (34), along with the generalized semi-major axis, $\tilde{a}$, defined in Equation (13).
- The components of the position vector in the equinoctial frame are determined via

$$\begin{aligned} X &= \tilde{a} \left[\alpha p_1 p_2 \sin \tilde{F} + (1 - \alpha p_1^2) \cos \tilde{F} - p_2 \right] \\ Y &= \tilde{a} \left[\alpha p_1 p_2 \cos \tilde{F} + (1 - \alpha p_2^2) \sin \tilde{F} - p_1 \right] . \end{aligned} \quad (37)$$

- Leveraging Equation (31), the elements q_1 and q_2 are used to determine the unit vectors that define the equinoctial frame, $[\mathbf{e}_X, \mathbf{e}_Y, \mathbf{e}_Z]^T$.
- The position vector is then found as

$$\mathbf{r} = X\mathbf{e}_X + Y\mathbf{e}_Y . \quad (38)$$

- The generalized and classical angular momenta, $\tilde{h}$ and h respectively, are determined next via the first three generalized elements

$$\tilde{h} = \left(\frac{\mu_{CB}^2}{\tilde{n}} \right)^{1/3} \sqrt{1 - p_1 - p_2} \quad (39)$$

$$h = \sqrt{\tilde{h}^2 - 2r^2U} . \quad (40)$$

- Next, the radial velocity, $\dot{r}$, is determined

$$\dot{r} = \frac{\sqrt{\mu_{CB}\tilde{a}}}{r} \left(p_2 \sin \tilde{F} - p_1 \cos \tilde{F} \right) \quad (41)$$

- Finally, the velocity vector is expressed as

$$\dot{\mathbf{r}} = \dot{X}\mathbf{e}_X + \dot{Y}\mathbf{e}_Y , \quad (42)$$

where,

$$\begin{aligned} \dot{X} &= \frac{\dot{r}X}{r} - \frac{hY}{r^2} \\ \dot{Y} &= \frac{\dot{r}Y}{r} + \frac{hX}{r^2} . \end{aligned} \quad (43)$$

The transformation from GEqOEs into Cartesian coordinates is utilized to visualize spacecraft trajectories in various frames and to assess the accuracy of propagating the GEqOE equations of motion against conventional methods.

GEqOE Time Derivatives

The process of obtaining the time derivatives for each GEqOE is detailed by Baù et al.³ In this section, the general form of the time derivatives that accommodates any perturbing forces is summarized. From Equation (5), recall that any perturbing force $\mathbf{F}$ comprised of conservative perturbations (derived from some potential energy, U) and non-conservative forces may be embedded into the GEqOEs. Projections of the total perturbing force and the non-conservative force term, denoted $\mathbf{P}$, into the orbital reference frame are given as

$$F_r = \mathbf{F} \cdot \mathbf{e}_r, \quad F_f = \mathbf{F} \cdot \mathbf{e}_f \quad \text{and} \quad F_h = \mathbf{F} \cdot \mathbf{e}_h \quad (44)$$

$$P_r = \mathbf{P} \cdot \mathbf{e}_r, \quad P_f = \mathbf{P} \cdot \mathbf{e}_f \quad \text{and} \quad P_h = \mathbf{P} \cdot \mathbf{e}_h. \quad (45)$$

Additionally, the angular velocity of the equinoctial reference frame, Σ_{eq} , with respect to the inertial frame, Σ , is projected onto the equinoctial axes as

$$w_X = F_h \frac{r}{h} \cos l, \quad w_Y = F_h \frac{r}{h} \sin l \quad \text{and} \quad w_Z = -F_h \frac{r}{h} \tan \frac{i}{2} \sin \omega + \theta. \quad (46)$$

Alternatively, these projections are expressed as

$$w_X = \frac{X}{h} F_h, \quad w_Y = \frac{Y}{h} F_h \quad \text{and} \quad w_Z = w_h = w_X q_1 - w_Y q_2. \quad (47)$$

Other quantities necessary to compute the time derivatives of the GEqOEs include the rate of change of the total energy of the system, that is evaluated as

$$\dot{\mathcal{E}} = \frac{\partial U}{\partial t} + \dot{r} P_r + \frac{h}{r} P_f. \quad (48)$$

Then, the general form of the time derivatives of the GEqOEs is given as

$$\begin{aligned} \dot{\tilde{n}} &= -3 \left(\frac{\tilde{n}}{\mu_{CB}^2} \right)^{1/3} \dot{\mathcal{E}} \\ \dot{p}_1 &= p_2 \left(\frac{h - \tilde{h}}{r^2} - w_h \right) + \frac{1}{\tilde{h}} \left(\frac{X}{\tilde{a}} + 2p_2 \right) (2U - rF_r) + \frac{1}{\tilde{h}^2} [Y(r + \tilde{p}) + r^2 p_1] \dot{\mathcal{E}} \\ \dot{p}_2 &= p_1 \left(w_h - \frac{h - \tilde{h}}{r^2} \right) - \frac{1}{\tilde{h}} \left(\frac{Y}{\tilde{a}} + 2p_1 \right) (2U - rF_r) + \frac{1}{\tilde{h}^2} [X(r + \tilde{p}) + r^2 p_2] \dot{\mathcal{E}} \\ \dot{q}_1 &= \frac{1}{2} w_Y (1 + q_1^2 + q_2^2) \\ \dot{q}_2 &= \frac{1}{2} w_X (1 + q_1^2 + q_2^2) \\ \dot{\lambda} &= \tilde{n} + \frac{h - \tilde{h}}{r^2} - w_h + \frac{1}{\tilde{h}} \left[\frac{1}{\alpha} + \alpha \left(1 - \frac{r}{\tilde{a}} \right) \right] (2U - rF_r) + \frac{r \dot{\alpha}}{\mu_{CB} \tilde{h}} (r + \tilde{p}) \dot{\mathcal{E}}. \end{aligned} \quad (49)$$

Any significant perturbations, whether conservative or non-conservative, that are essential for modeling spacecraft dynamics are, thus, directly incorporated into the generalized elements and their time derivatives.

