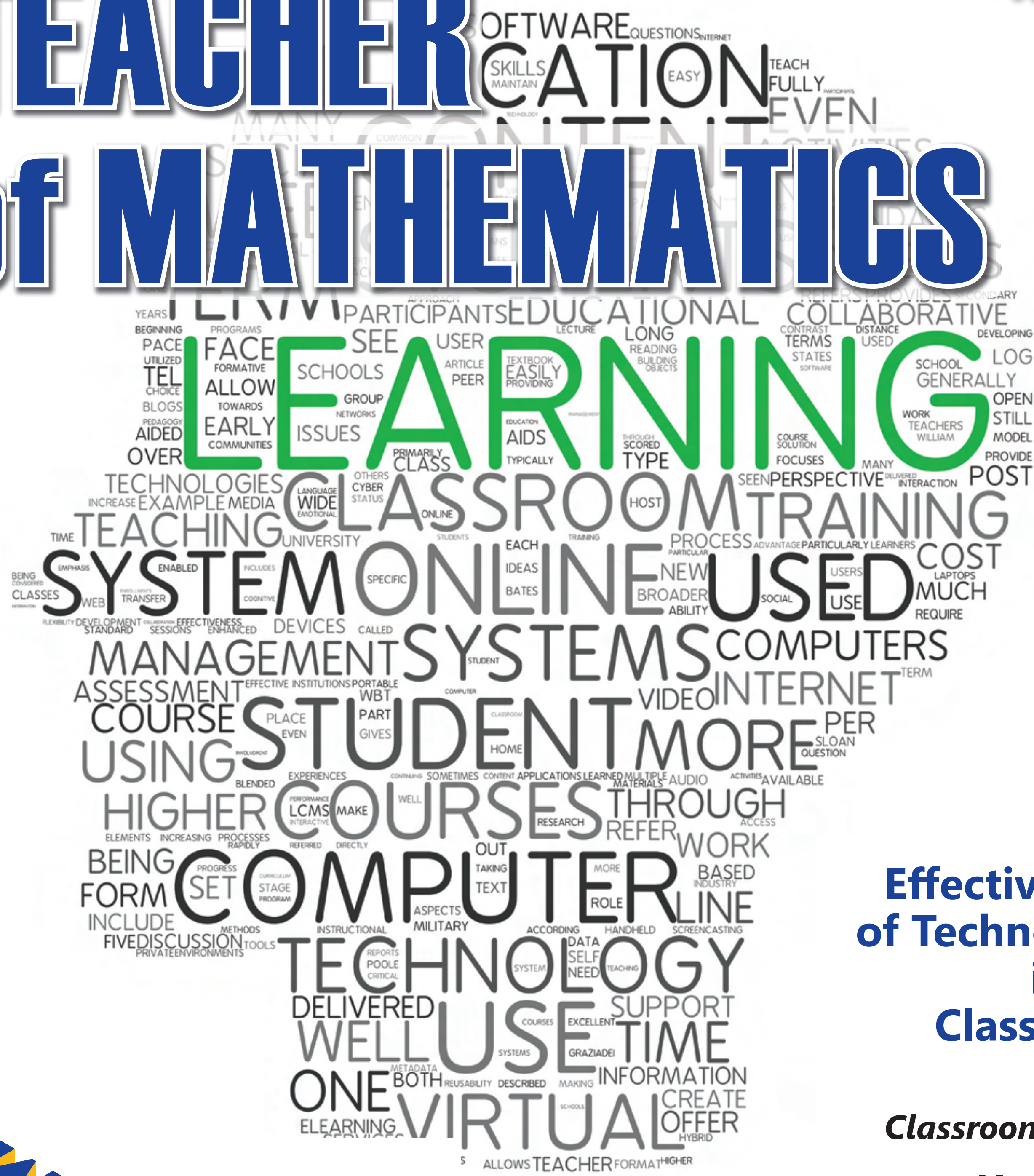
Wisconsin TEACHER of MATHEMATICS

NCTM Outstanding 2013 Publication Award



Effective Use of Technology in the Classroom

A Flip Classroom Model

Meaningful Measurement

Technology Tips

WISCONSIN MATHEMATICS COUNCIL, INC.

Volume 66 | Number 1 | Fall 2014

Wisconsin Teacher of Mathematics
Spring 2015 Journal

The Spring 2015 issue of the Wisconsin Mathematics Teacher will focus on how educators have embedded the Standards for Mathematical Practice into their classroom. The editorial panel would like to showcase examples of rich mathematics tasks and activities that engage students in the habits of mind that we seek to develop in students. We are interested in articles that address the following essential questions:

- What tasks or activities have you implemented that focus uniquely on one or Standards for Mathematical Practice?
- How do you help your students understand the meaning of the Standards for Mathematical Practice?
- How do you assess students' proficiency in the Standards for Mathematical Practice?

If you have ideas or questions for this focus issue or wish to submit an article for review, please visit the WMC website for more information (www.wismath.org/resources/). The submission deadline for the Spring 2015 issue is February 28,

MANUSCRIPT SUBMISSION GUIDELINES

- **Send an electronic copy of your manuscript to the Wisconsin Mathematics Council. Manuscripts may be submitted at any time for review.**
- **Manuscripts should be typed, double-spaced.**
- **Include all figures and photos in .jpg format; submit high resolution copies of figures and student work. Please do not place figures or photos within the document; rather indicate their placement in the document, e.g., Figure 1 here.**
- **All fractions need to be formatted as follows—2/3.** *Do not accept auto formatting of fractions.*
- **All manuscripts are subject to a review process.**
- **Include name, address, telephone, email, work affiliation and position.**

Direct all correspondence to:

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Wisconsin Teacher of Mathematics

The National Council of Teachers of Mathematics selected the Wisconsin Teacher of Mathematics to receive the 2013 Outstanding Publication Award. This prestigious award is given annually to recognize the outstanding work of state and local affiliates in producing excellent journals. Judging is based on content, accessibility, and relevance. The WMC editors were recognized at the 2014 NCTM Annual meeting.

President's Message

Technology + Strategic Planning = Better Teaching

Technology has definitely transformed our culture in many ways. For most workplaces in our society, technological tools are changing how people do their work. Regardless of the setting, the use of new technologies takes place after various degrees of development, planning, and training. The world of education is no different. For mathematics educators, the effective use of technology in the classroom requires thoughtful planning with a clear purpose for the use of any technological tool. Quality mathematical learning targets should always drive why and how different technologies can be used to improve learning. To successfully accomplish this in a climate of ever-changing technology options, educators must have access to quality training that goes beyond the basics of how to use the new technological tools.

For students, insufficient planning can make it appear that a lesson using technological tools lacks any real purpose beyond the integration of technology. This is especially true if the teacher puts all his/her effort into creating a learning activity, with the plan to just hand it off to the students for the learning to take place. In this case, the teacher put a lot into the planning, but needed to go even further. The teacher still needs to make specific plans to set the stage for the intended learning, as well as how he/she will assess and confirm the status of the learning in relation to the given intentions. For this reason, it is essential to explicitly share the learning intentions up front, and then revisit them whenever evidence of their success is demonstrated. Learners can analyze the benefits of using the technological tools if they begin with a clear understanding of the key learning expectations for the lesson. Student feedback and self-assessment can be vital components for determining whether the technological tools were pivotal in achieving the learning goals.

Many districts across the state have been allocating funds to acquire new technological tools in an effort to improve the learning in their schools. I am concerned that many teachers are being pushed to use a variety of new technological tools without any meaningful training on how the technology can be used to transform the learning environment. This may be inadvertently creating environments where teachers are using technological tools for the sole purpose of improving the efficiency of how they deliver content. This is a natural reaction when the funding for quality training does not accompany the funding for placing new technologies into classrooms. I encourage all educators to continually advocate for technology training that focuses on how a given tool can effectively transform learning in the classroom.

I would also hope we would make a push to provide educators with quality collaborative opportunities to develop learning plans where the integration of technology elevates the mathematical learning goals to a higher cognitive level. This may be idealistic, but if we want all students to have a solid foundation of mathematical understanding we need to fight for changes that will assist us in transforming the learning environment.

Good planning is a necessary component for providing successful learning opportunities on a consistent basis. This does not change when we are infusing technology into the learning environment. I would encourage all math educators to become more strategic in their planning when it comes to using technological tools in their classroom. We must continue to grow our awareness of the most effective ways to enhance or transform the learning of mathematics – which includes the integration of technological tools.

Doug

Doug Burge

WMC President, 2013-2015

Editors' Notes

We are excited to bring you this fall's issue with a mix of different articles focusing on technology, classroom activities, and advocacy. In her article, *A Flipped Classroom Model in Middle and High Schools*, Ismail provides a framework for creating a classroom environment designed to meet the diverse learning needs. Ismail outlines guidelines for a flipped classroom as well as practical advice for those interested in trying this model. Nickels describes a mathematics project that incorporates the use of Lego Mindstorms in her article, *Meaningful Measurement: Addressing Equity through STEM*. In this piece, Nickels discusses how this emerging robotics technology can be used to address equity through two mathematical tasks that integrate engineering design. In *Technology Tips: Digital Artifacts in One-to-One Classrooms*, the editorial panel discusses one role that one-to-one devices (e.g., iPads) can play within the classroom.

Several submissions showcase classroom activities that assist students in making relevant connections. With a focus on the elementary grades, Wickstrom and Wessman-Enzinger discuss strategies for connecting visual fraction models to probability in their article, *A New Spin on Fair Sharing*. Connecting mathematics and literacy is the focus of Ebert's article, *The Mathematics of the Fault in our Stars*. Likewise in their article *Examining Formative Assessment*, Hlas, Robach, Fiori, and Spear provide a review of the literature describing the benefits of ongoing formative assessment in the mathematics classroom.

The last two submissions focus on advocacy. In their article, *Computer Science Needs Mathematics Teachers,* Brylow, Gendreau, Kmoch, Kuemmel, and Magiera discuss an opportunity for current mathematics teachers to participate in a National Science Foundation funded project that is aimed at increasing the number of certified computer science teachers in Wisconsin. In a slightly different vein, Steele, in his article, *Let the Pilot Fly the Plane: Advocating for Our Work as Teachers*, discusses perspectives on professionalism within the profession of education, and offers suggestions for educators to engage with educational stakeholders.

