Wisconsin TEACHER of MATHEMATICS

WISCONSIN MATHEMATICS COUNCIL, INC.

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NCTM Outstanding 2013 Publication Award

Creating and Implementing Rich Tasks in the Classroom

Students Engaging in Mathematical Practices: As the Gear Turns

Making the Most of Mathematical Modeling

Dynamic Geometry Software Meets the Pythagorean Theorem

Wisconsin Teacher of Mathematics Spring 2016 Journal

The theme of the Spring 2016 issue of the *Wisconsin Teacher of Mathematics* will be **Unpacking Principles to Actions.** The editorial panel would like to showcase manuscripts that focus on two of the teaching practices discussed in *Principles to Actions* (NCTM, 2014), specifically, building procedural fluency from conceptual understanding and supporting productive struggle in learning mathematics. We encourage authors to submit articles that address questions such as:

- How can procedural fluency be built upon a conceptual foundation?
- How can educators provide students with opportunities to engage in productive struggle within the mathematics classroom?

If you have ideas or questions for this focus issue or wish to submit an article for review, please email Josh Hertel (jhertel@uwlax.edu) or visit the WMC website for more information (http://www.wismath.org/Write-for-our-Journal). The submission deadline for the Spring 2016 issue is February 26, 2016.

Wisconsin Teacher of Mathematics, the official journal of the Wisconsin Mathematics Council, is published twice a year. Annual WMC membership includes a one-year subscription to this journal and to the Wisconsin Mathematics Council Newsletter.

The *Wisconsin Teacher of Mathematics* is a forum for the exchange of ideas. Opinions expressed in this journal are those of the authors and may not necessarily reflect those of the Council or editorial staff.

Submissions are welcome!

- Manuscripts may be submitted at any time.
- All manuscripts are subject to a review process.
- Include the author's name, address, telephone, email, work affiliation and position.
- Manuscripts should be double-spaced and submitted in .doc or .docx format.
- Embed all figures and photos.
- Send an electronic copy of the manuscript to jhertel@uwlax.edu.

Direct all correspondence to:

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The submission deadline for the Spring 2016 issue is February 26, 2016.

Wisconsin Mathematics Council Distinguished Mathematics Educator Award

The **Distinguished Mathematics Educator Award** is the most prestigious award that the Wisconsin Mathematics Council bestows. The award recognizes individuals for their exceptional leadership and service to the state's mathematics education community.

Nominations are now open, and you may download the nominations form at www.wismath.org. Application deadline is January 31, 2016. The award recipient will be honored at the Thursday evening *Celebrate WMC* event at the Annual Conference.

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Wisconsin Teacher of Mathematics

The National Council of Teachers of Mathematics selected the Wisconsin Teacher of Mathematics to receive the 2013 Outstanding Publication Award. This prestigious award is given annually to recognize the outstanding work of state and local affiliates in producing excellent journals. Judging is based on content, accessibility, and relevance. The WMC editors were recognized at the 2014 NCTM Annual meeting.

President's Message

Opportunities for Productive Struggle

At the October Wisconsin Mathematics Council's *Mathematics Proficiency for Every Student* conference, I presented the following statement to teachers:

> *Mathematics classrooms that promote students' understanding "allow mathematics to be problematic for students"* (Hiebert, 2003, p. 54).

There were many comments that came to my mind before teachers discussed this statement. These comments included, "Sometimes we feel the need to reduce student struggle by funneling questions that guide them to the answer," and "Student frustration means I may have not given my students the tools they need to solve a task." However, the actual discussion focused on the detriment to removing student struggle as it will diminish students' opportunities to develop a deep understanding of mathematics. The authors of the recent publication by the National Council of Teachers of Mathematics, *Principles to Actions: Ensuring the Mathematical Success of All*, state that, "'Rescuing" undermines the efforts of students, lowers the cognitive demand of the task, and deprives students of opportunities to engage fully in making sense of the mathematics" (NCTM, 2014, p. 48). Equipped with these ideas, I posed the question, "How can we support students' productive struggle in mathematics by engaging them in tasks that help them make sense of and connect various mathematical concepts?" In short, we wondered how we could, in keeping with Hiebert's suggestion, create opportunities for mathematics to be problematic for our students.

In *Principles to Actions*, the writing team highlighted five critical teacher actions that could provide opportunities for engaging students in productive struggle. These five actions include:

- Use tasks that promote reasoning and problem solving; explicitly encourage students to persevere; find ways to support students without removing all the challenges in a task.
- Ask students to explain and justify how they solved a task. Value the quality of the explanation as much as the final solution.
- Give students the opportunity to discuss and determine the validity and appropriateness of strategies and solutions.
- Give students access to tools that will support their thinking processes.
- Ask students to explain their thinking and pose questions that are based on students' reasoning, rather than on the way that the teacher is thinking about the task. (NCTM, 2014, p. 49)

One strategy that I have found to be helpful in meeting these recommendations is to "open up"

Grade	Traditional	Open Ended Question
Elementary	What is 2 + 8?	There are 10 students on the playground. Some are boys and some are girls. How many boys and girls could be on the playground? Use pictures, words and numbers.
	Find the area and perimeter of a rectangle with a length of 3 inches and a width of 8 inches.	Find all the rectangles with an area of 24 square inches. What do you notice about their perimeters?
	What is 8 x 7?	Convince somebody in two different ways how you know that 8 x 7 = 56?
	In a relay, three kids ran a total of 1 mile. If each kid ran the same distance, what fraction a mile did each kid run?	In a relay, three kids ran a total of 1 mile. What fraction of a mile could each kid have run? Show two different ways using numbers and a visual fraction model.
Middle School	Find the volume and surface area of a rectangular prism.	Construct two rectangular prisms with the same volume but different surface areas.
High School	Solve: ¾ + x/10	Jenn was solving a proportional relationships task and came up with the following equation. ¾ + x/10 Explain two different ways to can solve the equation. Construct a story situation that Jenn could be solving.
	If $x = 45^\circ$, verify that $\sin(2x) = 2 \sin x \cos x$	Decide whether each of the following statements are true. Justify your reasoning: $\sin(2x) = 2 \sin x$ $\sin^5 x = \sin x^5$

traditional mathematical tasks so that my students will have the opportunity "for delving more deeply into understanding the mathematical structure of problems and relationships among mathematical ideas, instead of simply seeking correct solutions" (NCTM, 2014, p. 48). The table on the previous page presents a variety of modified tasks for different grade levels.

These tasks are designed to provide students with the opportunity to engage in high-level thinking by highlighting multiple entry points and solutions strategies as student problem solve and reason with mathematics. Throughout the school year, I challenge you to (a) create open tasks that will provide students with the opportunity to productively struggle and (b) share your results with other Wisconsin mathematics teachers by writing a short piece for the *Wisconsin Teacher of Mathematics*! I look forward to furthering the discussion about how to promote students' understanding through problematic mathematics.

Jenn

Jennifer Kosiak
WMC President, 2015-2017

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National Council of Teachers of Mathematics. (2014). *Principles to actions: Ensuring mathematical success for all.* Reston, VA: Author.

From the Editors

Editors' Notes

This issue of the *Wisconsin Teacher of Mathematics* focuses on implementing high-level tasks that engage students in developing the habits of mind outlined by the Common Core State Standards for Mathematical Practice (SMP). The tasks and activities discussed in the issue's six articles help "describe varieties of expertise that mathematics educators at all levels should seek to develop in their students" (NGA/CCSO, 2010, p. 6).

Three articles explore how to engage students in *making sense of problems and persevere in solving them* (SMP #1). In the article, *Dynamic Geometry Software Meets the Pythagorean Theorem*, Gilbertson and Lawrence explore how students can use GeoGebra, the free dynamic mathematics software program, to make conjectures and investigate their relationship to the Pythagorean Theorem. In *Curiosity and the Coin Sword*, Paape describes how to engage students in problem solving with an activity focused on noticing and wondering. Richards, in the article *Exploring Cognitive Demand Online with the WMSI Fellows*, describes an online course offered through the Brookhill Institute of Mathematics that provides teachers with a cyclical process to identify and increase the cognitive demand of mathematical tasks.

Two articles provide activities in which students will *model with mathematics* (SMP #4) as they "solve problems arising in everyday life, society, and the workplace" (NGA-CCSO, 2010, p. 7). In the article, *Making the Most of Mathematical Modeling*, Junko and co-authors analyze research to describe several recommendations for implementing this important practice standard in the mathematics classroom. Ebert's article *Engaging Our Students in Upper-Level Mathematics: The Power of Parabolas*, is the first in a series of articles that will focus on using real-world applications to engage students in meaningful problem solving. In this first piece, Ebert discusses several real-world applications of the parabola including headlights, the Olympic torch, and a mini-golf hole.

In the final article, *As the Gear Turns*, Reiten, Ozgur, and Ellis describe an activity that serves as a vehicle to introduce ratio concepts by reasoning with rates of change. In doing so, the authors provide insight into how to engage students in *reasoning abstractly and quantitatively* (SMP #2) *and constructing viable arguments* (SMP #3).

There are several ways that you can be involved in writing for the *Wisconsin Teacher of Mathematics*. We hope that you can share the exciting work that is happening in your mathematics classrooms by:

• Writing an article for the journal! We encourage submissions on a variety of topics including classroom innovations, teaching tips, action research, and reviews of technology. For example, if you implement mathematical tasks that are designed to engage students in doing mathematics as outlined by the eight Standards for Mathematical Practice, we would love to hear from you!

• Submitting a note from the field (~250 words) in which you provide feedback on journal content, sound off on current issues in education, or briefly highlight a classroom innovation. Notes From the Field can be submitted using the following link http://goo.gl/np0qpN

• Submit a piece that focuses on the use of technology in the classroom, a project that you have used in your teaching incorporating technology, or a review of technology that you use with students.

We hope you enjoy this issue and find the articles to be useful in your classroom practice. If you have an idea for an article or questions about submission, please contact us.

Joshua Hertel
Jennifer Kosiak
Jenni McCool

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Curiosity and the Coin Sword

By Adam Paape, Concordia University

On a recent trip to the Milwaukee Public Museum, I found myself captivated by one of the exhibits. This exhibit was one of many fascinating items that I would see on my trip, but this piece was different. It truly piqued my interest and I had a whole host of questions that immediately popped into my head. The item of my intrigue was a 19th century coin sword from China (Figure 1). I had never seen anything like it before, and it got me thinking.

Classroom teachers have a similar hope for their students. Teachers want our students to enter our classrooms and immediately be engaged and filled with wonder. We know that this kind of engagement and wonder does not happen by accident. We can help to foster the curious; we can create an itch that simply must be scratched. The coin sword was my itch, and I needed to scratch it.

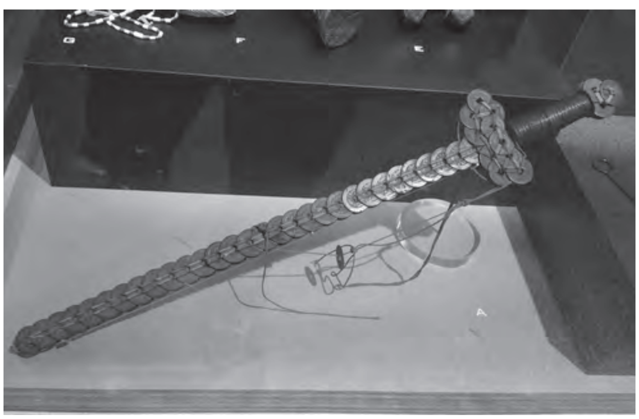

Figure 1. The coin sword.

What follows is my flow of curiosity caused by the sword. A wonderful characteristic of a "noticing and wondering" activity like this is that the learners lead the way for what they see as curious. This kind of activity is one where student-generated questions are central and puts the learner at the heart of the educational exchange. The development of student curiosity becomes the vehicle by which students develop conceptual understanding (Colin, 2011).

My First Noticing and Wondering

I noticed that the sword was made of coins. This is a simple observation. However, this knowledge led me to think about a follow-up wondering. I wonder how many coins make up the sword. Based on the picture, I made a couple of follow-up wonderings that would influence my estimation. Is the sword the same on both sides? Are all the coins the same size? They appeared to be identical coins throughout the sword.