CISLUNAR STATE PROPAGATION

As spacecraft and other objects traverse cislunar space, their complex dynamical motion is influenced by the gravitational forces of both the Earth and the Moon. Therefore, in addition to the dominant gravitational influence of one body, the gravitational perturbation caused by the other must be accounted for in trajectory design and prediction. While other perturbations, such as the oblateness effects of the Earth and Moon or the gravitational influence of the Sun, may also be significant, accounting for the gravitational forces of both the Earth and the Moon is essential for preliminary analysis. For cislunar state propagation employing the Generalized Equinoctial Orbital Elements, the conservative perturbations arising from third-body gravity effects are incorporated.

One dynamical model that serves as a point of comparison against third-body perturbed GEQOE solutions is the Earth-Moon Circular Restricted Three-Body Problem (CR3BP). The Earth-Moon CR3BP is a medium-fidelity dynamical model that takes into account the gravity of Earth and Moon on the motion of the spacecraft.⁶ The third body, assumed to be a massless spacecraft, does not influence the orbits of the Earth and the Moon, which orbit their mutual barycenter in circular coplanar orbits. These simplifying assumptions, i.e., that the orbit of the Moon is circular and has zero inclination, are also adopted in evaluating the GEQOE-propagated solution for a direct comparison against the CR3BP. The relevant equations of motion for the Earth-Moon CR3BP and the methodology for embedding third-body perturbations into the GEQOE set are provided below.

Cartesian Coordinates: Earth-Moon Circular Restricted Three-Body Problem

The Circular Restricted Three-Body Problem (CR3BP) is an autonomous dynamical model that incorporates the gravitational influences of any two planetary bodies, for example, a planet-moon or a Sun-planet pair.⁶ In this investigation, the Earth-Moon CR3BP is leveraged. The Earth and the Moon are assumed to be point masses orbiting their mutual barycenter in circular orbits. The third body models the spacecraft, which is assumed to be of infinitesimal mass and, thus, does not influence the orbits of the Earth and the Moon. A rotating frame based in the motion of the primaries is adopted to describe the behavior of the spacecraft governed by the Earth and the Moon. In this frame, represented $\{\hat{x}^R, \hat{y}^R, \hat{z}^R\}$, $\hat{x}^R$ is directed from the Earth to the Moon, $\hat{z}^R$ is parallel to the angular momentum direction of the system, and $\hat{y}^R$ completes the dextral orthonormal triad. The quantities l^* , m^* , and t^* , termed the characteristic length, characteristic mass, and characteristic time, respectively, nondimensionalize the equations of motion of the spacecraft in this model. The resulting scalar equations of motion that represent the motion of the spacecraft as expressed in the rotating frame are expressed as

$$\ddot{x} - 2\dot{y} = \frac{\partial U_{\text{CR3BP}}^*}{\partial x}, \quad \ddot{y} + 2\dot{x} = \frac{\partial U_{\text{CR3BP}}^*}{\partial y} \quad \text{and} \quad \ddot{z} = \frac{\partial U_{\text{CR3BP}}^*}{\partial z}. \quad (50)$$

The quantity U_{CR3BP}^* is a pseudo-potential function defined as

$$U_{\text{CR3BP}}^* = \frac{1 - \mu_{\text{CR3BP}}}{r_{\text{Esc}}} + \frac{\mu_{\text{CR3BP}}}{r_{\text{Msc}}} + \frac{x^2 + y^2}{2}, \quad (51)$$

where μ_{CR3BP} is the CR3BP system mass parameter evaluated as

$$\mu_{\text{CR3BP}} = \frac{M_M}{M_E + M_M}, \quad (52)$$

where the quantities M_E and M_M represent the mass of the Earth and the Moon, respectively. For the Earth-Moon system, this value is approximately equal to 0.0121506. Additionally, in Equation (51), r_{Esc} and r_{Msc} denote the nondimensionalized distances between the Earth and the spacecraft as well as the Moon and the spacecraft, respectively. The values $[x, y, z]$ correspond to the nondimensional position of the spacecraft relative to the system barycenter in the Earth-Moon rotating frame; similarly, $[\dot{x}, \dot{y}, \dot{z}]$ represent the velocity components of the spacecraft as viewed in the rotating frame. While the equations of motion in Equation (50) yield spacecraft trajectories directly in this rotating frame, transforming the trajectories into body-centered inertial frames provides further insight into the spacecraft's motion in cislunar space.

GEqOE Coordinates: Embedding Third-Body Perturbations

For parameterizing cislunar trajectories using GEqOEs, the addition of lunar third-body gravity perturbations is necessary. A general form of the associated potential energy is given as⁷

$$U_M = \mu_M \left(\frac{1}{r_{Msc}} - \frac{\mathbf{r} \cdot \mathbf{r}_{EM}}{r_{EM}^3} \right), \quad (53)$$

where $\mathbf{r}$ represents the position vector from the Earth to the spacecraft. Equation (53) may be rewritten as a convergent series of Legendre polynomials

$$U_M = \frac{\mu_M}{r_{EM}} \left[1 + \sum_{k=2}^{k=\infty} \left(\frac{r}{r_{EM}} \right)^k P_k(\cos \alpha) \right], \quad (54)$$

where α is the angle between the position vectors $\mathbf{r}$ and $\mathbf{r}_{EM}$, and P_k represent Legendre polynomials.⁷ The perturbing acceleration takes the form