Finally, the *Wisconsin Teacher of Mathematics* needs you! Over the last year we have created several new opportunities for involvement with the journal and we encourage members of the readership to seize on these opportunities.

- Write an article for the journal! We encourage submissions on a variety of topics including classroom innovations, teaching tips, action research, and reviews of technology. If you have an idea for an article or questions about submission, please contact us.
- Submit a note from the field (~250 words) in which you provide feedback on journal content, sound off on current issues in education, or briefly highlight a classroom innovation. We hope that this forum can help to promote open discussion about issues and topics in mathematics education within Wisconsin. Notes From the Field can be submitted using the following link http://goo.gl/np0qpN.
- Submit a piece for *Technology Tips* that focuses on the use of technology in the classroom, a project that you have used in your teaching incorporating technology, or a review of technology that you use with students.
- Sign-up to review submissions. Articles are reviewed by members of the editorial panel as well as teachers in the field. We are currently in the process of building a reviewer database and encourage experienced teachers to apply to be part of our reviewer pool.

We hope that you will consider joining us in helping to promote and showcase the wonderful work that is happening around the State of Wisconsin in mathematics classrooms!

Josh Hertel
Jennifer Kosiak
Jenni McCool

WMC Editorial Panel

A Flipped Classroom Model in Middle and High Schools

By Abir Ismail, Ramsey Middle School, Minneapolis, MN

Mathematics teachers usually find themselves between two extremes, struggling students who need more time and advanced students who are ready to move forward. Therefore, the goal is to focus on how to meet the needs of all students. From various experiences in different mathematics classes, teachers realize that every student has his/her own needs, readiness, preferences, and interests. Differentiated instruction enables teachers to plan strategically to meet the needs of each student in today's highly diverse classrooms. Scigliano and Hipsky (2010) noted many benefit for differentiating instruction such as: enhanced self-efficacy, increased content understanding, learner empowerment, increased academic achievement, and inclusion of each student in the learning process.

A flipped classroom is one instructional framework that helps to differentiate instruction and support student learning. The basic idea of the flipped classroom is that what is traditionally done in class is done as homework, and the work that is normally done as homework is done in class. Students watch the lecture on a video and fill in notes at home. The next day, they will do the mathematical practice questions in class. A flipped classroom allows the teacher to differentiate content and process while providing extra time in class to work with every student and meet their individual needs. Before I explain how I implemented a flipped classroom, I will present some research on the benefits of using the flipped classroom model.

Research on the Flipped Classroom

To date, there is no significant scientific research metric to indicate exactly how well flipped classrooms work. But in one survey of 453 teachers who flipped their classrooms, 67 percent reported increased test scores; 80 percent reported improved student attitudes; and 99 percent said they would flip their classrooms again next year (Flipped Learning Network, 2012). There were several benefits that were mentioned from implementing flipped classrooms in school districts across the United States. In the high school mathematics department of the Byron Independent School District in Minnesota, the mathematics teachers redesigned the curriculum and created their own lessons and materials through video lessons. They noticed that academic achievement was enhanced in the district following the change (Fulton, 2012). In Detroit, Michigan's Clintondale High School, failure rates in English and language arts were reduced by about two-thirds after implementing the flipped classroom (Rix, 2012).

Researchers studying the effectiveness of using the flipped classroom have noted that the flipped classroom approach provided an engaging learning experience, increased student achievement, effectively helped students learn the content, and increased student self-efficacy in their ability to learn independently (Defour, 2013; Goodwin, & Miller, 2013; Jaster, 2013; Talley, & Scherer, 2013). According to Stansbury's (2013) review on the benefit of the flipped classroom, many methods of learning incorporated into the flipped learning model are supported by years of research focused on increasing student's academic achievement such as active learning, the use of assistive technology, and constant feedback. At the same time, flipped classrooms allow students to learn at their own pace, provide a personalized learning environment, and reach different learning styles.

My Implementation of a Flipped Classroom

There is no one-way to implement the flipped classroom. In this section, I will share how I implemented a flipped classroom in two units: a Variable and Patterns Unit in a 7th grade mathematics class and a Solving Linear Equation Unit in a high school algebra class.

Preparing Students and Launching Expectations

Prior to teaching this unit, I asked the students to complete an inventory to make sure that all students either had Internet access at home, a computer without Internet access, or a DVD player with TV access. In the middle school there was no need for this inventory because every student received an iPad from the district with access. I used the Explain Everything App to record videos and uploaded them to YouTube as unlisted so that only a person with the URL could find the video. For students who had a computer at home without Internet access, I provided a DVD with the videos that could be accessed using a computer. For those students that did not have a computer at home but had a DVD player, I converted the videos to DVD player format by using Wondershare DVD Creator Software. Students received a packet that contained all the materials for the chapter. Each package included:

- "How to access the video lessons and note taking strategies" sheet (pink)
- an uncompleted note packet for the unit (blue)
- a detailed checklist (blue) for the unit
- warm-ups or entrance slips (blue)
- class work practices (blue)
- extra practices (yellow)

As is noted above, color coded paper was used to make it easier for the students to distinguish between the different materials. For example, all the minimum daily expectation assignments were printed on blue paper.

Prior to the launch of each unit, I explained what a flipped classroom was, and what the student expectations were during the unit. These expectations were written clearly on the board delineating what students would be required to do at home versus in class. Figure 1 is a diagram showing the classroom expectations.

☺ **Ask Questions Anytime** in any section

At Home

- Watch the Video and fill in the **Notes**

In Class (Ask Questions any time)

1. **Warm up**
2. Complete the blue **Class work sheet** (In your notebook)
   a. Check answers (Blue answer sheet)
3. Work on Yellow **Extra Practice sheets**
   a. Check answers (Yellow answer sheet)
4. Complete **checklist daily** and get teacher's Initials

Figure 1. Classroom expectations for the flipped unit.

To implement the flipped classroom approach, an online page "Glogster" was developed to include all the links for the video lectures for the unit. These online pages can be found at:

High school: http://abirismail.edu.glogster.com/solving-linear-equations/

Middle school: http://abirismail.edu.glogster.com/variables-and-patterns

I went over the Note Taking Strategies sheet (see Figure 2) because this was the first time students would have seen the flipped classroom. After that I modeled how the students would do their homework by watching the first lesson video and by completing the lecture notes for the first lesson with the students in class. The note packet included the problem, and the students had to fill in the steps on how to solve it.

The Five R's of Note-Taking:
Record * Reduce * Recite * Reflect * Review

1. Record. Write and fill in your note package

- Identify the learning target.
- Read the entire section once from your note package without taking notes.
- Watch the video and Write down information and Fill in your notes.
  - o You can pause the video at any time to write the information on the screen
  - o You can rewind any part of the video and listen to it again
  - o You can watch the video as many times as you prefer

2. Reduce. After the Video, summarize key/cue words and concepts

3. Recite.

- Review from memory what you have learned
- Create your own examples or solve some of the Class Work practice questions

4. Reflect.

- How does this relate to what you knew before?
- Write down any question you still have to ask your teacher in the next class.

5. Review. Review the notes you took:

- When you solve the Entrance Slip in class
- If you need help while working on the In-Class Worksheet
- At your next study session
- Before reading new material
- When studying for tests

Figure 2. Note taking strategies.

I also provided the students with a checklist page attached to the note packet. The checklist included the unit schedule and the expectations for the learning segment. As students moved through the flipped classroom unit, they had to get the teacher's initials in class after they finished their minimum daily work expectation. The minimum work expectation consisted of watching the video, filling in the lecture notes, completing the warm-up page, and finishing the blue class work practice sheet.

At the high school I offered two laptops with six headphones to any students that did not watch the video the previous night while all the other students were working on their class work. Students were also encouraged to come before school, during lunch, or after school to watch the video. I also communicated with the Success Center teacher to remind students to watch the video at the center. In the middle school, students used their iPad to watch the video in class while the rest of the class was working independently on their class work. I usually encouraged students to watch the videos at home by recognizing those who watched the video, bringing in a little treat for the independent learners, talking to students individually, or calling parents to remind their children to watch the video at home. In the first day of implementing the flipped class, half of the class did not watch the video at home, but the number decreased rapidly after the third class.

How the In-Class Lessons Ran

At the beginning of each class, students worked on their daily entrance slip or warm-up packet while the teacher checked if they filled in their lecture notes. The lesson was launched with an exploration where a real-life question was discussed. This question depended on the concept that the students had watched the day before. After the discussion, the students were asked about what they learned from the video and what questions they still had. One example of exploration discussion is an online manipulative that I used in the high school class to help students develop procedural fluency and conceptual understanding for solving multi-step equations. The manipulative, which can be accessed via http://www.mathplayground.com/AlgebraEquations.html, introduced a visual representation that used algebra tiles and a pan balance to model how to solve these types of equations. Students were asked to write an equation and solve the problem:

> *Looji had four packs of pencils and five extra pencils for a total of 13 pencils. How many pencils are in each pack?*

The students then came up with an equation, using the concept they had learned in the provided video to answer the question, and substituting the answer back into the equation to check their answer.

To build conceptual understanding, I wrote the equation that the students came up with on the board ($4x + 5 = 13$) and used the online manipulative to help students describe their steps for solving the equation. Students directed me to place 4 *xs* and 5 ones on one side of the scale to represent the four packs of pencils and five extra pencils. They then had me balance the scale with 13 ones, which represented the 13 total pencils. Thus, the online manipulative was used as a visual tool to represent the balance of the equation, and students had a great visual representation to build their understanding of what balancing an equation means. During this activity, I asked probing questions such as: How many variables do we have? What is the coefficient of x and what does this coefficient mean in the context of the problem? What is the constant that has been added to the variable side? What number on the right balances this equation? How would you solve this equation? Do we subtract 5 first or divide by 4 first? Why?

Throughout the flipped unit, these types of explorations served as a quick formative assessment for conceptual understanding for each lesson. Another formative assessment that was used was warm-up or Entrance slips. These quick assessments provided me with opportunities to assess the students' understanding, re-teach any concepts, and discuss any misconceptions the students might have. During the warm-up, I put the questions on the board and called on students randomly to answer them. I then discussed the key concepts with the students to be sure they were using the mathematical vocabulary and terms while answering the questions. Then I allowed a few minutes for the students to fill the entrance slip reflection that asked students to rate their confidence level for each of the learning targets represented in the homework video or warm-up. After class I read these students' reflections and provided them with feedback.