These initial wonderings led me to reach out to the museum to ask my initial question. How many coins are there in the sword? I got an immediate response through Twitter. The museum representative said, "We've checked with our experts, but can't give you an exact number. The number of coins varies from sword to sword. It's the thread on the swords, always red or yellow, that is used to impart the idea of wealth and fortune." At first I was dissatisfied with this response. However, as I thought about this within a classroom environment, my dissatisfaction began to subside. Too often in mathematics, we make it so that our students expect one exact answer. This one-answer philosophy disallows for students to express ideas, theories, and different perspectives. The open-ended nature of this activity encourages students to enter "into the mathematics conversation, from their vantage point, … to increase their confidence in doing mathematics" (Varygiannes, 2013, p. 278). The attributes of this sword allows students to think about methods and strategies for estimation that are useful in daily life. How often do our students ask the question, "When are we ever going to use this stuff?" Teachers need to show their students that estimation and general number sense are an everyday skill for the functioning members of society.

Since one of our goals as math teachers is to facilitate meaningful mathematical discourse among students, an open-ended answer to the coin count is wonderful (NCTM, 2014). As students express their ideas of the count of the coins, a teacher is given the opportunity to have students present a logical method for their estimation process. Kazemi and Hintz (2014) have as their fourth principle of classroom talk the idea of teachers making sure that all students are sense makers and that student ideas are valued. Therefore, if student A says the sword has 50 coins and student B says the sword has 1,000 coins, a conversation of method, justification, and reasonableness of answer should follow.

The Continued Chain of Wonder

I cannot help but think of some more questions in respect to this sword. I would ask students to come up with questions of their own for which the answers would satisfy their curiosity. Here are my questions of curiosity.

- How heavy is the sword?
- Was the sword designed to be art work?
- How big is the biggest coin sword ever made?
- Are all the coins the same kind of coin? Are there swords made up of multiple kinds of coins?
- How does the diameter of the coin relate to the height of the coin?
- What if I made coin swords out of pennies, dimes, nickels, quarters, and silver dollars, how would these swords relate to each other? How would the lengths of the swords relate? How would the weights of the swords relate? How would the value of the swords relate?
- How much yellow and red thread is woven through the sword? Is the length of the yellow and red thread the same? Is it different?
- How does one make a coin sword? Where and how does one start the process?

This is just the beginning of my list of curiosity questions. Do you have any questions?

Another Sword and the Curiosity It Created

I did a little additional research on coin swords and found a posting where an individual was looking to sell a Chinese coin sword. Here are the images. This sword was sold for $50. What a deal!

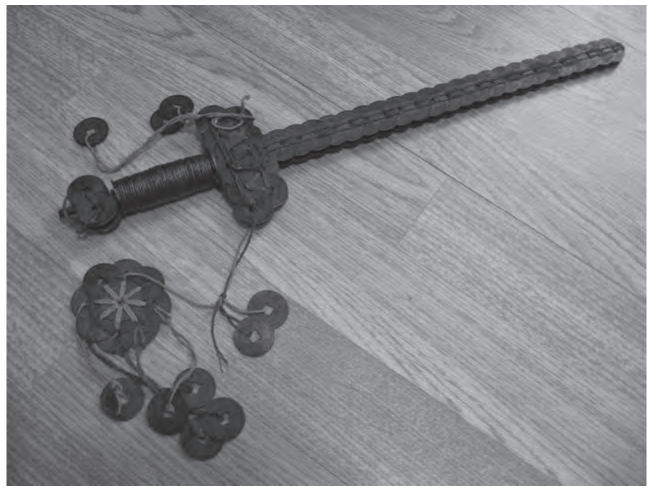

Figure 2. A different coin sword.

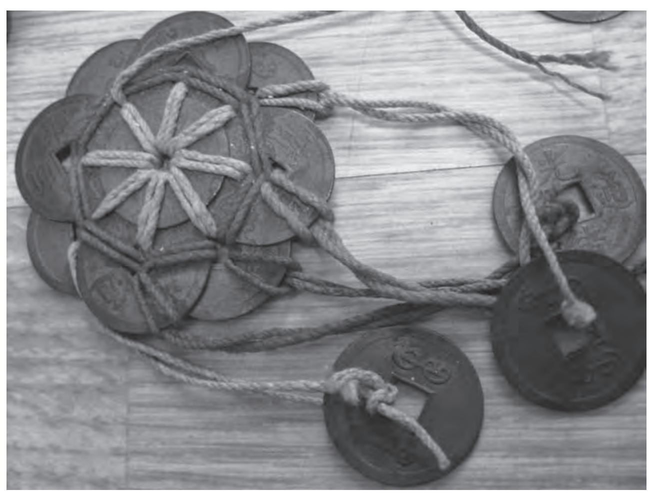

Figure 3. A close-up image of the second coin sword.

However, more wonderful questions came to mind.

- If the entire sword is worth $50, how much would each individual coin be worth?
- Is there any significance to the additional blue thread in this sword?
- In the close-up image, I can see that a square is cut out of the middle of the coins. How does this influence the weight of the coin?

What do I notice?

- I notice that the additional tassel piece in Figure 3 has some wonderful geometric attributes (symmetry, octagonal red pattern, eight central angles, the octagon is regular).
- It appears that the tassel piece forms a 1, 4, 4 pattern with the number of coins. Is there another single coin on the other side? Then the pattern is 1, 4, 4, 1.
- What if the tassel piece added another layer of coins? What would the pattern be then? How many coins in the tassel piece with another layer? Would the pattern become 1, 4, 6, 4, 1? Pascal's triangle?

A sword like this is a wonderful confluence of history, art, and mathematics. A certain power and authenticity come into play when multiple disciplines of content are found within an activity. As a result, I look forward to my next trip to the museum with my family to find more curious artifacts. I am willing to bet that the mathematical applications will be intriguing and wonderful.

Benefits of a Noticing and Wondering Activity

The benefits of an activity focused on noticing and wondering is that every student can participate, regardless of their current mathematical understandings. This sword and all of the mathematical possibilities it contains, allows all students to access the math. As Boaler (2015) has noted in reference to a successful math class, "(Students) talked of many different activities such as asking good questions, rephrasing problems, explaining ideas, being logical, justifying methods, representing ideas, and bringing a different perspective to a problem" (p. 67). This activity is filled with all of these qualities of an engaging mathematics classroom. Moreover, a noticing and wondering activity puts the student as the leader in the educational venture.

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Students Engaging in Mathematical Practices: *As the Gears Turn*

By Lindsay Reiten, Zekiye Ozgur, and Amy Ellis, University of Wisconsin-Madison

Ratios, proportions, and linear relationships are among the essential concepts that students study in middle school, making up the foundational ideas supporting students' progression into more advanced mathematical topics. However, many students experience difficulties in developing a robust understanding of these concepts and making connections between them. Research in mathematics education (e.g. Ellis, 2007; Smith & Thompson, 2007) suggests that students can develop a better understanding of algebraic relationships when they reason about real-world quantities. Quantities are individuals' conceptions of measurable attributes of objects or events, such as length, area, volume, or speed. Working with quantitatively rich contexts can allow students to explore a situation, investigate and make sense of the relationships between quantities, and test conjectures. The Common Core State Standards for Mathematics (CCSSM) (NGA/CCSSO, 2010) also emphasize modeling quantitative relationships with mathematics. In this article, we present examples from a classroom implementation of a unit called *As the Gears Turn*, in which students identified, explored, and represented linear relationships through studying gear ratios.

We also highlight some of the mathematical practices advocated in the CCSSM, which we observed during the unit's implementation. Specifically, we focus on the following three mathematical practices: (a) *Reasoning abstractly and quantitatively*, in which students make sense of quantities and their relationships in problem situations; (b) *Modeling with mathematics*, in which students identify important quantities in practical situations, analyze their relationships mathematically, and interpret results in the context of the situation; and, (c) *Constructing viable arguments and critiquing the reasoning of others*, in which students justify their conclusions, communicate them to others, and respond to the arguments of others.

The *As the Gears Turn* unit serves as an introduction for students to begin forming ratios, rates of change, and determining whether data are linear. Ultimately, the goal of this unit is to help students understand that a relationship is linear if the rate of change of one quantity compared to another is constant. The examples of student work that we share here come from an 8th grade mathematics classroom at a public school. The class worked on the unit for ten days. Students worked with physical gears, identified the relevant quantities (teeth and rotations), and explored the relationships between those quantities. They began with gears with simple teeth numbers: 8 teeth, 12 teeth, and 16 teeth. As students progressed, they began to rely less on physical gears and shifted to new problems that included more challenging gear pairs, e.g., gears with 5 and 7 teeth, as well as describing and representing different types of situations involving gear pairs. Below we share two tasks, Gears Task 4 and Gears Task 5, from the unit. These tasks occur at the beginning of the unit, therefore the examples we share demonstrate how students are being prepared to meet the ultimate unit goal. We hope that teachers will find these tasks useful for enriching their own linear functions units.

Prior Task Information

Prior to the tasks presented below, students spent time investigating the gears, exploring how the gears rotated, how to keep track of the number of rotations two connected gears make, and how to determine the relationship between the ratio comparing the number of teeth on the connected gears and the ratio comparing the number of rotations the gears make, i.e., the reciprocal relationship between the "teeth ratio" and the "rotation ratio." For example, the "teeth ratio" for gears with 8 and 12 teeth is 8:12, or 2:3, whereas the "rotation ratio" for this same gear pair is 3:2 (meaning that for every three rotations the smaller gear makes, the larger gear will make 2 rotations).

The main goal for Tasks 4 and 5 (see Figures 1 and 4) is for students to understand why the gear ratio, or the relationship between the numbers of rotations two connected gears make, is always constant. Sub goals of the tasks include: creating a ratio by forming a constant multiplicative comparison, understanding and generating equivalent ratios, and constructing a constant rate of change by thinking of infinitely many equivalent ratios. Additionally, these tasks begin to lay the foundation for students to recognize that a relationship is linear when either Quantity A is always n times Quantity B or the ratio of the change in A to the change in B is constant.

Gears Task 4

Task 4 presents students with two uniform tables (see Figure 1) and asks students to determine (a) whether all of the entries in each table are correct, and (b) how the two tables relate to one another. Notice that the entries in Dottie's table are created by doubling the previous entry whereas the entries in Jesse's table are incremental, in which an increase of one half a rotation for the small gear results in an increase of one third of a rotation for the big gear. Both tables have affordances. Dottie's table is helpful for noticing the constant ratio between the number of rotations of the small and big gears (i.e., 3:2), and Jesse's table is helpful for highlighting the covariation of the number of rotations that the small and big gears make (for every one half rotation of the small gear, the big gear rotates one third of a full rotation).

Students worked on the task in small groups first and then discussed their ideas as a whole class. During the small group work time, students developed and shared several different explanations for why they thought the entries all came from the same gear pair. Up to this point students had been engaging in quantitative reasoning by identifying the quantities involved in the task situation, i.e., teeth and rotations, and trying to make sense of the relationship between these quantities. For example, Gus recognized the pattern in Dottie's table, i.e., the entries are multiplied by 2 (see Figure 2), while Hope described the same pattern in terms of the relationship of the two gears spinning together, i.e., they covary. Hope said that the entries were correct "[b]ecause each time the small gear turned 3 times the big gear turns 2 times... and all of them [the entries in the table] reduce to 3/2" (see Figure 3). Although Gus and Hope recognized different patterns, and thus provided different explanations, their teacher encouraged both students to share their ideas with their peers.

Gears Task 4

1. While rotating two gears, Dottie came up with this table of rotation pairs:

Small	Big
3	2
6	4
12	8
24	16
48	32
96	64
192	128

Do you think every entry in the table is correct? If so, why? If not, which ones are not correct?

2. While rotating two gears, Jesse came up with this table of rotation pairs:

Small	Big
1	$\frac{2}{3}$
$1\frac{1}{2}$	1
2	$1\frac{1}{3}$
$2\frac{1}{2}$	$1\frac{2}{3}$
3	2
$3\frac{1}{2}$	$2\frac{1}{3}$
4	$2\frac{2}{3}$

How is Jesse's table related to Dottie's table?