$$-\nabla U_M = \frac{\mu_M}{r_{EM}^2} \sum_{k=1}^{k=\infty} \left(\frac{r}{r_{EM}} \right)^k \left[P'_{k+1}(\cos \alpha) \mathbf{e}_{r_{EM}} - P'_k(\cos \alpha) \mathbf{e}_r \right], \quad (55)$$

where $\mathbf{e}_{r_{EM}}$ and $\mathbf{e}_r$ are the unit vectors along $\mathbf{r}_{EM}$ and $\mathbf{r}$ respectively. By embedding the lunar third-body potential energy directly into the total energy of the system, leveraging the time derivatives of the GEqOEs, the element set is numerically integrated to obtain the spacecraft state history over time. Furthermore, after integration, the GEqOE set is transformed into Cartesian coordinates at each time step to obtain a direct comparison against the conventional CR3BP solutions.

To assess the accuracy of propagating lunar gravity perturbed cislunar orbits via the GEqOE equations of motion, sample orbits from the Earth-Moon Circular Restricted Three-Body Problem (CR3BP) are considered. The Cartesian solutions are propagated using the CR3BP equations of motion, while the GEqOE solutions are propagated using third-body gravity perturbed GEqOE equations. A summary of the methodology for propagating and comparing the solutions obtained via the two methods is provided below:

- **CR3BP Solution:**

- Propagate CR3BP initial conditions using the CR3BP equations of motion (Equation (50)) in the Earth-Moon rotating frame.
- Transform the orbit from the Earth-Moon rotating frame to the Earth- or Moon-centered inertial frame.

- Transform the body-centered inertial frame states into GEqOE coordinates at each time step.
- **GEqOE Solution:**
 - Extract the initial state of the CR3BP trajectory expressed in the GEqOE coordinates.
 - Propagate the initial state using the GEqOE equations of motion (Equation (49)).
 - Transform the GEqOE solution into Cartesian coordinates, i.e., into the Earth- or Moon-centered inertial frame at each time step.
 - Transform the body-centered inertial frame states to the Earth-Moon rotating frame.

This process is applied for several Earth-, Moon-, and libration point-centered orbits from the CR3BP. Solutions obtained via propagation in GEqOE coordinates are validated against their conventional Cartesian analogs in both sets of coordinates and, for completeness, in both the inertial and rotating reference frames.

RESULTS

A wide variety of cislunar orbits identified in the Earth-Moon CR3BP are relevant for missions to cislunar space. Orbits about the L_1 and L_2 libration points, for instance, provide localized access for missions near the lunar vicinity. There also exist orbits, such as sidereal resonant orbits, that traverse the entire cislunar domain, offering broader access to cislunar space while periodically passing close to both the Earth and the Moon. Thus, to evaluate the utility of GEqOEs as an option for cislunar state propagation, a wide variety of orbits that exhibit vastly different geometries in space are analyzed.

Earth-Centered Orbits

In the CR3BP, sidereal resonant orbits offer options for Earth-centered orbits that span the cislunar domain. These orbits are characterized by rotating frame orbital periods that are commensurate with the lunar sidereal period.⁸ In general, sidereal resonant orbits originate from Keplerian trajectories that exhibit a precise mean-motion resonance with the Moon around the Earth. However, in the CR3BP, the resonance ratio is not an exact integer. Despite this near-resonance, these orbits remain closed and periodic in the Earth-Moon rotating frame and follow the same distinctive geometry exhibited under Keplerian motion. Sidereal resonant orbits may be characterized by their resonance ratio, denoted $p:q$, where the spacecraft orbits the Earth p times in the inertial frame over approximately q lunar orbits.

One example of a sidereal resonant orbit is the planar 4:1 resonant orbit that appears in Figure 1. Solutions obtained by propagating the GEqOEs in time, and via the CR3BP equations of motion, appear in blue and orange respectively. Transformed into Cartesian coordinates, the orbit obtained via the GEqOE method closely follows the CR3BP solution in both the rotating and inertial frames, as shown in Figures 1(a) and 1(b). While the inertial frame view appears as a single “elliptical” orbit, the rotating frame view exhibits the distinctive geometry that is inherent to sidereal resonant orbits. Figure 1(c) illustrates the isochronous error in position and velocity between the two propagation methods over one revolution, showing that the errors are bounded and within acceptable tolerance, confirming the consistency of the solutions in Cartesian coordinates. While visualizing the solutions in Cartesian coordinates provides an intuitive understanding of the trajectory in configuration space,

the evolution of the two solutions in the GEqOE coordinates further validates their similarity and the accuracy of the GEqOE propagation. Figure 1(d) illustrates the time history for each of the six GEqOEs, with the blue curves indicating the solution obtained by propagating the GEqOE equations of motion, and the orange curves corresponding to the solution obtained by propagating the CR3BP equations. Over the course of one orbit period, approximately *27.3 days*, the two solutions follow closely in both Cartesian and GEqOE coordinates.

Similar analysis is repeated for other resonant orbits. As an example, an orbit from the family of spatial 3:1 resonant orbits is selected, similar to the orbit selected for the Interstellar Boundary Explorer (IBEX) extended mission.⁹ The period of the selected orbit is approximately *27.2 days*. Figures 2(a) and 2(b) illustrate the trajectories obtained via the GEqOE propagation (blue) and the CR3BP propagation (orange) in the Earth-Moon rotating and Earth-centered inertial frames. The trajectories appear overlaid in both the frames; however, the isochronous error in position and velocity is higher than that for the previous case. This error, plotted in Figure 2(c), arises from a phase shift that develops along the orbit over time. The time evolution of each GEqOE appears in Figure 2(d) and illustrates consistency between the two solutions in the generalized coordinates.