After the warm-up discussion, students worked individually to finish their blue class work practice sheet and check their answers from the blue answer sheet that I provided. The students usually asked me or another student questions they had during this time. After they corrected their blue class work, students could ask me to check their completion of the minimum expectation work and initial their checklist.

To differentiate content for advanced students and struggling students, students worked on the yellow extra practice sheets after they finished all of their blue minimum expectation worksheets. The blue class work practice sheets were a direct practice for the given section; however, the yellow extra practice sheets included correcting the error, a word problem from real life, an ACT question as well as extra practice problems for the concept. Students could then check their answer for the extra practice. During the class work time I was able to answer students' questions and clear up any misconceptions. At the same time, I could work with the struggling students to make sure they understood the concept while the advanced students were working on the yellow extra practice sheet. The students who had excessive absences were able to watch the videos in class and work at their own pace without delaying or interrupting the learning of the rest of the class.

Once students finished all the minimum requirement of class work and the extra practice sheets for the daily lesson, they could watch the next day's video in class and work on the next lesson. If students finished the entire unit requirement early, I provided these students with an extra project that guided them through discovering and investigating the use of the mathematics from the unit in real life.

Conclusion

The implementation of a flipped classroom allows the teacher to use multimodality in teaching the unit. The lesson is represented on videos where it will be easy for visual and auditory learners to interact with the material. Tactile learners benefit from many in-class activities that help students move around the class. These activities found in the middle school Glogster unit included the Mystery game and the jumping jack data collection activity. As such, students were active partners in their own learning, and at the same time, they were engaged and motivated to work while I was present to support every individual student.

If you are interested in applying this flip classroom approach you can start with a Glogster that I made, which includes the material for the middle and high school units. It also includes information on how to make your own video using Explain Everything App. The Glogster can be found at: http://abirismail.edu.glogster.com/my-flipped-classroom.

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Meaningful Measurement: Addressing Equity through STEM

By Megan Nickels, Illinois State University

A fundamental tenet behind NCTM's Equity Principle (NCTM, 2000) and the newly released, widely adopted Common Core State Standards for Mathematical Content and Practice (Common Core State Standards Initiative, 2010) is mathematics for all. This is a timely reminder that the discourse of equity has become normative in the field of mathematics education (Lawler, 2005). Concerns of equity within mathematics education, however, are often only concerned with broadening students' access to high quality mathematics activities and opportunities. Although this aim and the resulting initiatives are to be applauded in their own right, they nonetheless stop short of extending equitable notions to the nature and use of mathematics and the child's mathematical agency. Attention to these shortcomings and the distribution of equitable notions to each of them would bring the mathematics education community closer to a truly equitable mathematics education that emphasizes the child's authorship and authority for mathematical thinking and learning.

Recent STEM (science, technology, engineering, and mathematics) initiatives propose integrating engineering design as an alternative approach to teaching mathematics. These initiatives have the potential to provide a rich space in which students can synthesize and apply mathematical knowledge in a way that is integrated across academic disciplines, situated socially and culturally in a manner resonant with the child's sense of self, and supremely holistic. (International Technology Education Association, 2000; Vossoughi, Escudé, Kong, & Hooper, 2013). The central tenet of these propositions is that an engineering design can serve as a catalyst in the creation of mathematical environments and as a modeling tool with which children can think mathematically and flexibly. Mathematical environments are defined as a classroom culture elicited by rich contexts that naturally give rise to mathematical problems or questions and provides students with valuable tools to allow functional experimental activity to take place simultaneously with the act of formalization. Such thinking involves creative activity on the part of the learner, and it is suggested that such activity, which places the learner in charge of his or her learning, is inherently motivating for students. Engineering design activities also encompass hands-on construction that can promote three-dimensional thinking and visualization by applying mathematics skills and strategies to real-world problems that are relevant, epistemologically, and personally meaningful (Bers, 2008; Papert, 1980; Resnick, Berg, & Eisenberg, 2000).

This article describes a project designed to address a greater notion of equity through a mathematics lesson that integrated engineering design. Lego Mindstorms EV3 robotics were used to investigate angle and angle measurement with middle school students. Students were given a choice between two tasks. Task 1 was a traditional LOGO task wherein they would write a program that would result in the robot drawing a square, equilateral triangle, regular pentagon, regular hexagon, regular octagon, regular decagon, star polygon, and n-gon. Task 2 was a task in which students were asked to investigate the relationship between motor rotation and robot rotation by programming a robot to rotate at every angle on a unit circle mat. The graphical programming required of the robots exposed students' thinking and understanding of angle, angle relationships, angle measurement, ratio, and function.

The two tasks emphasize the role of robots as transitional and relational objects, which students can use to explore their ideas from their perspectives. Students were asked to complete these tasks after a sequence of introductory robotic activities on basic numeracy, decimal and fractional numbers, the relationship between diameter and circumference, and conversion between centimeters and inches. Figure 1 outlines the task components, description, and assessment guidelines.

Task Component	Objectives	Description
Write a program that would result in the robot drawing a: square, equilateral triangle, regular pentagon, regular hexagon, regular octagon, regular decagon, star polygon, and n-gon.	The main objectives were: to develop analytical and synthetic thinking; to support the acquisition of methodological and algorithmic thinking skills; to enhance student ability to solve problems using programming environments; to develop creativity and imagination.	The ultimate goal of this task is to have students generalize a formula for finding the sum of the angles in any n-gon, which implies understanding the relationship between interior and exterior angles and the number of sides of a regular polygon.
Investigate the relationship between motor rotation and robot rotation by programming the robot to rotate at every angle on a unit circle mat.		By programming the robot to rotate, students must be able to validate the amount of motor rotation required to produce rotation of his robot. No sensors are allowed in the solution. The task further requires students to estimate, calculate, and measure angles, understand ratio concepts and use ratio reasoning to solve problems.

Figure 1. Project components.

Michala's Approach: Task 1

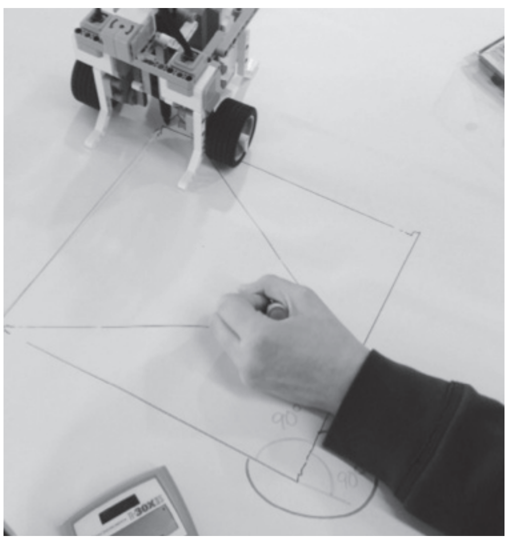

Figure 2. Michala's approach.

Michala's Initial Perceptual Organization of the Task

Michala chose to complete the traditional LOGO task of drawing polygons. She began by reasoning that she would need to build a two-motor robot to make precise turns. A one-motor robot can move forward and backward easily; however, it is difficult to turn. A two-motor robot design, on the other hand, does turn easily because one motor can be turned off while the other motor is turned on, forcing the robot into a curved turn. Alternatively, the two motors can be programmed to rotate in opposite directions, causing the robot to pivot around a tighter radius or even a pivot point. Michala's next step was to create a table on a poster board in order to keep track of information about each polygon because she did not know this information for regular polygons with more than four sides. Initially her categories included the name of the polygon and number of sides. By the end of this activity her categories were: Name of Polygon, Number of Sides, Sum of Angles, Actual Interior Angle, Exterior Angle, Reflex Angle, Total Number of Trials, Success Rate % (number of successful completions/number of trials) * 100 (see Figure 3a and Figure 3b).

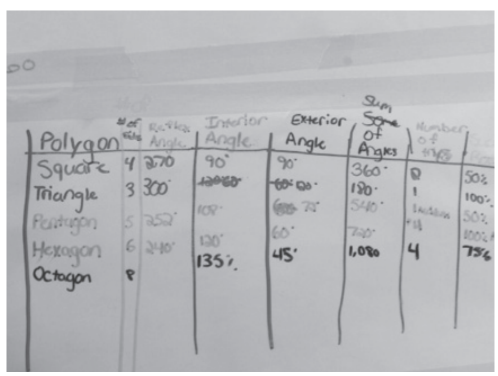

Figure 3a. Michala's inductive reasoning.

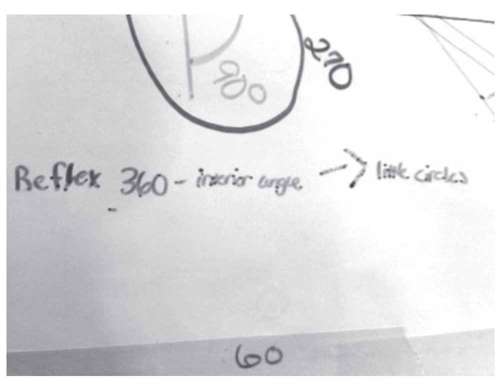

Figure 3b. Michala's inductive reasoning.

Michala's Inductive Reasoning

On day six of the task, after successfully drawing a square, triangle, pentagon, hexagon, and octagon through body syntonic reasoning (i.e., imagining what she might do if she were the robot) and increasingly sophisticated intuitions about angle measurement, Michala explained her new strategy for drawing additional polygons.