3. Come up with pairs of gear rotations that are NOT present in either Dottie's table or Jesse's table. Do any others exist, or is this it?

Figure 1. Gears Task 4.

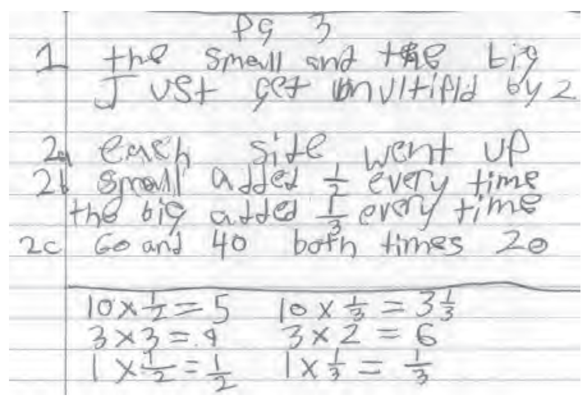

Figure 2. Gus's work for Gears Task 4.

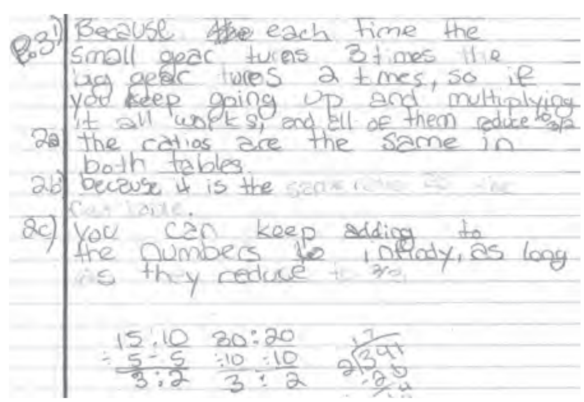

Figure 3. Hope's work for Gears Task 4.

When discussing how the tables related to each other (Question 2), several students commented, "The tables were the same, just different numbers." As demonstrated in the following excerpt, the teacher (Ms. L) encouraged students to *model with mathematics* and to *construct viable arguments* by pushing them to elaborate on their explanations and make connections to the gears context.

Hope: Seems like the same thing.

Ms. L: Well that's important to notice, though. Some people don't notice that.

Hope: Can I just write like it's the same thing?

Ms. L: You can but then you have to justify why you know it's the same thing.

Hope: Because this term's the same.

Ms. L: So what about the numbers can prove your opinion that it's the same thing? So when you say it's the same thing, what do you mean?

Later on,

Ms. L: It's not very clear about what is exactly the same, because it's not exactly the same. (Points to something on Hope's sheet.) Like these are different numbers.

Leo: The [ratio's the same].

Hope: [The ratio's the same].

Ms. L: Oh! OK. The ratio of what?

Leo: The fractions. Like the small to the-

Ms. L: -There you go.

In Question 3 students are asked to come up with other pairs of gear rotations that are not present in the tables. Asking students to generate equivalent ratios may help them understand that there are infinitely many equivalent ratios, thus encouraging students to reason abstractly. Teachers can then build on this understanding to help students understand that it is the constant rate of change that defines linear relationships. However, it is important to note that students who can generate some new rotation pairs may not necessarily understand that they can create infinitely many pairs, or that each pair will represent the same ratio. Therefore, eliciting students' ideas and asking them to justify their conclusions is an important part of deepening their understanding as well as supporting students' abilities to reason abstractly and construct viable arguments.

In Task 4, students know that each entry is supposed to represent a pair of rotations of the same gear pair, but there is a possibility that Dotty made an error when keeping track of the rotations. Therefore, in this task, students primarily develop a strategy to check each entry. For example, students may try to check the entries by actually rotating the gears the amount given in the tables, rather than trying to find a structural relationship. Although this strategy works for the first three rows, the remaining entries become cumbersome to rely on physically rotating the gears. Thus, Task 4 initially allows students to use concrete tools (gears) if needed, but also pushes students to move beyond thinking with the concrete tools to think more abstractly, which is needed for Task 5.

Gears Task 5

In an effort to further extend students' understanding of the constant gear ratio, Gears Task 5 (see Figure 4) presents a non-uniform table of rotation pairs, which encourages students to reason abstractly and quantitatively. Although evaluating whether all entries come from the same gear pairs (Task 5) and deciding whether each entry is correct (Task 4) require the same kind of reasoning, the prompts may have a different *feel* for students. Tasks 4 and 5 both encourage students to construct ratios, however, Task 4 enables students to initially use their gears (if they choose) before necessitating the need for constructing ratios, i.e., when students begin working with the larger numbers. On the other hand, there is more ambiguity in Task 5 because there is a possibility that some entries come from different gear pairs. The only way for students to know whether the entries come from the same gear pair is to determine if there is a constant ratio between the rotations that the small and big gear make. The prompt does not tell students to make a ratio, but the nature of the task encourages them to form a ratio each time in order to determine whether each pair comes from the same set of gears.

The following table contains pairs of rotations for a small and a big gear. Did all of these entries come from the same gear pair, or did some of them come from different gears altogether? How can you tell?

Small	Big
$7\frac{1}{2}$	5
27	18
$4\frac{1}{2}$	3
16	$10\frac{2}{3}$
$\frac{1}{10}$	$\frac{1}{15}$

Figure 4. Gears Task 5.

Because the teacher encouraged students to share multiple solution strategies in Task 4, students were able to draw on their own understanding of concepts as well as insights gained from the class discussion to engage in later tasks. For example, although not directed to do so, several students applied a strategy Lewis shared for Task 4 as they worked on Task 5. According to Lewis, for Task 4, he took "half of the smaller number and added it to the smaller" number and checked to see if it was equal to the larger number. Many students figured out that the entries in the table were from the same gear pair. For example, Neelam (see Figure 5) decided that each of the entries came from the same gear pair by determining whether the number of smaller rotations was 1.5 times the number of the bigger rotations. She checked this relationship by dividing the number in the big column by two and adding it to the original number in the big column to get the number of turns the small gear made.

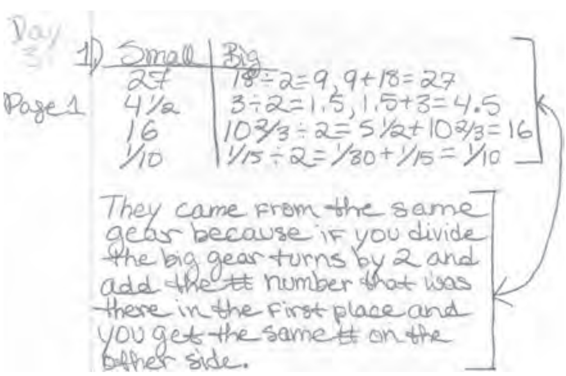

Figure 5. Neelam's work for Gears Task 5.

In contrast, rather than applying the strategy he shared before, Lewis "reduced" each gear pair to the ratio 2:3. Lewis divided the number in the big column in half (which gave him his "2" in the ratio) and multiplied by 3 (mentally) to see if it was equal to the number in the small column. Since each gear pair "reduced" to the ratio 2:3, Lewis concluded that the entries came from the same gear pair.

Throughout the unit, the teacher continually encouraged students to share their ideas and to reflect on the ideas their peers had provided. For instance, during a whole class discussion, after a student shared how she used Lewis's strategy, the teacher asked the class "Okay, why do you think? Why, can someone explain why they think this makes sense? Either explain why it makes sense or ask a question about it." The teacher then built upon the explanations students shared and asked if students had any additional explanations for determining whether the entries were coming from the same gear pair. Eventually this discussion led to students recognizing the reciprocal relationship between the teeth and rotation ratio (see Figure 6).

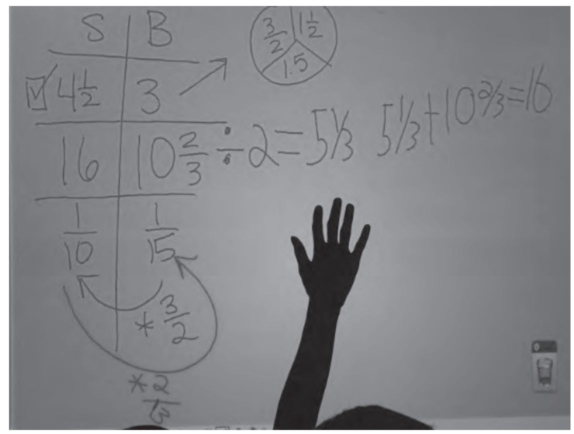

Figure 6. Students shared class discussion about Gears Task 5.

Progression of Tasks

Consistent with what others have advocated for (e.g., Day, 2015; Hiebert et al., 1997), the mathematically rich tasks in *As the Gears Turn* unit foster student reasoning by providing opportunities for: reflecting, communicating, and engaging in the mathematical practices advocated for by CCSSM. As the tasks progress, the prompts encourage students to rely less on the physical gears and reason abstractly as they work with more challenging gear pairs, e.g., gears with 5 and 7 teeth. The students were supported and encouraged (by the teacher and tasks) to model quantitative relationships with mathematics and to construct viable arguments. For example, students identified important quantities in the gears context and the relationship between them, represented and connected those relationships, and drew, shared, and justified their conclusions.

Additionally, the tasks build off each other by continually reinforcing and extending concepts as students progress from tasks focusing on forming ratios to tasks requiring the development of equations that represent various gear situations. For instance, the goal of Gears Task 20 (see Figure 7) is for students to understand how to determine the relationship between the rotations of the gear pairs for situations that can be modeled with y=mx and y=mx+b. Prior to this task, students are introduced to y=mx+b situations by exploring contexts in which one gear rotates a number of times on its own before the second gear joins it and then the gears rotate together. The task begins by having students determine whether all of the entries in the table come from the same gear pair, and then asks students to justify their conclusion. These prompts are similar to the ones given in Gears Tasks 4 and 5. However, to further extend students' thinking and build from previous tasks, students are now also asked to describe the gear situation(s) that might have generated the entries. Students may choose to describe the situation in words and/or algebraically, thus providing different possibilities for how to engage with the task. For students who are ready, the teacher may also encourage them to express the relationship algebraically.

The following table represents pairs of gear rotations:

A	B
2	0
10	5
16	10
18	$11\frac{2}{3}$

1. Do all of the pairs come from the same gear combination? How can you tell?

2. Describe the gear situation(s) that generated these pairs.

Figure 7. Gears Task 20.

Closing Remarks

The tasks in the *As the Gears Turn* unit require students to identify the quantities involved in the problem context and investigate and describe the relationships between the relevant quantities (the number of teeth on each gear and the number of rotations the gears make). With the support of the teacher, the students engaged in various mathematical practices that are advocated for by CCSSM. The students reasoned abstractly and quantitatively as they made sense of the quantities and the relationships between them, modeled with mathematics by analyzing the relationships between different gear pairs, and constructed viable arguments to justify their conclusions, communicating their ideas and responding to each other's arguments. We hope that mathematics teachers find the tasks and ideas shared in this paper helpful for their own classrooms. We invite teachers to visit the following website, https://sites.google.com/site/badgerellis/sparq, for the complete set of tasks, along with a description of the goals of each task, suggestions for implementations, and some alternative tables and tasks.

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Making the Most of Mathematical Modeling

By Ellen M. Junko, Kelsey M. Jensen, Nicole A. Brooks, Courtney L. Bliss, Andrew K. Bonlender, Peter Nugent, Jordon C. Stevens, Brent A. Witkowski, Meghan A. Ricci, Scott S. Spear, Christopher S. Hlas (PhD.), Sherrie Serros (PhD.)

According to Cai, et al. (2014), "Mathematical modeling itself is an essential skill that all students should learn in order to be able to think mathematically in their daily lives" (p. 5). Because of its connection to our daily lives, we believe that mathematical modeling develops and challenges the minds of students in both the real world and the mathematical world. The Common Core State Standards for Mathematics (CCSSM) also includes mathematical modeling in the requirements for all levels of mathematics. The CCSSM Standard for Mathematical Practice for modeling states, "Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace" (NGA/CCSSO, 2010, p. 7). Real world problems open students' eyes to the fact that mathematics is in the world around them.