Moon-Centered Orbits

In addition to the Earth-centered sidereal resonant orbits, orbits that appear centered on the Moon in the Earth-Moon rotating frame are considered as well. Examples of such orbits include the Low Prograde Orbits (LPOs) and orbits from the Butterfly family that bifurcate from the L_2 halo family in the CR3BP.¹⁰ Sample orbits from these families are constructed using the GEqOE equations to evaluate the effectiveness of the methodology across various orbit options in cislunar space.

Figure 3 illustrates an orbit from the eastern half of the LPO family, possessing a period of *13.8 days* in the rotating frame. The rotating frame view in Figure 3(a) shows the GEqOE (blue) and CR3BP (orange) solutions. In order to highlight the periodicity of the selected orbit in the Moon-centered inertial frame, two revolutions of the orbits appear in Figure 3(b). The isochronous error in position and velocity over one period plotted in Figure 3(c) confirms consistency between the two solutions in Cartesian coordinates. Finally, Figure 3(d) highlights the evolution of the GEqOEs obtained via both the methodologies. Note that because the selected orbit lies within the Earth-Moon plane, the q_1 and q_2 elements are both equal to zero.

Examples of spatial Moon-centered orbits include the butterfly orbits that bifurcate from the L_2 halo orbit family and can provide access to the lunar surface.¹¹ A sample orbit from this family with a period of approximately *26 days* is selected for analysis. The resulting orbit, propagated using the GEqOE (blue) and CR3BP (orange) equations, appears in Figures 4(a) and 4(b), as viewed in the Earth-Moon rotating and Moon-centered inertial frames, respectively. The error in position and velocity between the two methodologies is consistent with the observations from previous orbits, as shown in Figure 4(c). Finally, the evolution of the generalized elements over time via both methods appears in Figure 4(d). Despite the seemingly complex geometry of the selected orbit in the rotating frame, the GEqOE method is able to accurately follow the CR3BP solution.

Libration Point Orbits

The final sub-category of CR3BP orbits considered in the current work includes orbits about the CR3BP libration points. Due to their relevance for lunar missions and cislunar operations, orbits about the L_1 and L_2 libration points are considered. Figure 5 illustrates the results for an

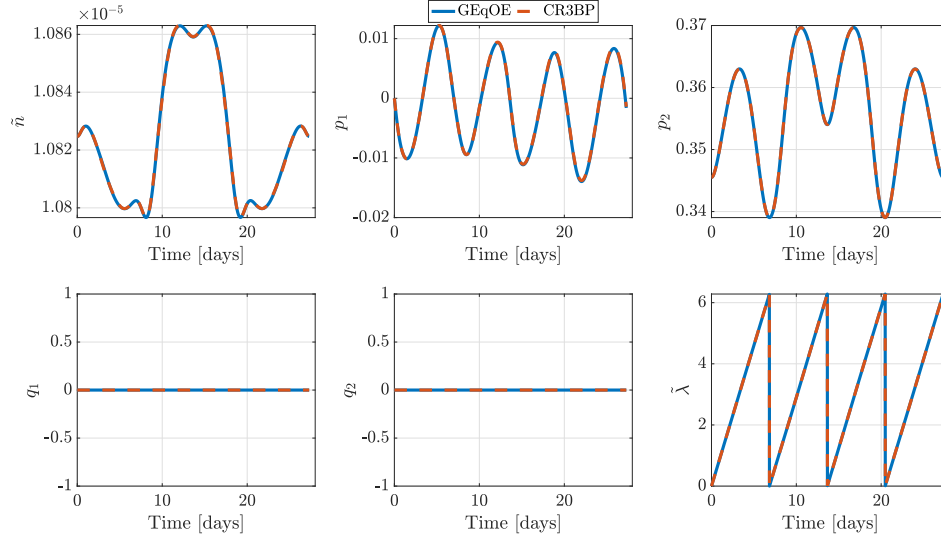
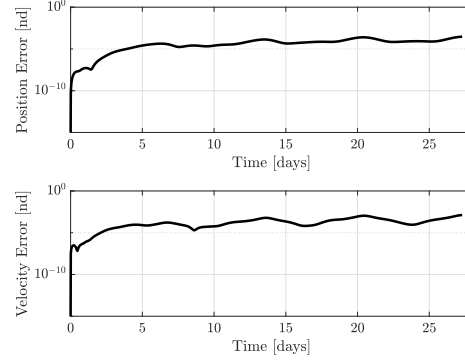
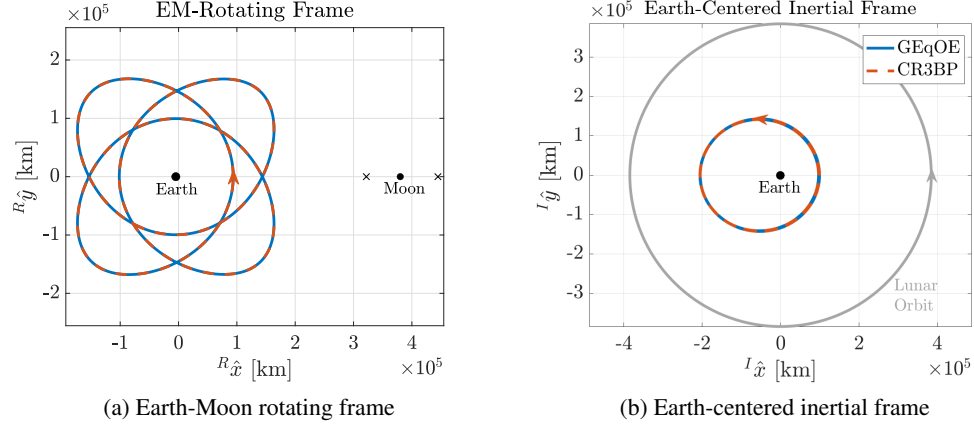


Figure 1. The 4:1 planar sidereal resonant orbit in ((a),(b)) Cartesian and (d) GEQOE coordinates obtained via the GEQOE (blue) and CR3BP (orange) equations of motion. The error in position and velocity between the two methods appears in (c).