> *Michala: So see I've been keeping track and um I write down the interior angle and the exterior angle and the reflex angle every time. So I was looking here and this column [interior angle] and this one [reflex angle] add up to 360. And but if you look here this column [interior angle] and this one [exterior angle] um they add up to 180 each time cause its like half the turn. You're making circles. These are the little circles. Each shape has as many little circles as it does sides and then one big circle.*
>
> *MN: Wait, so can you tell more about what you mean when um you say each polygon has as many little circles as it does sides?*
>
> *Michala: I don't know [laughs]. Umm, yeah so I guess it was easier for me to think about the angles as parts of a circle. Cause you're measuring the turn of your robot or I mean your robot is turning the amount of degrees you tell it to. Um, so yeah, um the angle is just how much it turned out of a whole circle. That's like how I got better at guessing because I started to have this idea about circles and so I drew the shapes, um polygons by hand and you can like tell right away that the interior angle is getting bigger and um at the same time the reflex is getting smaller because that interior one took up a bigger part of the circle. So I knew circles were 360 degrees so if I needed a bigger interior angle I guessed a number closer to 360.*
>
> *In her own words, Michala was beginning to formulate ideas pertaining to how an angle is measured with reference to a circle, and to consider the fraction of the circular arc between the points where the two rays intersect the circle.*
>
> *MN: Ok, I think I follow you. So, um you said something about a big circle too? Right?*
>
> *Michala: Yeah every one of these shapes has one. They fit into it. [Draws circles around her polygons.] And like the more sides you keep adding the closer you get to it. But um the circle gets made because um look at my robot here [places robot by hand on vertice of octagon] it starts driving here and then it makes all of its turns and stuff and then it [continues to drive robot around the vertices] ends up here exactly how you started it. So you gotta figure it turned through 360 degrees to end up there in the exact same spot.*
>
> *MN: So then was that helpful, I mean thinking about the big circles in terms of programming your robot?*
>
> *Michala: Yeah cause then you think about the big circle; the reason all the angles, um exterior angles add up to 360 is because you can put them all in a circle. So I count the turns. Like alright if you need to figure out the interior angles of a hexagon you take 360 and divide by 6 because your robot turns 6 times. That's 60. So that's the exterior angles but if you look that's what it actually turns through. So then 180 minus um what'd I say, um 20 that's 120. So the interior angles of a hexagon are 120. And you could write that as 180-360/t.*
>
> *MN: So could you make any n-gon now?*
>
> *Michala: Yeah, its always gonna work so I could make a lot more. Well probably hundreds but you're gonna get closer to a circle each time and that's it. That's the limit!*

Kasim's Approach: Task 2

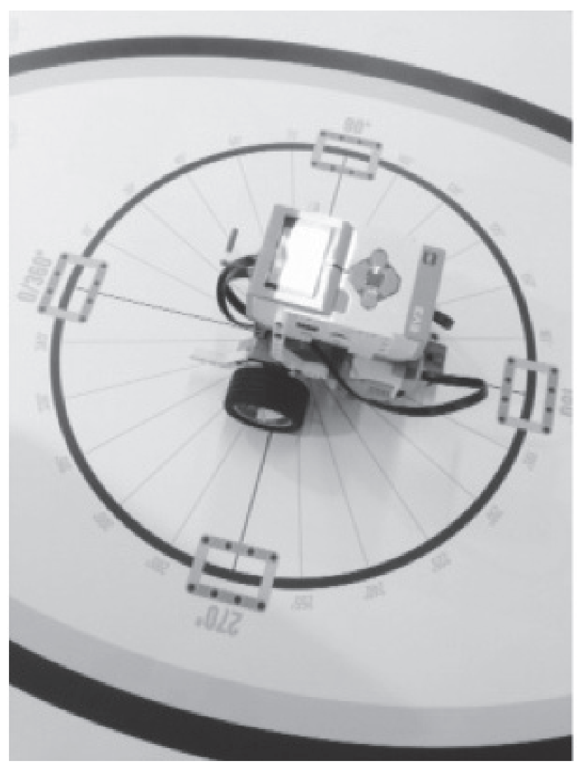

Figure 4. Kasim's approach.

Kasim's Initial Perceptual Organization of the Task

Kasim employed no methodological organization to begin his task. He simply began by guessing different programming commands for his robot. For example to turn 30 degrees, he set the motor rotation at 180. He was happy to continue in this way for several hours without making any significant progress or improving his guesses.

Kasim's Inductive Reasoning

> *Kasim: Oh man I was wrong, really wrong for like a whole day! I guessed a lot of crazy stuff but then I was like okay I remember the lesson on getting my robot to drive different lengths and I um had to figure out the circumference of the wheel. So that was like a huge clue to me- it has to do with the wheel!!! And then I was like ok why did you make me start by making circles. So yeah I just kept guessing degrees and I got it to make a circle with 707 degrees. But I know 360 degrees make a circle so um then this was like a ratio problem.*
>
> *MN: Can you describe that ratio to me?*
>
> *Kasim: Um yeah it's how much you have of degrees, I mean how many degrees you program the wheel to turn compared to how many degrees the whole robot turns. So I divided 707 by 360 and you get almost 2. [pauses and looks for notebook] I wrote it down, here, it's 1.96388889.*
>
> *MN: Um, so can you tell me what that number represents?*
>
> *Kasim: Uh, I think its how many degrees you have to make the wheel turn to make the robot turn 1 degree. Cause it works. I'm pretty sure I proved it with all of my turns.*
>
> *Kasim was further challenged to discover this ratio for other robots, each uniquely designed to draw circles with different circumferences. Kasim went on to discover that there was indeed a relationship between the circumference of the wheel and the amount that the robot turned.*
>
> *Kasim: OK so for example 90 degrees. You can't just tell it to turn 90 degrees because it's not the robot that gets told to turn it um your program tells the wheel which is attached to the motor how much to turn. The wheel turns. And you get two circles each time. You have the wheel's circle and the circle the robot drives. You can measure the circle your robot drives in it's tightest circle, but it works too if you look at how you build your robot and how far apart the wheels are underneath um from the like inside of the wheel and you use that, um it's your diameter for the circle the robot turns in and you want to know how long that circle is all the way around. That's um circumference. My robot's diameter is 11 cm so I timesed it by pi. And my wheel is 2 inches, um about 6 centimeters so its circumference is pi times 6. So that's a ratio too I think. Yeah it is, yeah it's wheel circumference compared to robot circle circumference.*

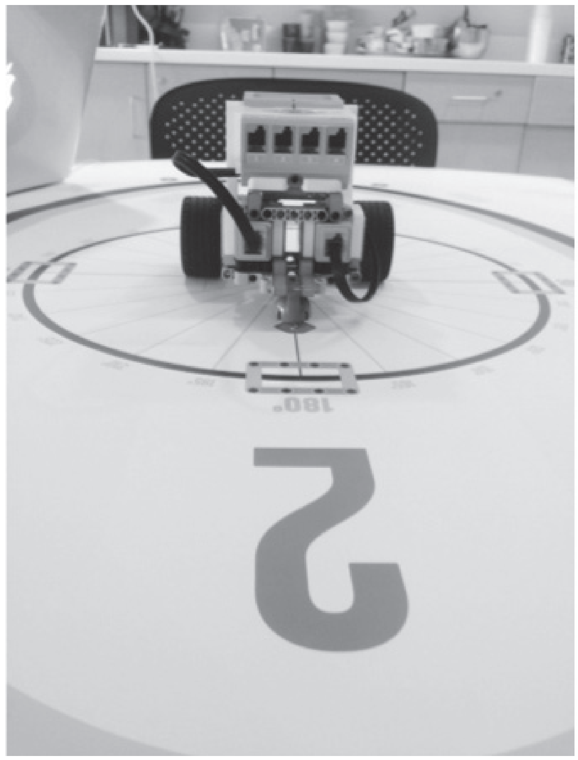

Figure 5. Kasim's approach.

Conclusion

The value of STEM education in general and robotics in specific, is that it has a particular role to play in helping students to develop effective ways of thinking about mathematics. These ways of thinking are often extremely personal and thus serve as more concrete foundations leading to formal understandings. For both Michala and Kasim the incremental and modular structure of the robotics and its graphical language provided powerful images ready for appropriation to formal Euclidean ideas. For example, even at the most elementary level, Michala reasoned the circumference of the circle is a limit of polygons. We can thus interpret Michala's case in the context of a very sophisticated Piagetian concern with the accessibility of ideas. In the original design of LOGO, Papert intended for sensorimotor constructions to play a leading role in children's learning—according to Piaget these constructions were established in the first two years of life. What this means is that robotics tasks, like LOGO, are designed in a way that it permits a sensorimotor interpretation. Michala modeled the movement of the robot from her description of bodily movements. In Papert's terms, her reasoning is ego-syntonic; in Piagetian terms, it is assimilable.

In the case of Kasim, his mathematical thinking can be described by another of Papert's principles:

Some of the most crucial steps in mental growth are based not simply of acquiring new skills, but on acquiring new administrative ways to use what one already knows (Minsky, 1988, p. 102).

Kasim had previously worked with ratios and circumference during prior robotics tasks, which provided him with the necessary ingredients for reasoning through his challenge. What became important was how he organized these bits of information to develop an understanding of the robot's function for turning.

These are instances of natural and thus equitable learning. In this kind of learning, a robotic command is learned as an everyday concept in which referential function and meaning function coincide. This means that the meaning of a robotic command is in the action that is produced in using its concept. It is only later that mathematical concepts acquire, in students' minds, a genuine or scientific meaning as part of the hierarchic system of all mathematics.

These powerful ideas for learning mathematics can easily be incorporated into classrooms to whatever extent each teacher feels comfortable doing so (e.g., a one day task or a week long unit). Michala's traditional LOGO task can be a meaningful activity to introduce or concretize concepts of angle and angle measure or the classification of two dimensional figures. Kasim's unit circle task, while also useful in investigating angle, is effective in developing functional thinking.

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Technology Tips: Digital Artifacts in One-to-One Classrooms

By Josh Hertel, Jenni McCool, and Jennifer Kosiak, University of Wisconsin-La Crosse

As prices for technologies such as tablets and laptops have fallen over the last several years, it has become feasible for some schools to provide every student with their own device. Within these one-to-one classrooms students have an electronic device with them all the time for every subject. The growing presence of one-to-one environments presents new questions for mathematics educators to consider. Although a number of issues must be resolved before these technologies can be used within the classroom (e.g., funding the initial purchase, maintaining and repairing devices), equally challenging is finding ways to help educators understand how to use the devices strategically and effectively in their instruction. Simply put, one-to-one technologies are not a panacea that can resolve all of the problems that exist in teaching and learning mathematics. In the spirit of encouraging more discussion on this issue (and perhaps prompting some debate), we discuss one role that one-to-one technologies have played within our classroom as a tool for creating digital artifacts. In what follows, we outline this role and highlight specific ways it can be incorporated into the classroom.