In addition, CCSSM states that "modeling is best interpreted not as a collection of isolated topics but in relation to other standards" (2010, p. 57). Here we see that mathematical modeling is given special status because CCSSM identifies it as both a *mathematical practice* and a *content standard*. What about mathematical modeling requires such special treatment? To address this question, we read and analyzed research about mathematical modeling as part of a directed studies course at the University of Wisconsin–Eau Claire (http://www.uwec.edu). This article details the results of this work and offers recommendations from our findings.

What is Mathematical Modeling?

Mathematical modeling is a cyclic process of using mathematics to represent and better understand real world situations (Blum & Ferri, 2009; Hodgson & Harpster, 1997; Munakata, 2006). The process begins when a question is asked about the world outside of mathematics (See Figure 1). After finding the essential features and variables of the situation at hand, this problem is then translated into mathematical terms and enters the "math world" (Ang, 2010). Next, mathematical computations and analyses are applied to the mathematised problem. Before bringing the model back into the real world for reporting, the solution must be validated. If the answer is not reasonable, then from the validation step, one can go back to any previous step to determine how to address the issues.

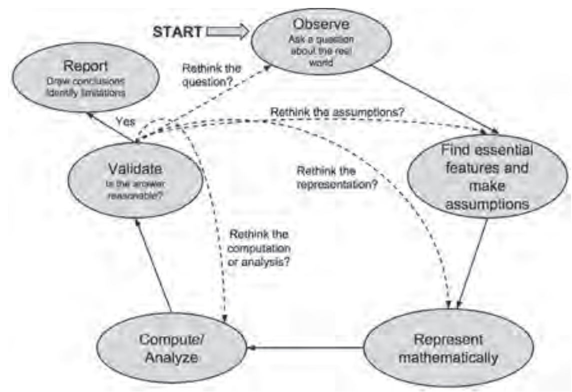

Figure 1. The process of mathematical modeling (based on class discussion).

The process described sounds like a combination of other mathematical techniques, such as problem solving and applications. As a result, one question our class asked was, "What are the differences between modeling, problem solving, and applications?" Applications can typically be found at the end of a chapter. They are carefully defined so students can apply recent computation techniques, typically in a forced context. Problem solving involves non-routine problems that are well-defined and typically have a specific solution. Modeling, however, involves open-ended problems that often do not have all relevant contextual information (see Figure 2 for examples).

	Key features	Examples
Exercises	Computation without context. Allows students to practice techniques for long-term retention.	Let x+5=7. Solve for x.
Applications	Exercises with guided context. Encourages students to find patterns in exercises with real world context to encourage near/far transfer.	Gladys and Prudence have seven watermelons together. If Gladys has 5 watermelons, how many watermelons does Prudence have?
Problem Solving	Non-routine problems. Provides students practice understanding problems without a specific solution technique. (Note: Once a student solves a problem-solving problem, it can become routine. Our example is likely routine for a student in calculus, but novel for a pre-algebra student.)	Three pumpkins and two cantaloupes weigh 32 lbs. Four pumpkins and three cantaloupes weigh 44 lbs. What is the weight of two pumpkins and one cantaloupe?
Modeling	Cyclic process of using mathematics to represent and better understand real world situations. Enables students to ask mathematical questions and find mathematical solutions to real world problems.	"The floor plans of airline terminals vary widely in design and are quite dissimilar. Which design is optimal for operations? Develop a mathematical model for airport design and operation. Explain how it would operate" (Munakata, 2006, p. 31).

Figure 2. Problem type descriptions.

Our conclusion is that mathematical modeling includes aspects of problem solving (the process) and applications (contextual situations), which is why it can be difficult to distinguish between the problem types (see Figure 3). Further, the modeling process allows students to expand their thinking in ways that general problem solving or application activities would hinder (Verschaffel & De Corte, 1997).

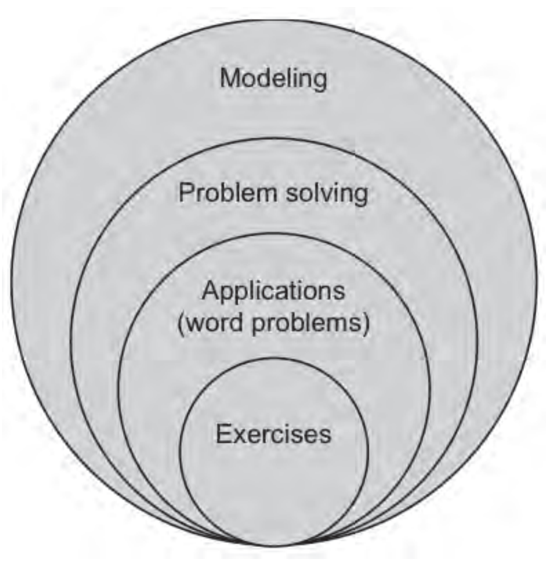

Figure 3. Relationships between problem types (based on class discussion).

Why is Mathematical Modeling Important?

In addition to the inclusion of mathematical modeling in CCSSM, there are several other reasons for implementing mathematical modeling. For example, modeling can promote students' curiosity through real world problems such as the following: "A school needs to replace two flagstaffs because they are aging. The two flagstaffs need to be measured (they are different lengths). Please provide a manual that tells us how to measure the length of any flagstaff" (Kang & Noh, 2012, p. 11). By posing real world problems that might relate to the students such as this, teachers have a less difficult time motivating students. Further, the test of the solution to the previous problem can be done by measuring the actual height of the flagstaff rather than checking the back of the book. Thus, the curiosity and real-life application that mathematical modeling elicits can create a better learning environment for students and teachers alike.

Along with this new curiosity, the fear of failure can also cause some students to shy away from modeling, or mathematics in general (Hodgson & Harpster, 1997). Failure in school is terrifying to students, teachers, and administration alike, but mathematical modeling encourages perseverance

(CCSSM, 2010, p. 6). Modeling encourages students to try many different approaches when one approach does not work, and it encourages students to work outside their comfort zones (Ang, 2010). If a modeling mentality is learned within mathematics classes, students may be less afraid to fail or try different approaches in other disciplines and after their schooling years.

Mathematical modeling also allows students to have more control over their learning (English, Fox & Waters, 2005) for several reasons. First, students are encouraged to ask their own questions. Problem posing creates a mind that is not told what to do, but instead a mind that is limitless with questions and possibilities (Ang, 2010). Second, mathematical modeling allows students to devise their own solutions, which promotes the valuable skill of drawing from one's own knowledge to find a solution.

Yet another benefit is that mathematical modeling can encourage interdisciplinary approaches and co-teaching (Cai, et al., 2014). For example, a mathematical modeling problem that prompts students to predict the frequency, velocity, and size of a hurricane requires knowledge of both hurricanes and the mathematical equations that go along with the involved science (English, Fox, & Waters, 2005). Incorporating mathematics with other subjects not only allows for the learning of two subjects at once, but it also shifts the focus from mathematics to another topic. For some students, this shift could make a task more interesting and less intimidating. Overall, the interdisciplinary nature of mathematical modeling leads to all varieties of thinkers being able to succeed (Kang & Noh, 2012).

Challenges for Teachers in Mathematical Modeling

Despite the many benefits, there are some barriers to using mathematical modeling in classrooms because modeling requires a lot of student involvement and input. Blum and Ferri (2009, p. 54) alluded to some potential difficulties a teacher might encounter in a modeling activity:

1. Maintaining a balance between student control and minimal instruction;
2. Knowing different methods of intervention for helping students throughout the modeling process; and,
3. Presenting concepts to students using various models.

Along with these difficulties, our class discussions prompted us to add more challenges to the list:

4. Maintaining consistent grading that still promotes independent thought from the students;
5. Choosing problems that are meaningful to the students, while still providing worthwhile mathematical content; and,
6. Planning how students will use technology, if needed, as a tool.

How Can We Overcome the Challenges?

Despite the many challenges modeling can pose, we believe it is worthwhile and can improve a student's education both in mathematics and other disciplines (Sole, 2013). Therefore, understanding how we can overcome the challenges and teach mathematical modeling is the next step. Before diving into a modeling activity, it is crucial for a *classroom dynamic centered around group work* to be established. According to a study done by Kaiser (2005), students value working in groups during the modeling process because they are able to critique their ideas and methods as a group, allowing them to move through the modeling process more fluently. Thus, developing a classroom that is welcoming to group work may make mathematical modeling easier for students.

The next step is to develop or find *mathematical modeling problems that are meaningful to the students.* Understanding who students are and what interests them is important to the modeling process. A teacher must ask questions that are meaningful to the students so they are motivated to find solutions. Ang (2010) provides an example of how a teacher can design a problem that is related to students' lives by constructing a modeling activity that asks if the flood protection mechanism at their school was efficient in casting off heavy rainfall or not. With mathematics, students were able to discover the best location to place the water catching mechanism. Finding problems similar to this one, where students can make a difference with their answer, is an important motivational tool that teachers can utilize when developing questions for mathematical modeling.

As the teacher, it is also important to *encourage different ways of going about problems.* As discussed earlier, mathematical modeling is a process designed for multiple answers and multiple ways of thinking. According to Blum and Ferri (2009), individual routes and multiple solutions should be encouraged, which requires the teacher to have a very thorough understanding of the content. If a teacher has one way of going about a problem ingrained in their mind, different methods could be disregarded. Thus, it is important for the teacher to keep a flexible mind and be prepared to see many different methods.

In addition, it is important for teachers to *relinquish some control to the students*. Blum and Ferri (2009) explain that, "When treating modelling tasks, a permanent balance between maximal independence of students' and minimal guidance by the teacher ought to be realised" (p. 54). Letting students create their own models while providing them with the tools to succeed is a delicate balance required for mathematical modeling.

As noted earlier, the cyclic nature of mathematical modeling is a key feature of the process. However, students are not likely to follow this cyclic format without guidance. Verschaffel and De Corte (1997) presented one approach to encourage validation of a mathematical model. Students were presented with four questions that involved the same arithmetic operation but yielded different answers due to the real-world context. In the end, students responded with comments such as, "The most important thing is that once you have finished the computational work, you still have to adapt the answer to the situation" (Verschaffel & De Corte, 1997, p. 591). This activity *encouraged cyclical thinking* by having students discover its importance themselves.

Encouragement for validating answers can also provide guidance in cyclical thinking. Questions such as "Does this make sense?" greatly impact a student's thought process by making them pause to question their work instead of just stopping at the answer. Being able to reflect on and pose questions about their work is an important skill for students to learn. As students become more comfortable with validation, they will begin to validate without being prompted (Kaiser, 2005). Another approach can arise from the idea of teaching failure. Kaiser (2005) notes that students should "experience the feelings of uncertainty and insecurity, which are characteristics for real applications of mathematics in everyday life and sciences" (p. 4). Not only are these feelings realistic, but they are important for being comfortable questioning, validating, and modifying models. One of the reasons validation is important is that it is not realistic to assume one's first attempt at a model is completely accurate. Therefore, students must become comfortable not getting everything right on the first try. One idea for building this comfort is engaging students in modeling activities that produce unexpected results. This, in turn, helps develop a cautious attitude in students (Hodgson & Harpster, 1997).

Assessment

Mathematical modeling can be a challenge in terms of assessment because many times there are not distinct right or wrong answers. To assess these problems for their value in a system, peer evaluation is helpful. For example, when students can describe their mathematical model to their peers, they are demonstrating a deeper understanding of the modeling process (Verschaffel & De Corte, 1997). In addition, teachers can work together with other colleagues in the department, or outside of the department, to provide collaborative feedback. Such feedback can help the models be more valuable to a child's learning (Munkata, 2006). Kang and Noh (2012) provide the following suggestions for rubric criteria:

- Accuracy - does the output of the model provide correct or near correct results based on the question and assumptions?
- Descriptive realism - are essential features and appropriate assumptions the basis for the model?
- Precision - does the model predict discrete numbers (or other definite kinds of mathematical entities: functions, geometric figures, etc.) or an imprecise range of numbers (or a set of functions, a set of figures, etc.)?
- Robustness - how does the model handle errors or outliers in the input data?
- Generalizable - does the model apply to a wide variety of situations?
- Fruitfulness - are the model's conclusions useful or do they encourage the use of other models? (p. 3)

For example, when looking at the airline terminal problem presented in Figure 2, the idea of *accuracy* may be the number of planes that are able to fit in the given area or the number of people that they can move in and out of the airport terminal. *Descriptive realism* may refer to the design the students choose to make their terminal. *Precision* may be being able to measure exactly how many planes the students can fit in their design. *Robustness* could be assessed by looking at whether planes might crash into each other entering or leaving.