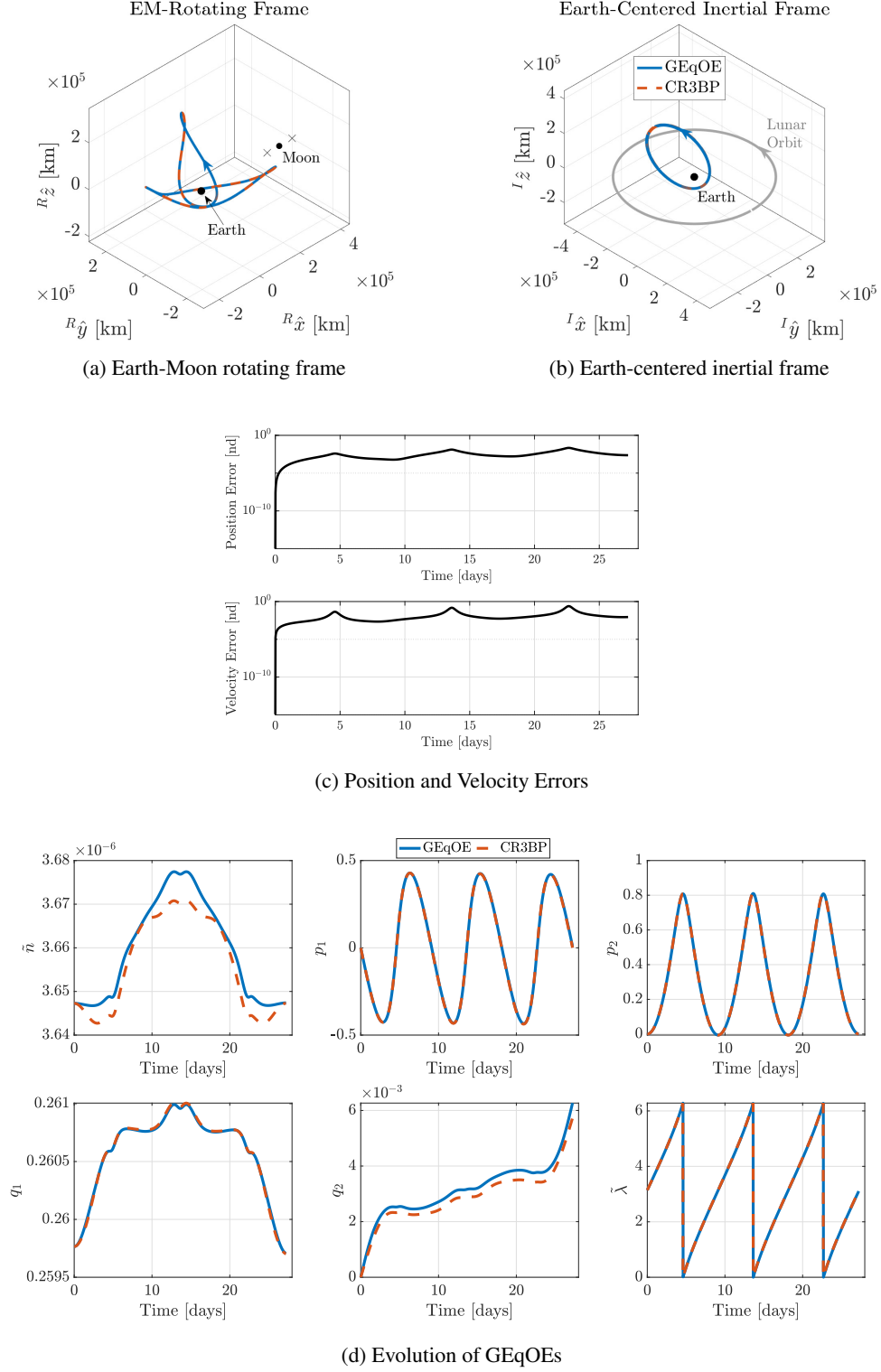


Figure 2. The 3:1 spatial sidereal resonant orbit in ((a),(b)) Cartesian and (d) GEqOE coordinates obtained via the GEqOE (blue) and CR3BP (orange) equations of motion. The error in position and velocity between the two methods appears in (c).