We use the phrase digital artifacts to refer to objects created by students as part of their learning of a particular content. This is intended to be a broad category that includes audio and video content, slide show presentations, and interactive online content. Once created these artifacts can serve a variety of purposes including a framework for student presentations, an assessment of what student has learned about a particular content, or a reference resource for use later. There are a host of different programs that provide students with the capabilities to create digital artifacts using one-to-one technologies. For this article, we have chosen to discuss four that we believe are particularly useful in the classroom: Padlet, Show Me, Educreations, and Explain Everything.

Padlet is a free web-based application that allows users to post messages, images, audio, or video files on an interactive webpage (http://www.padlet.com). The appeal of Padlet is the simplicity of the application. A teacher creates a Padlet wall and then shares a URL for the wall with students. Students can post on a Padlet page using any device that supports a web browser (desktop, laptop, tablet, smartphone, etc.). Likewise, anyone with an active internet connection and the correct URL can access the webpage and view content. Only the creator of the page has the ability to remove items. Within a one-to-one environment Padlet pages can serve a variety of purposes including a means to gather data and a tool for brainstorming.

As a data gathering tool, students can be given a specific task and then provided with exploration time to collect data and post on a Padlet wall. For example, students might be asked to find examples of squares around the school and conduct a 10-second interview in which they ask someone to define a square. In our own experience, these Padlet pages fill quickly and can become rich sources for classroom discussion and further investigation. Students are given many opportunities to grapple with the mathematics under consideration as well as a greater awareness of the presence of mathematical ideas in their daily life. As a tool for brainstorming, a Padlet page can be used by a group of students or the class as a whole. A teacher can create individual Padlet pages for different groups of students and then instruct groups to brainstorm together using the page. In this way, the page can serve as a repository for the group members as well as evidence of their work for the teacher. Similarly students might use the Padlet wall as a place to brainstorm their definitions of key academic language. As students use the virtual wall to share their definitions, the whole class can construct a more concise definition of a key term or concept such as triangle (see example in Figure 1), factor, ratio, or slope. In addition to use within the classroom, Padlet pages can also be shared with people outside of the class to highlight activities and student work.

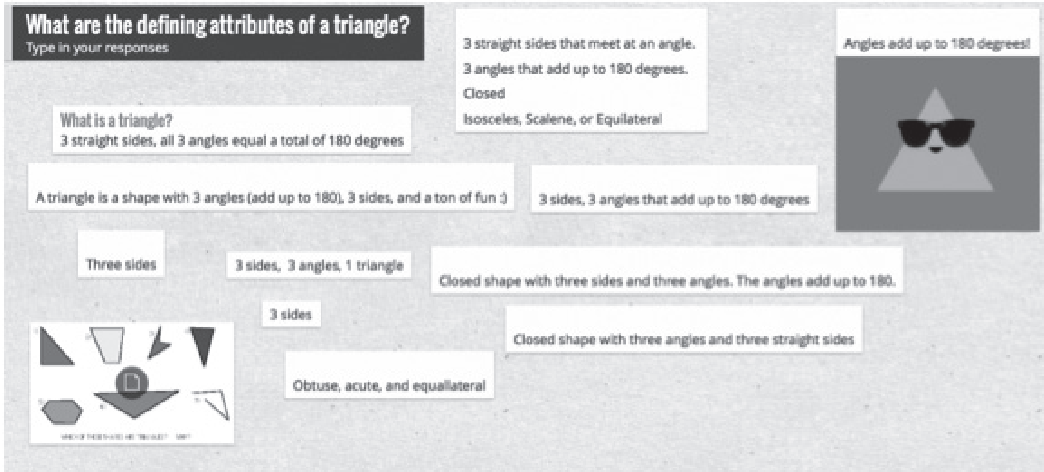


Figure 1. Padlet wall for attributes of a triangle

ShowMe (http://www.showme.com/) and Educreations (http://www.educreations.com/) are free applications for iPads that capture students' voices as well as written work. The resulting recording, commonly referred to as a screencast, can be uploaded to a teacher's account and then shared with others using a specific link. Pictures can also be incorporated into these screencasts. Both ShowMe and Educreations are especially friendly for younger students because the key features of the programs are visible to students from one page. This allows students to easily locate and select the tools they need without looking in drop down menus. Students can start and stop recording as many times as needed to complete a project; however, neither of the applications allow the user to edit the content of a video once it has been created. These apps can also be used by a teacher to create tutorials for students and/or parents, explaining concepts and strategies students are using in their class.

Another application for creating and sharing screencasts is Explain Everything (http://www.morriscooke.com/). In contrast to the two previously discussed apps, Explain Everything is available for both iPads and Android tablets and has more robust features that extend its functionality. Most importantly, the app provides editing capabilities that allow the user to pause, rewind, and record over previously saved audio or video. Additionally, it is possible to split up a screencast into several different parts. These parts can be uploaded as one continuous screencast or several individual videos. With this functionality it is possible to break up a long discussion into several smaller parts. Moreover, the ability to break up a screencast into parts can be used in conjunction with the editing features to replace portions of a screencast.

Within a one-to-one environment, screencasts can be made by students for a variety of purposes. For example, after reading Shel Silverstein's poem Shapes, elementary students can create an illustration for the poem and then generate a screencast in which they describe or compare the attributes of the given shapes (see Figure 2). Middle school students might document their process for data collection and discuss the resulting displays. Students in high school might create screencasts in small groups with each group focusing on a particular mathematical idea under investigation. Thus, whether a screencast is created for commentary about a mathematical concept, discussion of the mathematical method, evidence of a worked out solution, or some other reason, one-to-one devices make the generation of these digital artifacts an easy practice to incorporate into the classroom.

Figure 2. Screencast of Shel Silverstein's Shape poem.

We believe that professional resources are essential in helping educators stay ahead of the growth of one-to-one technologies. This article has focused on one specific role that these technologies can play and highlighted a few specific programs within a sea of applications. We are interested in hearing from other teachers using one-to-one technologies in the classroom and encourage readers to submit pieces detailing their own experiences, challenges, and successes working within this new environment.

A New Spin on Fair Sharing

By Megan Wickstrom, Montana State University and Nicole M. Wessman-Enzinger, Illinois State University

Students often have difficulties making connections between rational number concepts and their relationships to other mathematical applications and real world situations (Johanning, 2008). Researchers have advocated that students should experience using rational numbers with multiple and varied models integrated into context (Empson & Levi, 2011). In this article, we discuss a lesson that drew upon probabilistic reasoning as a means to help students connect rational number reasoning to real world situations. Probabilistic situations act as an extension to rational numbers in that they often involve fractional models and encourage students to reason through topics, such as part to whole relationships and fractional equivalence. Even though probabilistic reasoning is often clouded with misconceptions, it involves the ability to integrate rational number reasoning into a context with discussion and justification rooted in rational number thinking (Jones et al., 1997).

The Common Core State Standards for Mathematics suggest rational number equivalence should be addressed in the third and fourth grades. While working with a fourth grade classroom, we thought probabilistic comparisons might be an ideal context to elicit students' conceptions about fairness and rational number equivalence. We wanted to draw on students' knowledge of fair sharing in relation to their probabilistic reasoning. A fair sharing problem involves a number of items that need to be shared among a given number of people or groups (Empson & Levi, 2008; Wilson et al., 2012). We wanted to see if students' understanding of fraction equivalence would translate into their understanding of probability and fairness.

Keeping these ideas in mind, we began to plan the lesson and decided to create a scenario that centered on winning a game. We generated several spinners that each represented the same chance of winning but were composed of different size pieces and also arranged in different ways (see Figure 1).

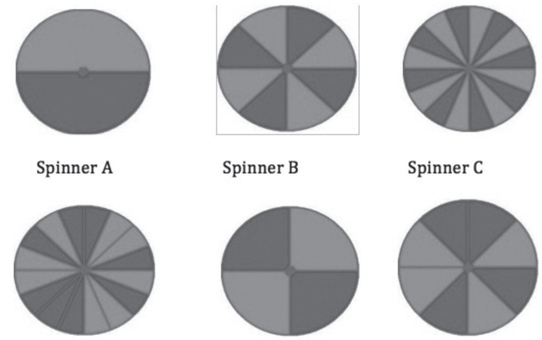


Figure 1. Spinners.

Researchers have indicated that it is important for students to see multiple representations of fractions beyond the circle model, like set models, fraction bars, area models, and number lines (Petit, Laird, & Marsden, 2010). Although we recognized that multiple models are important, we decided to focus on a singular fraction model for this lesson. We thought that one fractional model, specifically the circle model, would be best to help draw the students' attention to comparison and equivalence. We decided to make the spinners all varying representations of one-half utilizing the circle model and, depending on the results, we could explore other fractions as an extension.

Below we present this two-day lesson that aimed to introduce and elicit students' reasoning about fractional equivalence through the probabilistic concept of fairness.

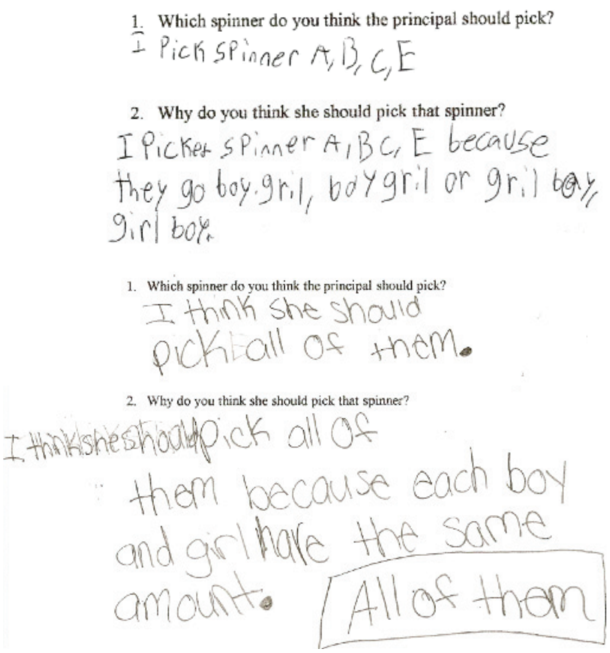
Lesson Day 1

On the first day of the lesson the students began with an introduction to the problem:

> *The boys and girls in the class are playing a game against each other. If the spinner lands on blue the girls get a point and if the spinner lands on red the boys get a point. Which spinner or spinners would you choose for the game?*

Before we gave the students the spinners to test, they were asked to explain which spinner or spinners they would choose and why. We gave them this prompt to see what initial conceptions or misconceptions they might have to help us guide the lesson. Of the student responses, half of the students picked Spinner A as the spinner they would use. This was primarily because they thought that the boys and girls had what seemed to be more area for the spinner to land on. Other students also picked A because they felt it was the best representation of equal.