Generalizable may show that the students have planned for more than one situation that could occur in the terminal. *Fruitfulness* would be if solutions could be applied to other airport terminals. There are many ways that the students could go about thinking of this problem. These are just a few of the options that could be included when looking at the problem and could be used as the basis for a rubric to ensure completeness in a student's model.

Conclusion

There are several ways that mathematical modeling can be taught. There are different techniques that may work better than others. To summarize and extend the above text, we have created the following quick tips (Figure 4).

In conducting our research, our group had many discussions regarding what is and what is not mathematical modeling. In the end, we decided that two criteria would separate mathematical modeling from other types of mathematics problems. First, mathematical modeling is a cyclic process that requires iteration (or revision) of ideas. Students must not think that their first attempt necessarily yields their final answer. Second, it is important that the process begins with student questions about the real world.

Having addressed the benefits and challenges, the final big question for our group was to think about how modeling might look in a classroom. When and how to structure modeling activities within a lesson or unit depends on the structure of the course and the goals of the modeling activity. Interdisciplinary courses, or school-wide thematic units in which mathematics and other courses are fully integrated, appears to be a natural course structure to promote modeling (Cai, et al., 2014). If interdisciplinary collaboration is not possible, then integrating real-life problems into each lesson could be used to motivate mathematical content (e.g., Coxford et al. 2003). Less integrated forms of modeling include: an "end of class" idea where modeling is used to review and apply recently learned mathematics material, including small projects within a class that seem totally separate from regular curriculum, and separating mathematics and modeling into distinct courses.

Through the semester, we created a Google Doc to collect real-world problems that might make good modeling problems. We end with some possible prompts to encourage mathematical modeling in your classroom:

- Should UW–Eau Claire build a parking ramp?
- If you open a gallon of milk before its "sell by" date, how long before the milk goes sour? Does it depend on how many times you remove it from the refrigerator or the temperature of the refrigerator?
- How far away should trash cans be? (Walt Disney reportedly did this for Disney Amusement Parks)

Quick Tips

- Encourage group work.
- Help students understand the scenario that is put in front of them.
- Find problems that are relevant to students.
  - Group students by similar interests.
  - Find scenarios that are related to the school or other things in the community.
- Encourage multiple strategies and multiple answers.
- Relinquish some control to students.
- Do not give students the answers.
- Encourage cyclical thinking.
  - Ask students if something makes sense.
  - Display incorrect work on the board, letting students know it is okay to make mistakes.
  - Explain why checking the validity of an answer is important.
  - Check a model in a different scenario to make sure it applies to a wide variety of situations.
- Use technology as a tool, not as an answer key.
- Use peer assessment to judge students' models.
- Encourage student creativity by avoiding the use of descriptive rubrics.

Figure 4. Quick tips.

• How can you predict whether or not there will be room for you on the bus?

• How are sale prices computed at a grocery store?

• How fast could Ebola spread in the US?

• What should you pick for paper, rock, and scissors? (best two out of three)

Good luck, and happy modeling!

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Dynamic Geometry Software Meets the Pythagorean Theorem

By Nicholas J. Gilbertson & Kevin Lawrence, Michigan State University

Two middle school students from different classes were having a conversation near a classroom door about their recent lesson on the Pythagorean Theorem from different teachers. Charlie was telling Horace, "It's pretty simple, it's just $a^2 + b^2 = c^2$." Horace inquired, "Doesn't it matter what *a*, *b*, and *c* are?" Charlie responded, "No, you just need to find the answer, as long as you know square roots, it's easy." This brief conversation really illuminated some common student issues. This made us wonder if Charlie had any sense of how to apply the Pythagorean Theorem except in neatly packaged practice problems, and whether he had any intuition of why the Pythagorean Theorem works beyond the numerical calculation. Most troubling, Charlie's perspective on this theorem is likely more typical in mathematics classrooms—that the Pythagorean Theorem is simply a way to practice calculations that are more sophisticated than linear equations.

The Pythagorean Theorem holds a unique place in mathematics. It is one of the earliest examples in the middle grades curriculum where students have the opportunity to engage in a complex algorithm that has been around for thousands of years. It is a beautiful example of the relationship between numeric quantities, and geometric lengths and areas. Young students can understand and apply the relationship, yet extensions are sufficiently rich to engage mathematicians for centuries (see for example Wiles, 1995). Its presence is so ubiquitous in society that numerous references exist in pop culture from the Scarecrow in the Wizard of Oz (Leroy & Fleming, 1939) to a Shaquille O'Neal press conference (Getz, 2011). Yet, testing data (National Center of Education Statistics, 2009, 2011) and classroom practice have highlighted issues with students' ability to apply and understand this theorem appropriately. In this article, we share some of the issues that we have noticed in our own teaching of middle grades and high school mathematics, and explore one activity that proved useful in developing a stronger understanding of the Pythagorean Theorem.

Issues with Understanding the Pythagorean Theorem

It always came as a surprise to us that after students had solved problems in the unit or lesson on the Pythagorean Theorem that they were less consistent in future units with being able to (a) identify situations where the theorem could be correctly applied, and (b) correctly apply it. These types of issues were mostly related to misapplication of the formula. In some cases, this meant that students would not recognize that given two sides of a right triangle, that the third side could be found using the Pythagorean Theorem. A second related issue is confounding the variables by treating a leg as the hypotenuse in the formula, such as in the student error given in Figure 1 below.

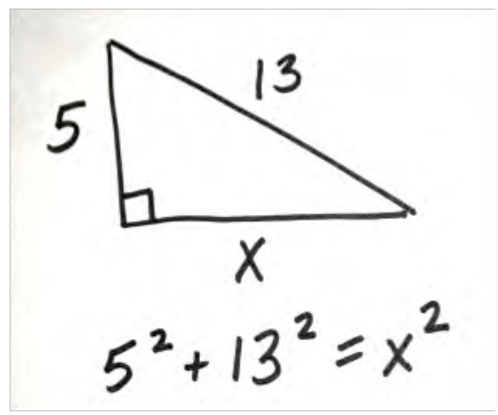

Figure 1. A common student error.

A second set of issues relate mostly to students understanding the Pythagorean Theorem and its limitations. For example, some students have a difficult time understanding the relationship between a geometric representation of the theorem and its algebraic-symbolic representation (see Figure 2) that goes beyond simply being able to calculate values. Some students may also overgeneralize the theorem to include all triangles, not just right triangles. These last two issues are particularly problematic if students are going to understand meaningfully a proof of the Pythagorean Theorem, which relies on a right triangle, and understanding areas of squares in relation to side lengths. These ideas are also important for understanding the converse of the Pythagorean Theorem, that is, if ΔABC (where $a < b < c$) has the relationship that $a^2 + b^2 = c^2$, then ΔABC is a right triangle.

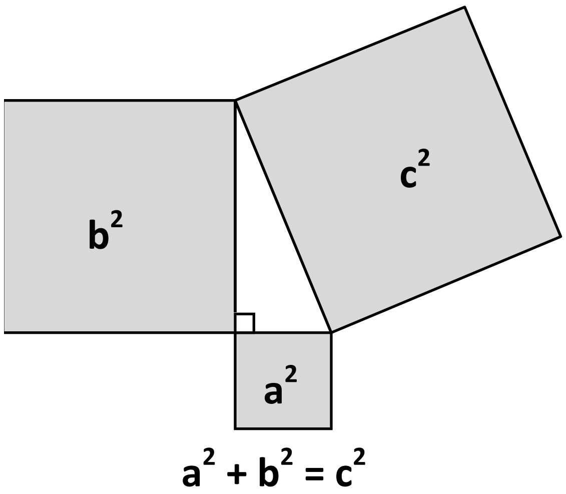

Figure 2. Relating the geometric and symbolic representations

To limit the difficulties related to understanding and misapplication of the Pythagorean Theorem, the first author created an activity for his students to investigate this topic. In relation to mathematical practices, this task focuses primarily on the first Standard for Mathematical Practice (NGA-CCSSO, 2010): *Make sense of problems and persevere in solving them*. In this task, students actively consider the relationship between the general theorem and special cases by conjecturing and investigating their conjectures. In relation to content, the broad goals for students were to have some enduring understanding of the Pythagorean Theorem where they can (1) identify a situation and know how to apply it correctly and (2) have an intuitive sense of why the theorem works. By "enduring" we mean that students are able not only to solve problems within a particular unit where the Pythagorean Theorem is the focus of a lesson, but also perform well on problems involving this topic in future units and coursework. Put together, these goals will hopefully position students to restrict the theorem to only right triangles and better understand a proof of the theorem.

The Task

The mathematical task described in this section uses GeoGebra (GeoGebra.org), a free online dynamic geometry software program . There are many benefits in using GeoGebra in this problem context. First, the software can help students see multiple cases quickly and efficiently. Second, it allows students to focus on relationships they notice without being burdened by calculations because GeoGebra will calculate values such as areas of squares. Third, it provides a venue for students to independently explore and extend what was discussed as a whole class.

This lesson makes use of the Launch-Explore-Summarize (Lappan et al., 2014; Shroyer & Fitzgerald, 1986) format of mathematics instruction. In this format, the teacher *launches* the problem by introducing students to the context with a whole-class discussion that promotes student talk about both the mathematics and the context. Next students independently, or in small-groups, *explore* the problem. During this time the teacher is monitoring students as they work, observing how they are solving the problem and asking probing questions (Smith & Stein, 2011). Finally, the teacher *summarizes* student thinking by conducting a whole-class discussion based on how students thought about and solved the problem. In this lesson, GeoGebra is used in two ways, as part of the whole-class discussion and as part of independent exploration .

Launch

The lesson begins with students looking at the following image. The class is asked to be as descriptive as possible with all relevant information they see on the screen in Figure 3. For example, students may notice the particular areas of the three squares, or the measure of the angle. They may also see the text above the figure that describes the combined areas of the squares. Other students might notice the dashed semi-circle or the white triangle that shares a side with each of the squares.

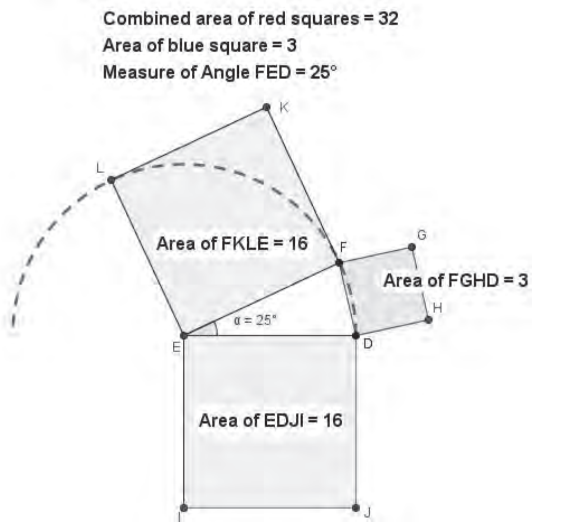

Figure 3. Launching the problem.

Once students have described as much as they can about the figure, point F (the upper left corner of the blue square) can be moved along the path of the semi-circle. Ask students, "If we move point F counterclockwise a little bit, what do you think will change about the figure and what will stay the same?" Students will have a variety of conjectures. Figure 4, shows this movement of point F. Of note is that the measure of ∠FED increased from 25° to 40° and the area of square FGHD increased to approximately 7.49 square units. The areas of the other two squares remained the same, however, because EF and ED are both radii of the semi-circle.