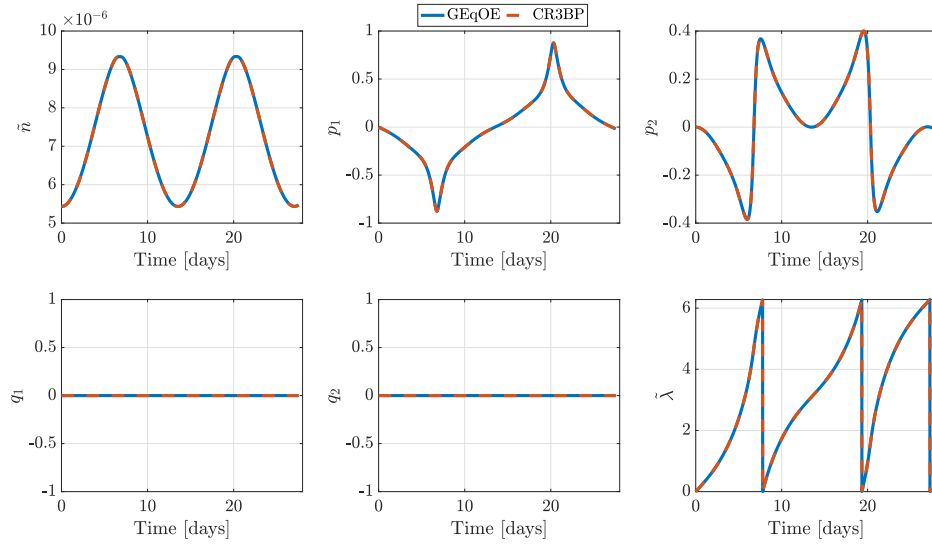
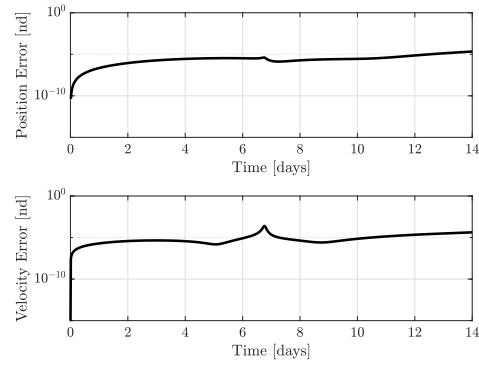
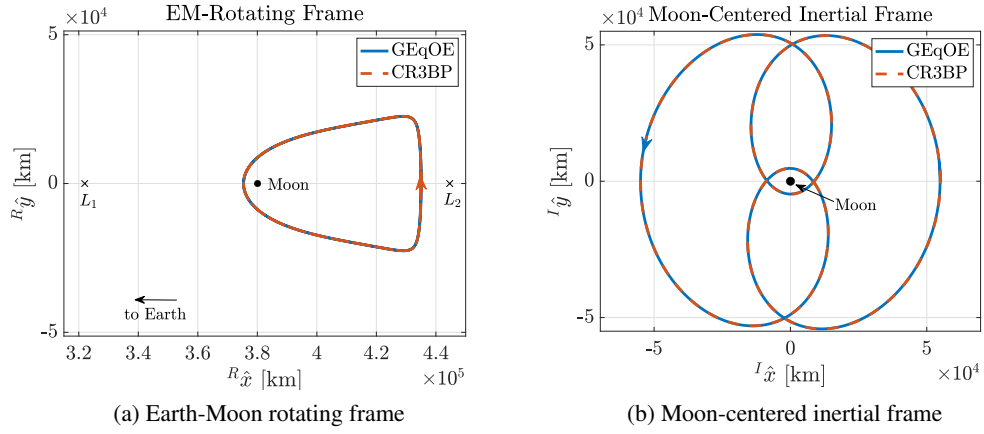


Figure 3. Low-prograde orbit in ((a),(b)) Cartesian and (d) GEqOE coordinates obtained via the GEqOE (blue) and CR3BP (orange) equations of motion. The error in position and velocity between the two methods appears in (c).

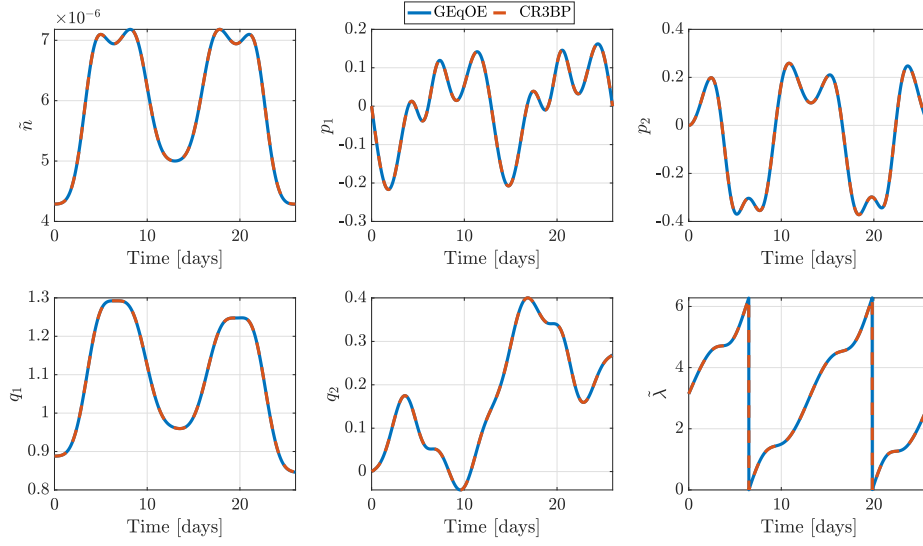
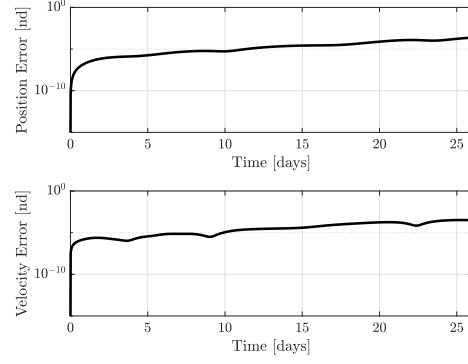
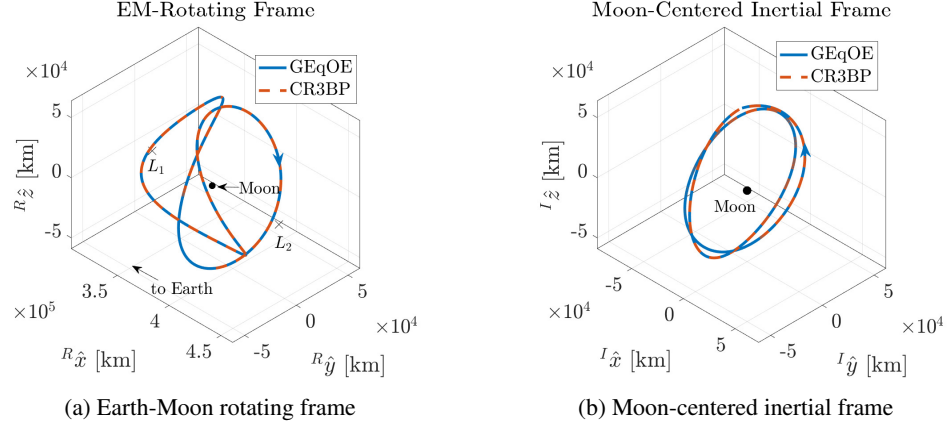