Some students were concerned with the order of the sectors on the spinners. They indicated that they should use spinners that had sectors that alternated colors (i.e., Spinners B, C, and E) otherwise it wasn't fair. Only two or three students initially responded that all of the spinners would work because they recognized that the spinners each represented one half even though they were different in appearance. Examples of their work are shown below in Table 1.


1. Which spinner do you think the principal should pick?
I Pick spinner A, B, C, E

2. Why do you think she should pick that spinner?
I Picked spinner A, B, C, E because they go boy.gril, boy gril or gril boy, girl boy.

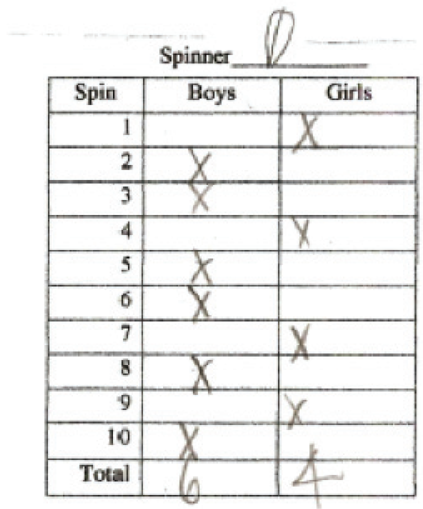
1. Which spinner do you think the principal should pick?
I think she should pick all of them.

2. Why do you think she should pick that spinner?
I think she should pick all of them because each boy and girl have the same amount. All of them

Following this reflection, we had each of the students spin each of the spinners ten times and record their findings to determine who won for each spinner (See Figures 2, 3 and 4). The students took turns spinning the spinners and exchanging them with classmates. Testing the spinners took the remainder of the time for mathematics and the lesson concluded with the students submitting their results to us.

Figure 2. Students collecting data.

Figure 3. More students collecting data.


Spinner D

Spin	Boys	Girls
1		X
2	X	
3	X	
4		X
5	X	
6	X	
7		X
8	X	
9		X
10	X	
Total	6	4

Figure 4. Student recording table.

Lesson 1 Reflection

Following the first day of the lesson, we realized that spinning the spinners only ten times was not enough. The students needed experience with spinning the spinners many times. We decided that we would compile the students' results and bring in the Law of Large Numbers to direct the students' focus to the layout of the spinners. The Law of Large Numbers states that the more times an experiment is performed the closer the results will be to the expected value. In our case, the greater the number of spins the closer the numbers would be to girls winning half of the time and boys winning half of the time. We heard several of the students mention the word fairness in the lesson, so we decided to begin the second lesson with a discussion about the fairness of the spinners. We felt that this would help the students to begin to focus on rational number equivalence.

Lesson Day 2

On the start of the second day of the lesson, the students were told that we compiled all of the spinner results so that we could see what happened if the spinners were each spun around 200 times (see Figure 5). Without showing them the results, we asked the students what they expected to see. We noted that the word "fairness" had come up in conversation several times the day before and asked the students what they thought the word fair meant. The students responded that they thought fair meant that each person would win the same amount of times. We then directed their attention to the spinners, and asked what a spinner would look like if it was fair and what results would we see from a fair spinner. Several of the students said that fair for the spinners would mean that there was a 50/50 chance of winning. When we asked the students to explain, they stated that each person should win half of the time or nearly half of the time. One of the students stated that if the spinners were fair and we spun the spinner 20 times, we should expect boys to win around 10 times and girls to win around 10 times. He said that 50/50 meant that the boys would win about 50% of the time and that the girls would win about 50% of the time. All of the students agreed that this was a good way to think about fairness for the spinners. Next, we asked the students to think about if all our spinners were fair and what they thought the results might look like for each of our spinners. After the students had pondered this question, we revealed the results on the overhead projector (see Figure 5).

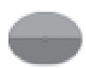
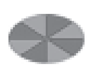
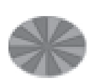
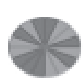
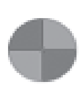
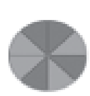
	Girls	Boys
Spinner A 	101 Spins	96 Spins
Spinner B 	89 Spins	91 Spins
Spinner C 	82 Spins	80 Spins
Spinner D 	89 Spins	92 Spins
Spinner E 	98 Spins	94 Spins
Spinner F 	82 Spins	81 Spins

Figure 5. Complied results presented to students.

Many of the students seemed surprised with the results, especially for spinners D and F. After the students viewed the results of 200 spins, we asked them:

> *How could all of these spinners look different, but the boys and girls won about the same number of times?*

The students were asked to jot down ideas about this question for a few minutes, and then the students shared some of their reasons why each of the spinners was different but yielded similar results. Several explanations arose from brainstorming.

Two of the explanations that the students came up with related to the area of the circle. Several of the students seemed to use spinner A as a benchmark spinner to compare the other spinners to. In one of the explanations, the student imagined the sectors of other spinners melting together and becoming Spinner A. In the second explanation, the student imagined breaking the spinners apart by their sectors and rearranging them to make Spinner A. In either case, both students pointed out that the sectors in each of the spinners could be rearranged to represent A or another spinner.

Other students focused on the number of pieces. Some of the students focused on the number of sectors for boys and girls on each spinner, such as comparing the ratios of girl and boy. The students referred to the number of sectors as the number of chances. One student said that the number of chances is equal for each spinner because spinner A has 1 chance for the girls and 1 chance for the boys and spinner B has 4 chances for the girls and 4 chances for the boys. Some students took this further and focused on the size of the sectors. They stated that not only did the students have the same number of chances but the pieces were the same size.

At this point, we decided these were good transitional explanations into fractional equivalence. We asked the students if they had heard of same size pieces before in mathematics. The students responded that they had discussed same size pieces when learning about fractions. We then asked the students:

> *How can you use fractions to describe the fairness of the spinners mathematically?*

The students began by pointing out that in spinner A the chance of winning for a girl or boy was 1 out of 2, in spinner B it was 4 out of 8, and it spinner C it was 8 out of 16, etc… We then asked them to explain further so what would make these the same. How could 1 out of 2 be the same as 2 out of 4 or 8 out of 16? One student said that they are all equivalent fractions. Knowing that this word was not commonplace in the classroom, we asked the students to describe what they thought equivalent meant. Many of them said that it meant that the fractions were the same but looked different. We asked them how they knew they were the same. The students pictorially showed with the spinners that the pieces could be put together to make one another and others began to use symbolic expressions (see Figure 6).

We also asked the students if they could create another spinner that was fair. Students were able to create spinners composed of six pieces as well as ten pieces that were fair and equivalent to the spinners they investigated.

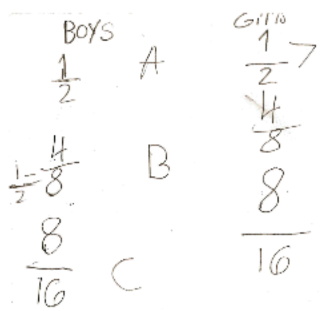


Figure 6. Probability of boys and girls winning.

Lesson Wrap-Up and Reflection

Knowing this was an introductory lesson, we wanted to find out where our students were and what we still needed to address. We asked the students to write a letter to the teacher using the following prompt:

Using your results and the results your classmates found and discussed, please write a note to the teacher telling her which spinner(s) are fair and why.

In many of the letters (See Figures 7 and 8), students discussed cutting, breaking apart, or melting the spinners to show that each of them were the same. Students also discussed the idea of fairness in that both the boys and girls had an equal chance to win.

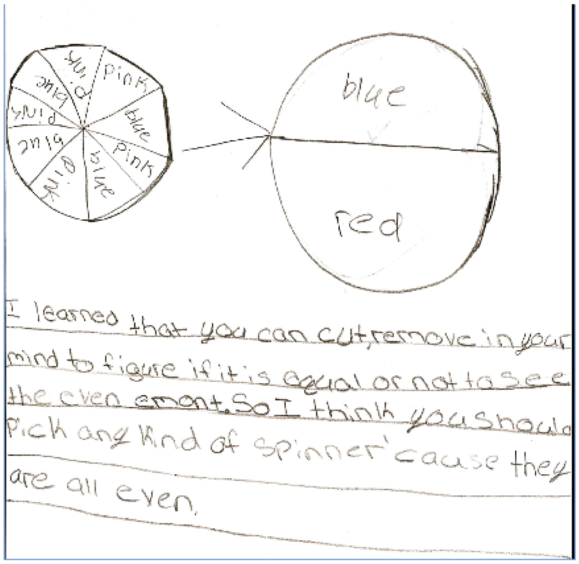
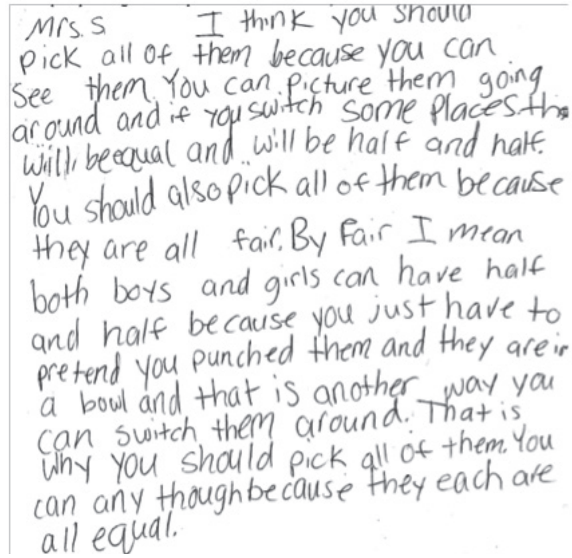


Figure 7. Sample student letter #1.