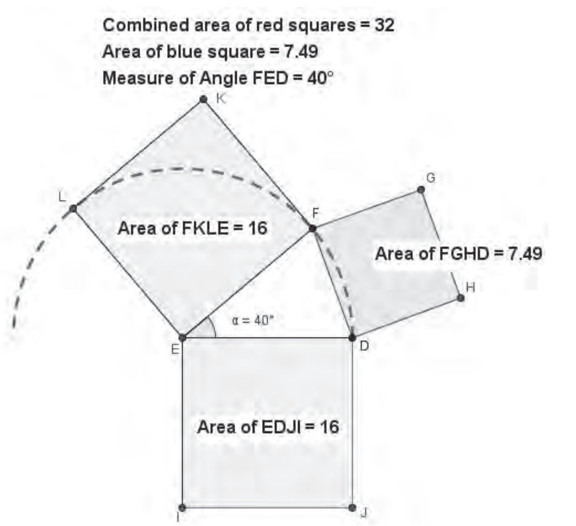

Figure 4. Moving point F counterclockwise.

Next, ask students what would stay the same and what would change if point F is moved further to the left along the path of the semi-circle. Again, students will have a variety of conjectures. Figure 5 below shows this movement. Similar to the last case, the two red squares have the same areas, the measure of ∠FED increased, and the area of the blue square increased. At this point, students should have a general sense of what aspects of the figure will stay the same and what will change when point F is moved on the semi-circle.

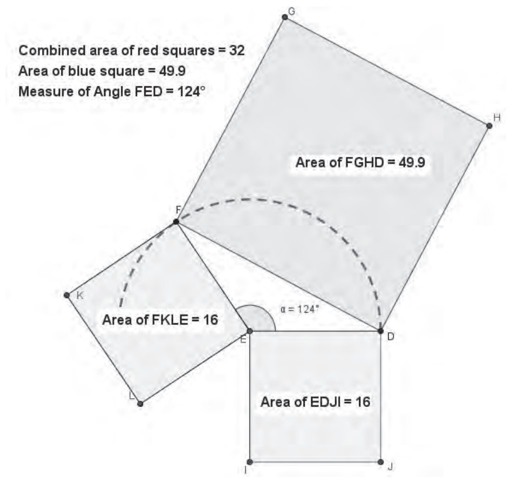

Figure 5. Moving point F further counterclockwise.

An important relationship to notice at this point is that the area of the blue square is less than the area of each red square when the measure of ∠FED was smaller (such as in Figures 3 and 4). In contrast, the area of the blue square is greater than area of each red square for a larger measure of ∠FED (such as in Figure 5). One question to have students explore at this point is: *Is it possible for all three squares to have the same area? If so, where would this occur?* This equivalence occurs when ΔFED is equilateral as in Figure 6.

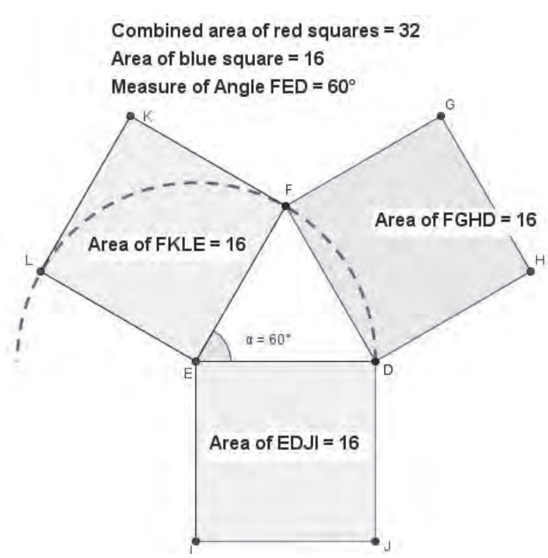

Figure 6. Equal areas of squares.

The purpose of this question is to support students as they explore relationships between areas of the squares. When the first author taught this lesson, students were able to determine quickly that there would be only one situation where the areas of the three squares were identical. Students were then asked, "As point F moved along the semi-circle, the combined areas of the two red squares has always been 32 square units. Is there a location for point F, where the area of the blue square will also be 32 square units? If not, why not? If so, where should point F be located, and how many different locations will result in an area of 32 square units?" These follow-up questions are less intuitive, but more important for understanding the Pythagorean relationship.

These questions generated a fair amount of discussion as students talked with their peers in small groups and with the whole class. Although some students conjectured that there was no way to guarantee that a square with area of 32 square units existed, a typical conjecture was that the square should exist because the area grew continuously as F moved around the semi-circle. Additionally, the area of the blue square was occasionally (as in Figure 6) less than 32 square units, and was at other times (as in Figure 5) greater than 32 square units. Only a few students had any idea where equality might occur, with even fewer students conjecturing that the relationship would hold true when ∠FED was a right angle. Point F was moved slowly around the semi-circle until students saw the relationship given in Figure 7.

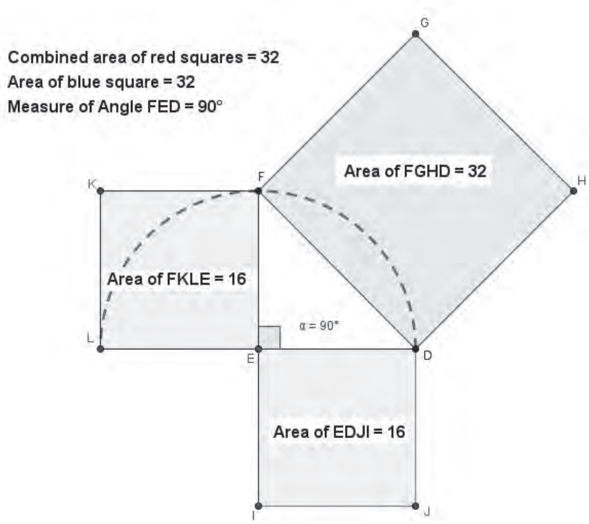

Figure 7. The right angle case.

At this point, it may be easy to think that the lesson is finished, because students have discovered the Pythagorean Theorem. What can be generalized from this figure, however, is not that clear. Figure 7 indicates that the relationship holds true in the case of an isosceles right triangle where both legs are 4 units long. Students were asked if they thought that there were other triangles where this same relationship holds true (i.e. the combined areas of the red squares equals the area of the blue square). Conjectures varied from believing this one triangle was uniquely special, to saying the property holds only for right triangles, only for isosceles triangles, or only for isosceles right triangles. Many students were unsure about whether this held true in any consistent way.

Explore

The question of whether other triangles would satisfy the relationship was the motivation for the exploration portion of the lesson. At this point in the semester, students were familiar enough with GeoGebra that constructing a triangle and squares while having GeoGebra calculate areas was straightforward. Students investigated their own conjectures and those of their classmates to verify empirically if any types of triangles maintained the Pythagorean relationship. After about 25 minutes of exploration time, the class came back as a whole to discuss what they found.

Summarize

Some students had determined that the relationship held true for all isosceles right triangles. This is an interesting case to begin with because it is correct, but is not as general as possible. Students showed examples that verified their results. Other students then argued that being isosceles did not matter and having a right triangle was sufficient. They also showed examples of non-isosceles right triangles that verified their results. The main question that came out of this discussion was why non-right triangles could not work. One student argued that it was for a similar reason why the relationship held true only when point F was moved so the resulting triangle was a right triangle. If F is moved so that ∠FED is acute or obtuse, and two of the squares (built off the legs) keep the same area, the third square's area will have to change.

Reflecting on the Task

Put together, the insights about the cases where the Pythagorean Theorem worked and when it did not supported students in better understanding the relationship and deductive arguments that established the general nature of the theorem for right triangles, which were a focus of subsequent lessons. Having taught lessons on the Pythagorean Theorem several times, what was noticeably different about using this task in a dynamic geometry environment was the lasting effects on student understanding of the relationship in future units. First, when students encountered situations that required them to apply the Pythagorean Theorem, their language describing the numerical process was anchored frequently in geometric language. Instead of saying, "well a is 3, and b is 4, and I have to find c", students would say, "the area of this square is 9, plus the area of this square is 16, which means the area of this other square is 25." This shows some evidence that students were able to re-present the image mentally when needing to apply this theorem. Second, because students explored all different types of triangles, it was rare to see students apply the Pythagorean Theorem in non-right triangle cases. This is perhaps due to the fact that the exploration focused on the three results of the Pythagorean relationship not just the right triangle case (Figure 8). As a result of engaging in this task, students seemed more proficient in being able to apply and understand the theorem, and gain valuable insight into understanding the geometric nature of a proof of this theorem.

For ΔABC (with $a \leq b \leq c$), if …

$a^2 + b^2 > c^2$, then ΔABC is an acute triangle

$a^2 + b^2 = c^2$, then ΔABC is a right triangle

$a^2 + b^2 < c^2$, then ΔABC is an obtuse triangle

Figure 8. Pythagorean equality and inequalities.

Using dynamic geometry software such as GeoGebra proved useful in supporting students' ability to generalize this important theorem, but not at the expense of over-generalization. Students actively engaged in sense-making and persevering as they worked through the problem in determining the types of triangles for which the Pythagorean relationship held true. As in most engaging problems, students often engage in multiple mathematical practices at once, such as "constructing viable arguments and critiquing the reasoning of others" (Standard of Mathematical Practice 3) showing the inherent interconnectedness of these standards.

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Exploring Cognitive Demand Online with the WSMI Fellows

By Paige Richards, Mathematics Program Specialist, Brookhill Institute of Mathematics

As noted in *Principles to Actions: Ensuring Mathematical Success for All* (NCTM, 2014):

> *Tasks with high cognitive demand are the most difficult to implement well and are often transformed into less demanding tasks during instruction. (Stein, Grover, & Henningsen 1996; Stigler and Hiebert 2004)*

This past spring, 32 members of the inaugural 2015 WSMI Fellows cohort put this claim to the test. The WSMI Fellows grew out of the Wisconsin Statewide Mathematics Initiative (WSMI, www.wsmi.net), a program of the Brookhill Institute of Mathematics. Since 2012, Brookhill has offered high quality, one week math courses to teachers through its WSMI summer Institutes. By the end of the summer of 2014 there were a number of teachers who had taken between two and four WSMI courses and were looking for a new challenge. In response, Brookhill designed a course that would take the content learning from WSMI to the next level, the classroom.

This first cohort of the WSMI Fellows, educators from all across Wisconsin, participated in an online semester long course. This impressive group of educators has teaching experience ranging from 9 to 20+ years. The Fellows collaborated as they identified, created, and implemented high cognitive demand tasks; examined student work; and made instructional decisions based on the student work.

The online course consisted of both synchronous and asynchronous sessions. There were four live sessions (synchronous). The asynchronous sessions consisted of weekly assignments and discussions that participants could join anytime during the week. Both formats allowed for small and large group discussions.

The project goals for WSMI Fellows were:

1. Continue developing teachers' understanding of the content and pedagogy that was introduced in the WSMI modules.
2. Deepen teachers' knowledge of effective teaching practices through the use of high cognitive demand tasks to increase student success in mathematics.
3. Form a collaborative network of mathematics instructional leaders in Wisconsin to engage in the process of inquiry and experimentation.

Rather than presenting a comprehensive picture of the work of the Fellows, this article focuses on the second project goal, specifically the use of high cognitive demand tasks. In what follows, I begin with a definition and discussion of cognitive demand. I then explore the collaborative process that was the heart of the Fellows experience. Finally, insights from the Fellows about implementing tasks with high cognitive demand are shared.

What is Cognitive Demand?

To understand cognitive demand, Brookhill used the work of Smith and Stein (1998), who have delineated characteristics of mathematical tasks into four levels of cognitive demand. These four levels are:

- Lower-level demands (memorization)
- Lower-level demands (procedures without connections)
- Higher-level demands (procedures with connections)
- Higher-level demands (doing mathematics)

For more details on these levels see Smith and Stein's (1998) *Selecting and Creating Mathematical Tasks: From Research to Practice.*

Stein, Smith, Henningsen, and Silver (2009) describe cognitive demand as, "The kind and level of thinking required of students in order to successfully engage with and solve the task" (p.1). The course focused on cognitive demand because students experience the highest learning gains when they are engaged in mathematical tasks that require high levels of reasoning and thinking (Stein & Smith, 1998). To help translate this research into practice, Brookhill used NCTM's *Principles to Actions* (2014).

Principles to Actions describes eight mathematics teaching practices, one of which requires that teachers *implement tasks that promote reasoning and problem solving*. This mathematics teaching practice is focused on cognitive demand and is grounded in the work of Smith and Stein (1998). For that reason, the practice of "reasoning and problem solving" drove the WSMI Fellows coursework.