Figure 4. A butterfly orbit in ((a),(b)) Cartesian and (d) GEqOE coordinates obtained via the GEqOE (blue) and CR3BP (orange) equations of motion. The error in position and velocity between the two methods appears in (c).

L_1 halo orbit via the GEqOE (blue) and CR3BP (orange) methods. The selected orbit possesses a period of 10.4 *days* and appears in the rotating and inertial frame views in Figures 5(a) and 5(b) respectively. Similar levels of error in position and velocity are seen as the other orbits considered in this investigation. The time history of the GEqOEs, shown in Figure 5(d), confirms consistency between the two methodologies in the generalized coordinates as well.

Another libration point orbit of interest is the 9:2 Near Rectilinear Halo Orbit (NRHO), selected as the operational orbit for NASA’s Gateway lunar outpost.^{12,13} Note that the resonance ratio indicates synodic resonance with the Moon, a characteristic that allows eclipse feasibility for both Gateway and the Orion spacecraft.¹⁴ Figure 6 illustrates the GEqOE and CR3BP solutions for this orbit over one revolution, approximately 6.5 *days*. The Cartesian representations of the solution that appear in Figures 6(a) and 6(b) confirm consistency between the two methodologies, which is further validated by the position and velocity error plots in Figure 6(c). Finally, the generalized elements for this orbit appear in Figure 6(d), where the CR3BP solution transformed into the GEqOE coordinates demonstrates agreement with the GEqOE-propagated solution. While the results in Figure 6 span one revolution of the orbit, the GEqOE propagation remains consistent with the CR3BP solution for up to 9 revolutions.

Limitations of the GEqOE Methodology

The GEqOE methodology is proven to be effective for various orbits selected from the Earth-Moon CR3BP. However, inherent limitations in the current formulation of the GEqOE set constrain its broader applicability. In particular, the current formulation is only valid in cases where the total energy of the system, defined in Equation (6), is negative. Additionally, similar to the classical equinoctial orbital elements, the GEqOE set is singular for retrograde orbits. As such, modifications to the GEqOE set may be necessary to guarantee their applicability to all orbit types.

CONCLUDING REMARKS

Cislunar dynamics are governed by, among other forces, the gravitational influence of both the Earth and the Moon. As such, traditional Keplerian orbital elements are not suited for applications in this domain. The current work assesses the applicability of the Generalized Equinoctial Orbital Elements (GEqOEs) for cislunar state propagation. These orbital elements are characterized by the total energy of the system and may be embedded with necessary perturbations. In the current work, in addition to the gravitational influence of the Earth, the GEqOEs are embedded with lunar gravity and evaluated for cislunar state propagation. To provide a comparison against conventional Cartesian propagation, the performance of the GEqOEs is compared against orbits constructed via the Earth-Moon Circular Restricted Three-Body Problem (CR3BP). Various orbits from different families, characterized by different geometries and energy characteristics, are successfully constructed using lunar-gravity perturbed GEqOE equations. The GEqOE set is demonstrated to provide the minimum level of dynamical model fidelity necessary for cislunar applications.

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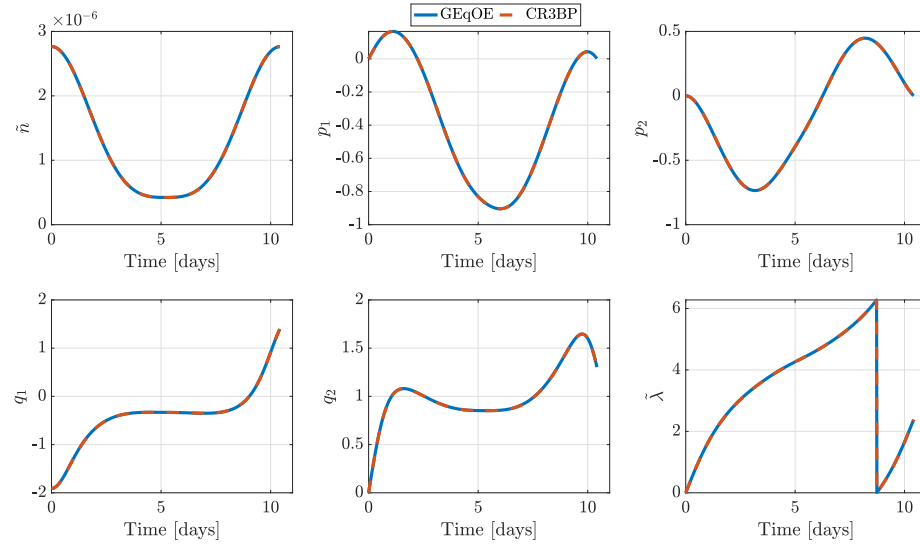
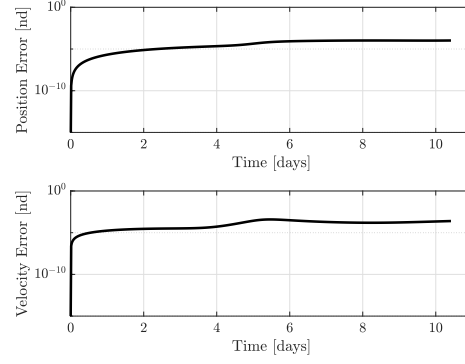
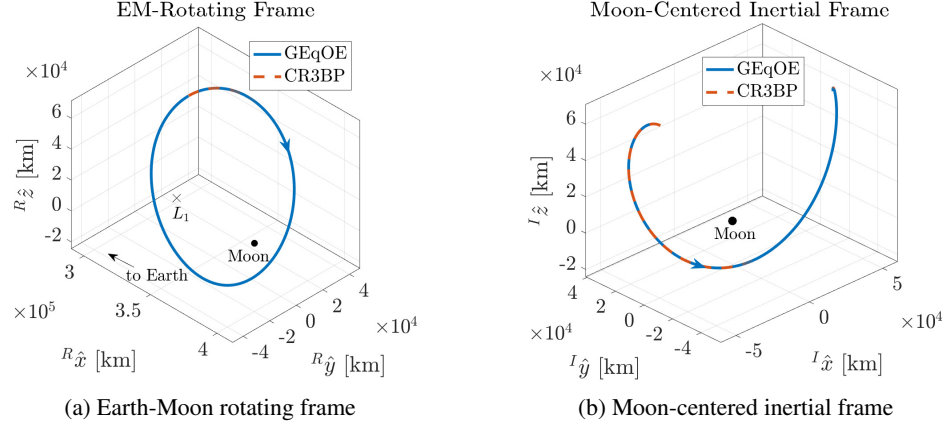