Mrs. S I think you should pick all of them because you can see them. You can picture them going around and if you switch some places they will be equal and will be half and half. You should also pick all of them because they are all fair. By Fair I mean both boys and girls can have half and half because you just have to pretend you punched them and they are in a bowl and that is another way you can switch them around. That is why you should pick all of them. You can any though because they each are all equal.

Figure 8. Sample student letter #2.

Conclusion

At the end of the lesson we, as teachers, had several realizations. We initially believed that probability would easily lend itself to the study of rational numbers. Students love to play games and often engage with tools like spinners or dice. As research (Johanning, 2008) indicated, it was not an easy task for our students to apply their rational number reasoning in a new context. The appearance and the arrangement of the spinners swayed their decisions. By allowing the students to interact with the spinners, collect data, and discuss, they were able to use prior rational number reasoning to help explain the phenomenon that they observed.

Probabilistic reasoning and the concept of fairness also allowed students to further define and visualize what it means for fractions to be equivalent. In the fourth grade, according to the Common Core State Standards for Mathematics, students are expected to explain fractional equivalence through visual models. During this activity, students were able to visualize the spinners melting or breaking apart to help further define, for themselves, what it meant for fractions to be equivalent. To further examine students thinking, next time we might ask students to design their own spinners to add to our set and describe why the spinners are fair.

When we integrate different mathematical content domains together, we have to juggle students' misconceptions, superstitions, and understandings within multiple content areas. It often seems easier to focus on one mathematical concept at a time. This lesson highlights that cross-conceptual mathematics lessons are important because they can help extend students' understandings by examining ideas and concepts in new or different ways.

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The Mathematics of *The Fault in Our Stars*

By David Ebert, Oregon High School, Oregon, WI

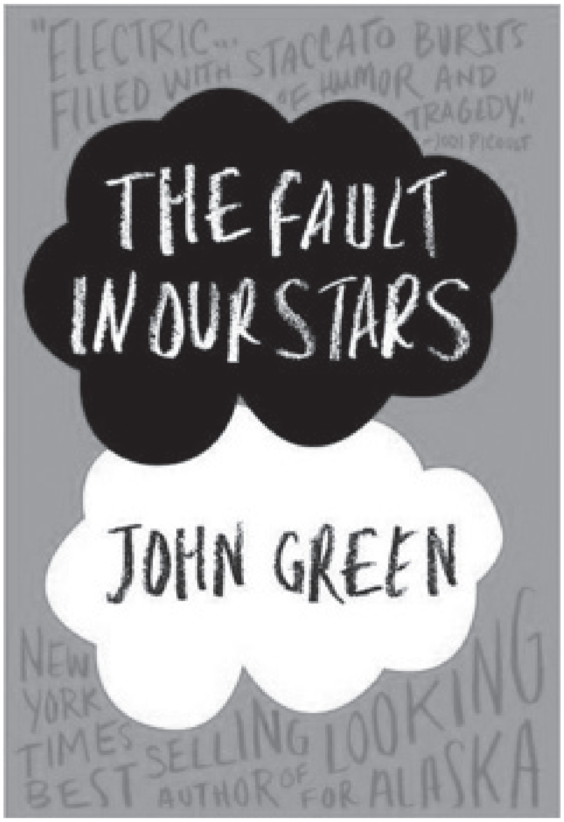

Figure 1. Fault of Stars

The Fault in Our Stars, by John Green, is a young adult novel that was published in 2012. This book immediately rose to #1 on many bestseller lists, and over one million copies of the book are in print. The movie adaptation opened in theaters in June 2014.

During the past year I noticed many of my students reading this book, and I was intrigued enough to read it as well. When I did, I was surprised at the mathematical references sprinkled throughout the book, especially references to infinity. After reading the acknowledgements at the end of the book, I was no longer surprised by the mathematical references. One of the acknowledgements is for Vi Hart, the popular video blogger. If you haven't marveled at her videos, check them out at vihart.com.

The book is about two teenagers fighting cancer who fall in love. One of the two, a young girl named Hazel, says,

> *I am not a mathematician, but I know this: There are infinite numbers between 0 and 1. There's .1 and .12 and .112 and an infinite collection of others. Of course, there is a bigger infinite set of numbers between 0 and 2, or between 0 and a million. Some infinities are bigger than other infinities. (p.260)*

Hazel is partly correct and partly incorrect, and her observation gives mathematics teachers a wonderful opportunity to share some advanced mathematics with our students.

Georg Cantor was a German mathematician who invented set theory in the late 1800s. He is perhaps best known for proving that some infinite sets, such as the number of real numbers, are infinitely larger than other infinite sets, such as the number of integers. Two sets are said to have the same cardinality if they can be put in a one-to-one correspondence. For example, the infinite set of whole numbers and the infinite set of even numbers have the same cardinality, as shown by the following one-to-one relationship:

$$\begin{aligned} 0 &\longrightarrow 0 \\ 1 &\longrightarrow 2 \\ 2 &\longrightarrow 4 \\ 3 &\longrightarrow 6 \\ 4 &\longrightarrow 8 \\ 5 &\longrightarrow 10 \end{aligned}$$

Because there is a one-to-one relationship between each whole number and each even number, these sets have the same cardinality, and the infinite set of whole numbers has the same number of elements as the infinite set of even numbers. Similar arguments can be used to prove that the whole numbers, even numbers, odd numbers, integers, and rational numbers all have the same cardinality, and therefore all have the same infinite number of elements.

Hazel is incorrect in stating that "There are infinite numbers between 0 and 1…(and) there is a bigger infinite set of numbers between 0 and 2." We can use a demonstration of one-to-one correspondence to show that the size of the infinite set of numbers between 0 and 1 is the same as the size of the infinite set of numbers between 0 and 2 (or between 0 and any number).

The first column represents all numbers between 0 and 1. The second column is two times the first column and is therefore all numbers between 0

.1 ⟶ .2
.11 ⟶ .22
.112 ⟶ .224
.7 ⟶ 1.4
.876 ⟶ 1.752

and 2. Since the first column is the list of all numbers between 0 and 1, and the second column is the list of all numbers between 0 and 2, there is a one-to-one relationship between the sets. They have the same cardinality, and therefore the same number of elements.

Hazel is correct, however, in stating that "some infinities are bigger than other infinities". Cantor was criticized by his contemporaries upon proving this in the late 1800s, but his proof is widely taught today as part of many college-level number theory courses.

To prove this, let's assume that the set of whole numbers and the set of real numbers between 0 and 1 have the same cardinality. There would then be a one-to-one relationship between the members in these sets. The left column is the set of whole numbers, and the right column is the set of all real numbers between 0 and 1.

We are assuming that the column on the left contains every whole number, and the column on the

0 ⟶ 0.3982582...
1 ⟶ 0.7591432...
2 ⟶ 0.2264855...
3 ⟶ 0.1239005...
4 ⟶ 0.9114102...
5 ⟶ 0.0344494...
6 ⟶ 0.7428760...

right contains every real number. However, we are able to construct a new real number that is not in the right-hand column. To do this, highlight the digit in the tenths place from the first number, the hundredths place from the second number, the thousandths place from the third number, and so on (see below).

Then, create a new real number that differs

0 ⟶ 0.3982582...
1 ⟶ 0.7591432...
2 ⟶ 0.2264855...
3 ⟶ 0.1239005...
4 ⟶ 0.9114102...
5 ⟶ 0.0344494...
6 ⟶ 0.7428760...

from each of these highlighted digits, such as 0.4670201…. because the numbers in each place value of this new number are not the same as any number in the right-hand column, this number cannot possibly be included in this list. Therefore the right-hand column cannot possibly contain every real number between 0 and 1. In fact, there are infinitely many ways to create the new real number that differs from each highlighted digit. So the number of real numbers between 0 and 1 is infinitely greater than the number of whole numbers. This is the idea of what is known as Cantor's diagonalization argument. Although this is a specific attempt at defining a one-to-one correspondence between the integers and the real numbers between 0 and 1, Cantor proved that no matter what mapping is defined, a number between 0 and 1 can be constructed in the same way as above that is not in the list. For a good concise explanation of this, visit http://www.youtube.com/watch?v=A-QoutHCu4o.

This is one example of mathematics appearing in our students' lives in a surprising way. As teachers, we always strive to make connections between the mathematics we teach and our students' lives. We should also be striving to make connections between our students' lives and interesting, beautiful mathematics.

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"How to Count to Infinity – YouTube" http://www.youtube.com/watch?v=A-QoutHCu4o

"Vi Hart" http://vihart.com/

Examining Formative Assessment

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Introduction

Formative assessment is a term that is often used in education, but what is it? Is formative assessment when student work is not graded? Is formative assessment when students receive immediate feedback? Is formative assessment when teachers make instructional changes based on student responses? These questions indicate a lack of consensus when discussing formative assessment, which motivated a group of college students to read and analyze research about formative assessment during a directed studies course at the University of Wisconsin–Eau Claire. This article details the results and recommendations from their findings.

What is Formative Assessment?

There are many definitions of formative assessment. For clarity, we have categorized them in the following ways: teacher-focused, student-focused and a broad, more encompassing definition.

Teacher-focused formative assessment occurs when the assessment is used to change the teacher's actions or instruction (Black, Harrison, Lee, Marshall, & Wiliam, 2004; Black & Wiliam, 1998; Boston, 2002; Ginsburg, 2009; McIntosh, 1997). Black and Wiliam (1998) found that formative assessment, like teaching, is interactive and that formative assessment uses information to adapt instruction to student needs. Such assessments may be of three types: on-the-fly, planned, or embedded (Shavelson, et al., 2008). On-the-fly assessment is unplanned and occurs when a teacher unexpectedly recognizes a "teachable moment" happening in the classroom. This type of formative assessment will usually involve unplanned questions and observations of students (Ginsburg, 2009). Planned formative assessment is when teachers can exert the most control by planning key questions ahead of time, preparing follow-up questions, and trying to use higher-order questions to promote thinking. Embedded-in-the-curriculum formative assessment refers to formal assessments that curriculum writers place at important spots within a curriculum. This assessment comes ready to use for educators to implement in their classroom, but often lacks flexibility.