Framing the Work of the WSMI Fellows

The central focus of the cohort during the first few weeks of the course was to establish an understanding of cognitive demand. The Fellows engaged in deep conversations about the thinking required of students using six common mathematical tasks. These math tasks were studied and discussed, then participants examined the Smith and Stein levels, and finally participants re-examined the tasks with the levels side by side. Participants had discussion groups that focused on the tasks and engaged in a cyclical examination of the levels of demand, constantly analyzing the tasks in relation to the Smith and Stein levels.

This cyclical process coupled with deep conversation using evidence from the tasks to ground the discussion deepened the understanding of cognitive demand. Without this necessary and challenging first step, the rest of the work would not have been possible. When working in isolation the level of demand can seem deceptively simple to identify. Through discussion with colleagues and opportunities for student trial, the understanding of cognitive demand deepened.

The Fellows were ready for the next step, selecting tasks to use with their students. Through the online platform, participants were placed in small grade band groups to create support networks and grade level expertise. Participants each selected a task they planned on using in their classroom, determined the level of demand, and shared the task and determination in their small group. Then group members weighed in and had discussions about the task and the level.

Figure 1 is an excerpt from a discussion thread where a participant, Charlie*, who is a math coach, posted his task.

Example First Grade Task

There were some bunnies in a yard. Three bunnies hopped away. There were four bunnies left in the yard. How many bunnies were in the yard at first?

Maria:

"Compare problems are so tough for our younger students! I'm wondering what are your thoughts on supporting students to access the problem while still maintaining the cognitive demand of the task?"

Charlie:

"One way I think we can maintain the cognitive demand of the task while still supporting them to access the problem is by keeping the focus on the process they are using to make sense of the problem. The other way is to focus on making sure they are modeling their thinking clearly and that their model actually matches up with how they solved the problem."

Stephanie:

"I am wondering what takes this situation to the cognitive demand level of 'doing mathematics' vs. procedures with connections? How do you support the students while working on this problem and keep the cognitive demand level at this level? I liked Mike's idea of using the actual students and chairs as 'tools.'"

Charlie:

"I really struggled with rating this problem. I initially rated it as procedures with connections. Then after talking with the classroom teacher, we decided to rate it as doing mathematics because we felt the students have not been given any explicit pathway or approach to solving the problem. The only explicit instruction that they have been given is only to make sense of a problem and persevering in solving it. Again, not sure how confident I am in rating this as doing mathematics."

**Participant names have been changed*

Figure 1. Discussion thread from WSMI Fellows.

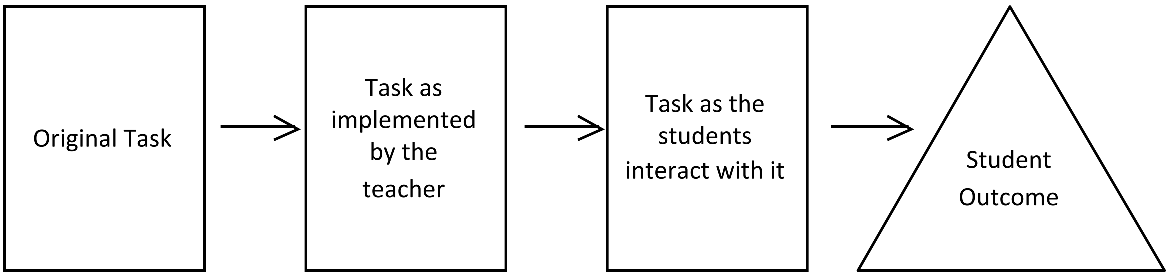

Figure 2. The Mathematics Tasks Framework adapted from Stein and Smith (1998).

To assist participants as they planned for task implementation, Brookhill created a Task Template (Figure 2) based on a framework from Stein and Smith (1998). Stein and Smith's *The Mathematics Tasks Framework* is a representation of how mathematical tasks unfold during classroom instruction.

The template for the Fellows was designed to bring awareness to each stage of the implementation process. Below are the three stages in the template.

1. Before task implementation

2. During task implementation

3. Reflection after task implementation

For each of the three stages of task implementation, Brookhill's template included questions and statements for the Fellows to consider. The intent of the questions was to provide prompts for the detailed thinking that is necessary for teachers to engage in before each of these three stages of implementation. This thinking should address questions such as, "What is the learning intention?", "How will you know when a student is successful?", and "What are the anticipated challenges?"

Mathematical Tasks (Before Task Implementation)

Each year there are more and more resources that provide teachers with access to high quality mathematics tasks. Many of these different resources were used by the WSMI Fellows. Yet the further into the course they went, the more the Fellows developed an understanding of the four levels of cognitive demand and found that they didn't have to search as hard for these high cognitive demand tasks. They could change the level of demand of any task in front of them.

The Fellows found that the time spent making sense of and debating the levels of cognitive demand for different mathematics tasks empowered them to analyze other tasks. This, in turn, allowed the Fellows to look at other tasks that involved a lower level of demand and to see the potential to increase the cognitive demand of any task. What was important was the understanding of what makes a task cognitively demanding.

As one participant noted:

> *I think since the beginning of the teaching profession, teachers have always been looking for those 'perfect' problems that would invoke deep student thinking. I know I have been on an unending quest of searching, buying, and creating these types of problems. I never felt quite satisfied with my results. I now finally realize (lightbulb moment) that even though I continually was searching for higher cognitive demand problems, I didn't really have a strong grasp of what that meant. After taking this class and doing more reading on it, I think I am beginning to understand what this truly means.*

Another participant said when describing altering worksheets in the mathematics program she uses:

> *With examining and prioritizing the mathematics content and mathematical practices I can comfortably look at what the purpose of the worksheet is and turn the concept and purpose into a task that promotes reasoning and problem solving...I can take the concept from the worksheet and organize and orchestrate a high cognitive demand task.*

Here a participant describes the evolution of the Fellows over the course of the semester:

> *The first time we had to choose a task, many teachers in the course just found a task that was already high level. But, as the course went on, teachers took it a step further and made modifications to existing tasks used in their*

classrooms. We found activities that were on the lower end of cognitive demand (memorization or procedures without connections) and changed components of them to make them higher level tasks (procedures with connections and doing mathematics). This was so powerful. As teachers, we realized we didn't have to reinvent the wheel or throw away all of our current teaching materials. We could take what we have and just make modifications that both increase the level of cognitive demand and engage our students. … This wouldn't have been possible without the rich discussions that were going on throughout the course. We gave feedback to each other on how we could improve our tasks before we implemented them through the Task Template.

Based on this information, it would seem that it is not simply finding, modifying or creating tasks that is valuable but really taking the time upfront to deepen the understanding as to what makes some tasks more cognitively demanding than others. The practice that was found to be most valuable for this work was to collaboratively look at mathematics tasks while going back to the levels of cognitive demand. This collaboration surfaced the understanding that *Principles to Actions* describes, "In determining the level of task, it is important to consider prior knowledge and experiences of the students who will be engaged in the task." (p. 22) It was these discussions and conversations that deepened everyone's understanding.

Maintain and Implement (During Task Implementation)

The next part of Brookhill's Task Template was titled during task implementation. This portion of the template asked teachers to describe how they set up the task and to rate the level of demand. Then the teacher described how students interacted with the task and again, rated the level of demand. Many participants noted that they had not taken into consideration the teacher's role in terms of altering or maintaining the cognitive demand of a task.

Some further thoughts from two participants:

The challenge that continues to jump to the front of my mind is ME. I have been trying to incorporate more tasks for a while and as I read the articles for class and participate in the activities, I continue to see how some of the tiniest things I am doing continue to reduce the cognitive demand. I love the idea of videotaping or having a partner teacher observe on a regular basis. Recognizing the problem is the first step and now it is time for real change.

Finding the tasks isn't that hard - they are literally all over the place. You can find them on the internet, in textbook series, down the hallway, on YouTube, etc. What IS challenging is how you implement the tasks, how you modify it for your students to pique their interest while setting them up for just the right amount of productive struggle. Then when implementing the task, the teacher has to tie his/her hands behind his/her back and ask Socratic questions tailored to each small group's needs while at the same time maintaining classroom discipline and differentiating for the students who arrive at a viable conclusion earlier.

It turns out that the during task implementation part of the template was probably the most valuable. The act of anticipating and reflecting on how the task was set up and the interaction with students during the task forced participants to pay careful attention to the role of maintaining, increasing, or decreasing the level of cognitive demand.

Reflection as a Part of the Process (Reflection After Task Implementation)

Stein and Smith (1998) remind us that reflection is a critical factor in teachers' professional growth and that "cultivating a habit of systematic and deliberate reflection may hold the key to improving one's teaching as well as to sustaining lifelong professional development." (p. 268) These thoughts led Brookhill to make reflection after task implementation the third component of the Task Template. This portion of the template provided participants with an opportunity to reflect on different parts of the task implementation and offered the richest reflection focused on the level of cognitive demand.

Here is an excerpt from a Task Template Reflection:

What did you do that led to task maintenance or task decline?

Task maintenance – I was conscious not to give away solution pathways or answer questions too readily. I tried to answer questions with questions.

Task decline – I selected a problem that provided a pathway. I should have modified that before implementing. I think I was doing too much "teaching" during the task. When students didn't have the procedural skills down and I tried questioning or giving clues, but they still didn't get it, I stepped in.

Many of the Fellows' reflections were similar. As the course progressed, the reflections became more celebratory. This became a space to acknowledge that the cognitive demand was maintained! The more the Fellows planned for and anticipated their role in maintaining the cognitive demand, the more successful they were in this goal.

Conclusion

The work of the 2015 WSMI Fellows cohort reinforces the importance of giving professional educators the freedom and opportunity to think in a collaborative environment with challenging tasks. The experience showed that implementing tasks that promote reasoning and problem solving is a bigger challenge than it appears. The act of making that happen is a high cognitive demand task. Teachers need ample time to really dig in and make sense of cognitive demand. They need to integrate knowledge of cognitive demand into their everyday practice.

Time to collaborate matters! To deeply understand cognitive demand, teachers need to talk about what cognitive demand IS and build that understanding collaboratively.

One of the WSMI Fellows summarized the course experience as follows:

Another big component of this course was analyzing how we could keep a high level of cognitive demand throughout the entire implementation of a task. Principles to Actions suggests that teachers should "support students in exploring tasks without taking over student thinking" (NCTM, p. 24). Through course discussion forums, we all agreed that this was hard for us! As teachers, we tend to do anything possible to help our students find success within a task. However, that often leads to us taking over student thinking by providing too many scaffolds, or worse, taking their pencil away (I was guilty of that!). We realized that this was something we needed to address if we really wanted to keep a high level of cognitive demand. To combat this, we were given the Task Template which had us reflect on student misconceptions and how we would address them before we taught the lesson. This specific type of preparation taught us the importance of scaffolding only when students need it (more "on the spot" scaffolding) rather than giving them too much help and bringing down the demand of the task. Also, we realized that it's important to prepare questions ahead of time that can be used to help students persevere when they were stuck. This truly transformed how I teach math. I realized that my students had way more "ah ha" moments when I questioned them rather than when I gave them hints when they were stuck. The cognitive demand stayed high throughout the task because I made sure the thinking was being done by the students. Through these course activities, we (WSMI Fellows) were able to support students without taking over their thinking.

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Engaging Our Students in Upper-level Mathematics: The Power of Parabolas

By Dave Ebert, Oregon High School

In the Spring of 2014, the National Council of Teachers of Mathematics (NCTM) released *Principles to Actions: Ensuring Mathematical Success for All*. This publication calls for moving "from 'pockets of excellence' to 'systemic excellence' by providing mathematics education that supports the learning of all students at the highest possible level" (p. 3). In order to achieve this goal, all teachers must look for ways to engage their students through the discovery of mathematical concepts and the application of mathematics in our world.

When planning a unit, all teachers need to focus on the needs of our students. Do the lessons engage our students? Are our students actively involved in the discovery of new material? Are our students exposed to the applications of the mathematical concepts in other subject areas and/or in the world around us? Teachers can teach any mathematical topic in a teacher-centric manner, or in a student-centric manner that answers all of these questions and leads to student engagement and retention of subject material. This article will focus on ways to engage our students in a unit on parabolas.