Figure 5. An L_1 halo orbit in ((a),(b)) Cartesian and (d) GEqOE coordinates obtained via the GEqOE (blue) and CR3BP (orange) equations of motion. The error in position and velocity between the two methods appears in (c).

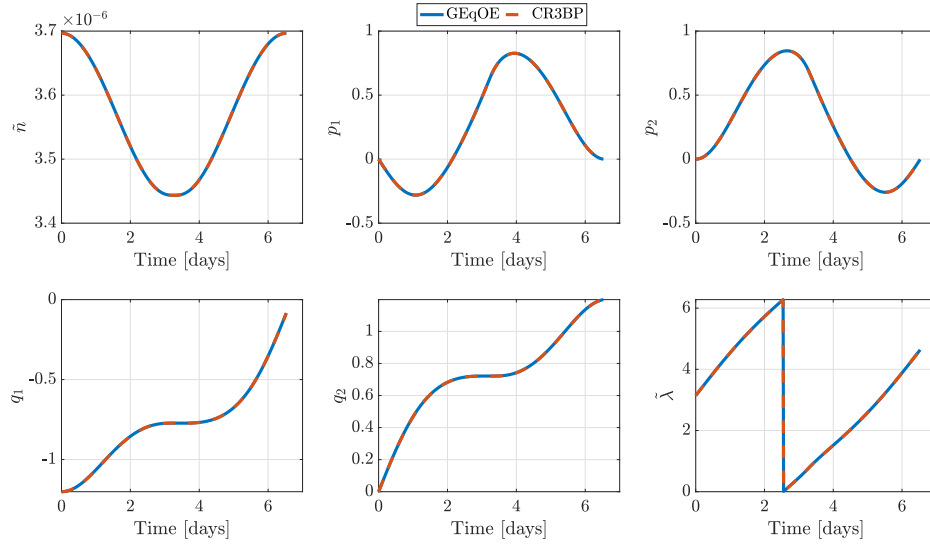
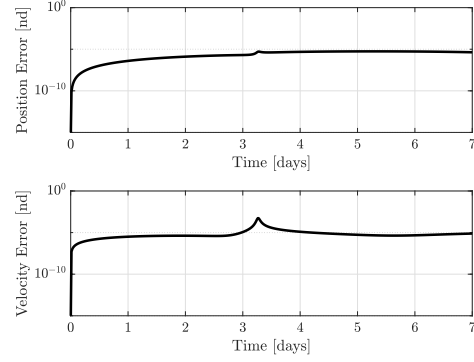
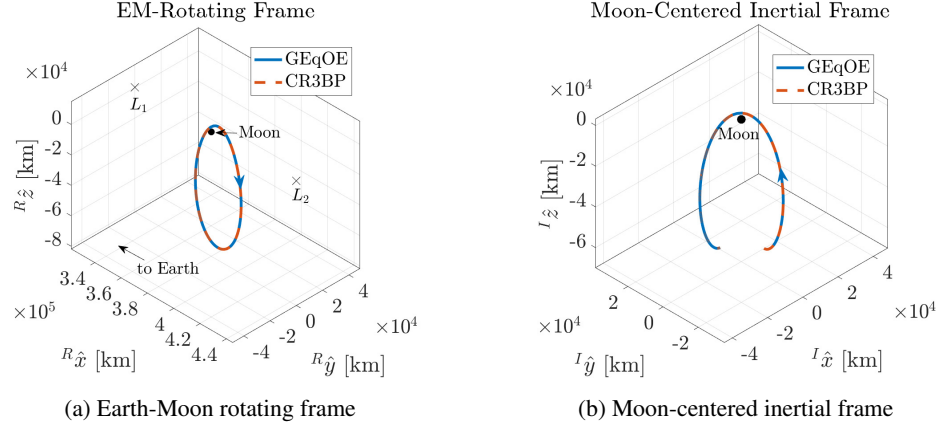


Figure 6. The 9:2 NRHO in ((a),(b)) Cartesian and (d) GEqOE coordinates obtained via the GEqOE (blue) and CR3BP (orange) equations of motion. The error in position and velocity between the two methods appears in (c).

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