The Common Core State Standards for Mathematics (NGA/CCSSO, 2010) includes many examples of content standards related to parabolas that students must know. These content standards, which are included in the Number and Quantity, Algebra, Functions, and Geometry conceptual categories, include:

- Solve quadratic equations with real coefficients that have complex solutions. (N-CN7)
- Factor a quadratic expression to reveal the zeros of the function it defines. (A-SSE3a)
- Complete the square in a quadratic expression to reveal the maximum or minimum value of the function it defines. (A-SSE3b)
- Use the method of completing the square to transform any quadratic equation in x into an equation of the form $(x - p)^2 = q$ that has the same solutions. Derive the quadratic formula from this form. (A-REI4a)
- Solve quadratic equations by inspection (e.g., for $x^2 = 49$), taking square roots, completing the square, the quadratic formula and factoring, as appropriate to the initial form of the equation. Recognize when the quadratic formula gives complex solutions and write them as $a \pm bi$ for real numbers a and b. (A-REI4b)
- Graph linear and quadratic functions and show intercepts, maxima, and minima. (F-IF7a)
- Derive the equation of a parabola given a focus and directrix. (G-GPE2)

Standards on their own do not inspire and excite our students to want to learn more; that is the role of the teacher with a well-designed unit plan. By connecting the mathematics of parabolas to the students' individual experiences, students are intrinsically motivated to extend their experiential and theoretical knowledge of parabolas.

Real-World Applications of Parabolas

Without realizing it, many teachers and students use a three-dimensional paraboloid daily. The cross section of a headlight of a car is a parabola. Rotating the parabola around the axis of symmetry gives a paraboloid. There is a light bulb at the focus, and the paraboloid is a mirrored reflector. Although the light from the bulb disperses in every direction, the parabolic reflector will focus the light in an outgoing beam (Figure 1).

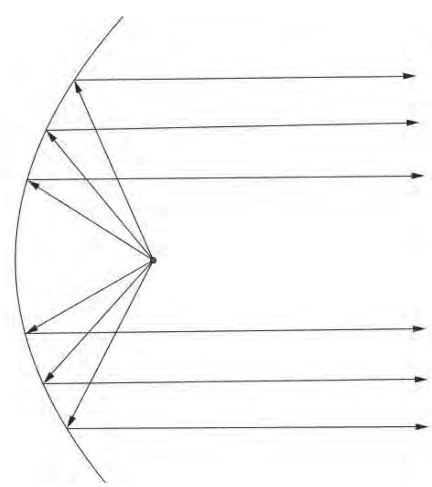

Figure 1. The parabolic reflector within a headlight.

A similar idea works with a satellite dish. Parallel waves are transmitted from a satellite to the dish, focused onto a receiver, and translated into a television signal. With both of these examples, the proper location of the focus is imperative.

Likewise, the Olympic Torch begins its journey every four years by being lit using a parabolic reflector (Figure 2). From Wikipedia:

> The Olympic Torch today is ignited several months before the opening ceremony of the Olympic Games at the site of the ancient Olympics in Olympia, Greece. Eleven women, representing the Vestal Virgins, perform a celebration at the Temple of Hera in which the torch is kindled by the light of the Sun, its rays concentrated by a parabolic mirror. The torch briefly travels around Greece via short relay, and then starts its transfer to the host city after a ceremony in the Panathinaiko Stadium in Athens.

Figure 2. Lighting the Olympic Torch. (source: Wikipedia)

A brief video of the torch lighting can be viewed on YouTube at the following URL: https://youtu.be/U2gbuOGcpzg.

Parabolic solar cookers work in a similar manner. The sun's rays reflect off the parabolic surface to a focus, where water is boiled in a pot or food is cooked in a pan. There is a 15-meter diameter solar bowl in Auroville, India that can cook enough food for 1,000 people per day. At least two humanitarian organizations, the Jewish World Watch and Solar Cookers International, provide parabolic solar cookers to people in need throughout the world.

Parabolic troughs are also used to harness the power of the sun. A parabolic trough is a three dimensional shape with parabolas on two ends and a curved rectangular surface connecting these parabolas. The focus is extended parallel to the curved rectangular shape. Power plants, such as the AREVA Kimberlina Solar Facility in Bakersfield, California, use hundreds of mirrors tilted along a parabolic surface to reflect the sun's rays onto a pipeline that lies at the focus of the parabola. A liquid is pumped through this pipeline, and the reflected solar energy heats the liquid to very high temperatures to generate electricity. A brief video of this solar facility can be viewed on YouTube at the following URL: https://youtu.be/XA9RiNu5ZnI.

Applications in the Classroom

Students learn by doing, so teachers should strive to have their students experience the mathematics they learn to lead to long-term retention. To demonstrate the focus of a paraboloid, obtain an old radar dish and cover it with reflective tape (Figure 3). The author announced to his classes that the first student to bring in an old radar dish would receive a candy bar, and one was brought in the very next day. Once covered, take the radar dish outside on a sunny day. Students can place their hands near to the dish without feeling any heat; once their hands are at the focus, the heat is unbearable. In fact, a dry leaf held at the focus will start to burn.

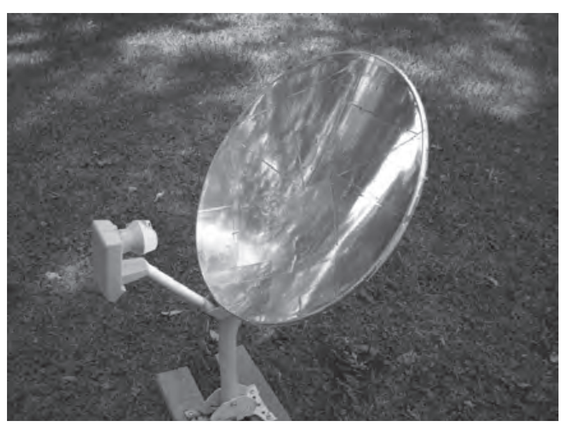

Figure 3. An old radar dish covered with reflective tape.

Another easy way to demonstrate the focus of a paraboloid is with a Fresnel Lens. If one were to take an upward-opening paraboloid and slice it horizontally, then compress those slices onto a flat surface, the result would be a Fresnel Lens. These are relatively inexpensive and are used by farsighted people to magnify text, or by backpackers to light a campfire. Students love using these to light paper on fire, and they demonstrate the focus of a paraboloid very well.

The author has also built a parabolic mini golf hole to demonstrate the focus of a parabola. This was built by drawing a point and a line on a piece of wood, and finding numerous points equidistant from the point (focus) and the line (directrix). These points were connected with a smooth curve, and another piece of wood was cut in this shape. Any golf ball hit parallel to the axis of symmetry will reflect off the parabolic curve and into the hole (Figure 4).

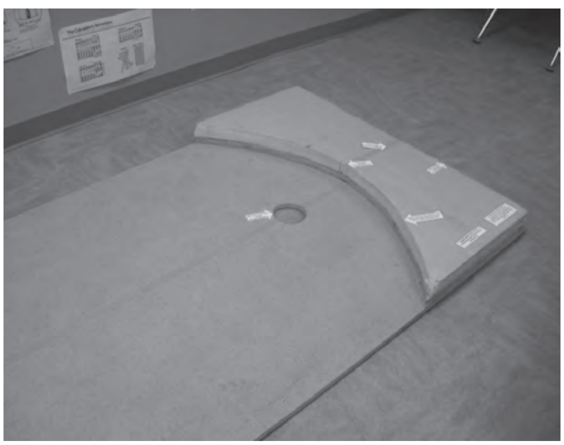

Figure 4. A parabolic mini golf hole.

The best way to have students apply their knowledge about parabolas is to have them build a parabolic hot dog cooker (Figure 5). Given some brief instructions, students can complete this work outside of class time with materials they can find around the house. Here are the steps to build a parabolic hot dog cooker.

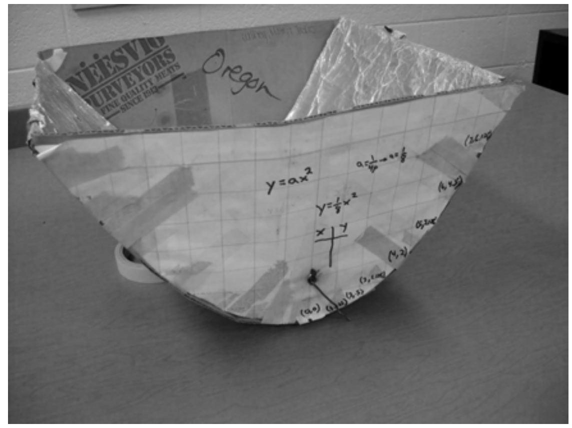

Figure 5. A parabolic hotdog cooker.

1. Determine the distance from the vertex to the focus. Any distance will work, but a distance around two inches works best.
2. Use this distance to find the equation of the parabola. If p is the distance from the vertex to the focus, the equation of a parabola with vertex at the origin is $y = \frac{1}{4p}x^2$.
3. Use the equation of the parabola to generate a table of points, then plot these points on large graph paper, or create a grid on which to plot the points.
4. Glue the graph to cardboard and cut out. Cut out a similar shape for the two ends of the parabolic trough.
5. Cut out a rectangular piece of cardboard using the length of the parabolic piece and a width of about two hot dogs. Score the cardboard by lightly slicing along the grain, and curve to fit between the parabolic pieces. Attach using duct tape.
6. Staple aluminum foil to the curved section.
7. Poke a hole in each parabolic section at the focus. Straighten a metal coat hanger and place it through this hole to use as a skewer for the hot dogs.

Once completed, pick a day to have the students bring in their parabolic hot dog cookers. It is recommended that the class gather in a prominent place so that students can explain the mathematics to any people walking by. Having the teacher provide the hot dogs and the students provide buns, condiments, chips, and beverages gives the perfect opportunity for a class party to celebrate a memorable learning experience.

By connecting the study of parabolas to real-world applications, and having students actively experience the mathematics they learn, students are naturally engaged in their learning. Teaching in a student-centric manner engages all our students and leads to the long-term retention of mathematical concepts.

References

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WMC PUZZLE PAGE

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PERSEVERE
PRACTICE
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PROFICIENT
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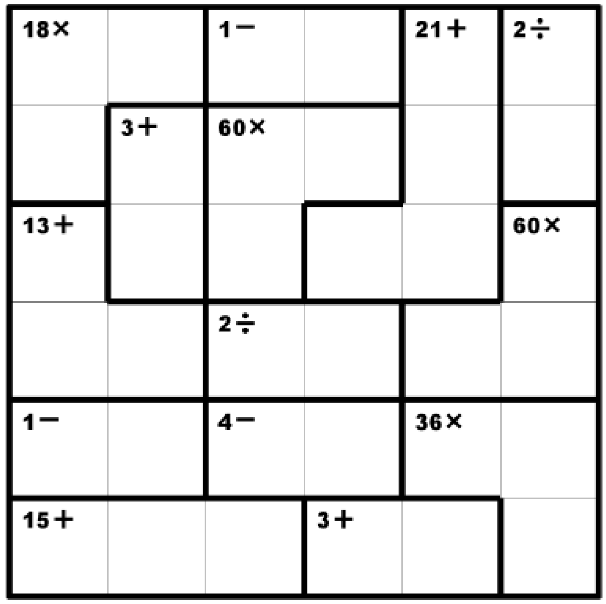
State Mathematics Competition

The following problem is from the 2015 High School State Mathematics Contest. For additional questions and solutions, visit wismath.org/contests

> How many points with integer coordinates are in the interior of the circle centered at (20,15) with radius 3?
> Note: do not include points on the circle itself.

Ken Ken

Fill in the blank squares so that each row and each column contain all of the digits 1 through 4. The heavy lines indicate areas that contain groups of numbers that can be combined (in any order) to produce the result shown with the indicated math operation.



Sudoku

Fill in the blank squares so that each row, each column and each 3-by-3 block contain all of the digits 1 through 9.

4	1		6	8		2		5
			1	5		6		
		5	4					
6	8					9		4
		1				8		
2		9					5	1
					4	5		
		6		3	5			
3		4		1	6		8	2

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