Student-focused formative assessment is also known as "assessment for learning" (Black & Wiliam, 1998; Hodgen, 2007; Nicol & Macfarlane-Dick, 2006). As Black and Wiliam (1998) stated, "assessment for learning is any assessment for which the first priority in its design and practice is to serve the purpose of promoting student's learning" (p. 2). Another source defines assessment for learning's purpose as informing teachers and students of the gap between what the student can do and what the student should be able to do with immediate feedback (Shavelson, et al., 2008). As one can see, these definitions focus on the use of assessment for a student's own progress.

Researchers have attempted to use the phrase "formative assessment" to reference a teacher point of view and the phrase "assessment for learning" to focus on student learning; however, such distinctions are not consistent. A broad definition is often used in research articles that encompass both teacher changes to instruction and student improvements in learning (Looney, 2011; National Council of Teachers of Mathematics, 2013; Shavelson et al., 2008; Wilen, 1991). For our purposes, we will use “assessment for learning” when solely focusing on student improvement. Otherwise, we will use “formative assessment” in its broadest sense to be inclusive of the definitions we found.

Why is Formative Assessment Important?

Formative assessment is important because students vary in instructional needs, and teachers need to be aware of how to best adapt their instruction to meet these needs. The National Council of Teachers of Mathematics states, "Effective formative assessment has a positive impact on student achievement and how they perceive themselves as learners" (NCTM, 2013, p. 2). In regards to achievement, research has found that the “typical effect sizes of the formative assessment experiments were between 0.4 and 0.7” (Black & Wiliam, 1998, p. 2). Such effect sizes indicate that improvements in formative assessment techniques can have achievement gains for students. Further, formative assessment can have an impact on student perceptions and help students take control of their own learning. For example, "learners who are more self-regulated are more effective learners" (Nicol & Macfarlane-Dick, 2006, p. 205).

Formative feedback is one critical component of formative assessment. Studies have shown that a grade is not enough for improvement, there must also be feedback given for how the student can improve (Black & Wiliam, 1998). This leads to the question: Why is comment-based feedback important? When students know the feedback they receive is being used to better their learning instead of just assigning them a grade they are likely to learn more (Black, et al., 2004). Further, comment-based feedback is crucial for student motivation. Although the need to motivate students is evident, it is often assumed that offering such extrinsic rewards as grades, gold stars, and prizes is the best way to do it. There is, however, ample evidence to challenge this assumption. Students who are told that feedback "will help you learn" achieve more than those who are told that "how you do tells us how smart you are and here are the grades you'll get" (Black, et al., 2004). In a competitive system, low achievers attribute their performance to lack of ability; high achievers, on the other hand, attribute performance to their effort. In comparison to a task-oriented system, students attribute performance to effort, and learning is improved, particularly among low achievers. A comprehensive review of research of feedback found that feedback improved performance in 60% of the studies. In the cases where feedback was not helpful, the feedback turned out to be merely a judgment or grade with no indication of how to improve (Black, et al., 2004).

Questioning also plays a vital role in formative assessment. Questioning has traditionally been considered the essence of effective teaching because of the multiple functions that questions serve. It has been suggested that questions serve two major purposes: to ascertain whether students remember and understand what has been taught, and to have students apply what they have learned (Wilen, 1991). To achieve these purposes, teachers can ask questions that require different levels of thinking from the student. Low-order questions focus on recalling of information while high-order questions require application of content knowledge. Wilen (1991) describes convergent questions as testing "basic knowledge, skills and understandings" (p. 13). Divergent questions, on the other hand, require critical thinking (see Table 1).

	Convergent	**Divergent**
Low order	What is the quadratic equation? (This type of question simply asks students to recall information without much thought. Answers are expected to be the same.)	Why does the quadratic formula work? (The solutions to this problem are similar but will still vary to some degree.)
High order	Solve $x^2 + 4x + 4 = 0$. (This problem has an application aspect to it but the answers still expected to be the same.)	Create a problem using the quadratic formula. (This type of question allows for many diverse answers. Students may write problems that focus on: applications, definitions, properties, etc.)

Table 1: Example questions with classifications and justifications

Teachers typically ask questions that require students to recall basic facts and memorized information (Wilen, 1991). To make questioning more formative for students, teachers might try to shift away from the teacher-dominated initiate-response-feedback pattern of low-order and convergent questioning (Hodgen, 2007). One way that teachers can do this is by trying to understand why students answer questions the ways they do and then interpreting their answers with more thought. Another way to shift discussion from being teacher-centered to being student-focused is to extend wait time after asking a question to at least three seconds. Teachers may also try to ask more high-order divergent questions, as they are the most useful in prompting critical thinking (Wilen, 1991).

How Can Teachers Use Formative Assessment?

There are many different ideas and techniques for how to implement formative assessment in the classroom and we have created a list of suggestions below. To assist in organizing the discussion, we have grouped ideas from research and our own brainstorming using three broad ideas: (a) How to create an atmosphere in which formative assessment is beneficial, (b) How to give feedback, and (c) How to incorporate assessment for learning.

How to Create an Atmosphere in which Formative Assessment is Beneficial

In order for formative assessment to be well received by students, teachers will need to help students be more active in their role as learners. Here are some suggestions from research articles to develop an environment where students can learn from their mistakes and the mistakes of others.

• *Encourage teacher and peer dialogue around learning* (Nicol & Macfarlane-Dick, 2006). Unfortunately, whole class discussion is not a practical way to achieve such dialogue (Hodgen, 2007). Instead, we recommend small group discussions.

• *Encourage positive motivation and self-esteem* (Nicol & Macfarlane-Dick, 2006). For example, encourage students to have a growth mindset (a belief that almost anything can be learned with effort and guidance) instead of a fixed mindset (some people are smart and others are not) (Dweck, 2007).

• *Incorporate formative assessment with summative tests.* A teacher can do this by having students help generate and answer their own questions for exam preparation. When students help generate exam questions they are given an opportunity to "think about what makes a good question for a test and in doing so need to have a clear understanding of the subject material" (Black, et al., 2004, p. 15).

• *Use student interviews to uncover student thinking* (Ginsberg, 2009; McIntosh, 1997). Although it would be nice to interview all students, it might be more practical to interview a random sampling of students during work times instead of focusing on one type of student, e.g., low-performing students.

• *Incorporate diagnostic testing as a type of pre-assessment to find out what students know before the instruction begins* (Ginsberg, 2009; McIntosh, 1997).

How to Give Feedback

Feedback is arguably one of the most important aspects of formative assessment if not the most important. However, the feedback that teachers give and receive has particular characteristics that educators must take into consideration. Feedback that teachers give to students is crucial to the learning experience. Black and Wiliam (1998) provide some tips on how to give the best possible feedback. Feedback on *how* students can improve must be included. This type of commentary can be accomplished through giving exemplars (good examples of past work or problems with uniqueness). Another way to provide feedback on how students can improve their work is through rubrics. A problem of this type of feedback is that students may see the rubric as a checklist and the students might lose creativity. Unfortunately, students tend to ignore comments when grades are also given. Students just given a grade see it as a way to compare themselves to others. On the other hand, students that are given ideas on how to improve themselves, rather than just seeing grades as a competition, tend to learn more (Black, et al., 2004).

According to Nicol & Macfarlane-Dick (2006), there are three necessary conditions for feedback to be useful. First, students need to know what good performance is by accepting the goal or target standard. Unfortunately there are usually mismatched goals between teachers and students. For example, teachers want students to understand the content, whereas the student may just want to get the assignment finished. Second, students need to be able to compare current performance to good performance. Teachers may present a model assignment to show students what good performance is, but such models often come before a student has had a chance to try the assignment on his or her own. Third, students need opportunities to close the gap between current performance and good performance. The most common example is to allow students to resubmit the assignment or to provide earlier feedback on a work in progress.

In order to collect the best feedback, it is important to gather feedback from all students rather than a select few (Shavelson, et al., 2008). Some ways to do this are exit slips (problems given to students towards the end of the class period to complete before leaving), mini white boards (students individually work on problems on a personal white board and hold up their work so that only the teacher can see their work), and hand signals such as a thumbs up or down from each student in response to how they feel about their current understanding of the content.

How to Incorporate Assessment for Learning

Using appropriate questioning tools is also helpful in formative assessment. Educators should stay away from low-level questions with one-word answers and instead move toward questions that are thought provoking and require more time to answer. There are various ways to move towards questions that require critical thinking such as phrasing questions clearly and increasing wait time to several seconds after asking questions (Black, et al., 2004; Wilen, 1991). Some exercises that ensure a more equal distribution of input from students are:

• *Incorporate Think-Pair-Share activities*. This activity is done by students thinking independently about a question or topic raised. The students then pair up with others or get into small groups and share their ideas with one another. After this the class comes back together as a whole and discusses the brainstormed ideas.

• *Apply the Predict-Observe-Explain assessment* (Shavelson, et al., 2008). Students explain their preconceived ideas or predict an outcome. Next, students observe if their ideas are correct or incorrect (to create cognitive dissonance). Students and teachers work together to fix the gaps between predictions and observations.

• *Provide opportunities to close the gap between what a student knows and what the student needs to learn* (Nicol & Macfarlane-Dick, 2006). For example, peer reviews give students time to correct their work before handing it in, are useful in saving time for the teacher, and students are more likely to listen to other students. However, peer reviews can be problematic when students are not as critical as they could be and do not provide useful feedback. Therefore, when completing a peer review the students must be especially aware of the goals of the lesson (Shavelson, et al., 2008).

• *Provide students with opportunities for self-assessment.* Self-assessment is also important for formative assessment to work, but "self-assessment will happen only if teachers help their students to develop the skill [of self-assessment]" (Black, et al., 2004). For example, students can learn how to score assignments in a manner similar to the teacher.

• *Provide students with opportunities for peer assessment.* Peer assessment is an important complement to self-assessment. The National Council of Teachers of Mathematics stated, "formative strategies embedded in instruction provide opportunities for students to make conjectures, incorporate multiple representations in their problem solving, and discuss their mathematical thinking with their peers" (2013, p. 2). Peer assessment is uniquely valuable because students may accept criticisms of their work from one another that they would not take seriously if offered by a teacher. Peer work is also valuable because the interchange is in a language that students themselves naturally use and because students learn by taking on the roles of teachers and examiners (Black, et al., 2004).

• *"Mathematize" by casting work in an explicit, mathematical form* (Ginsburg, 2009). In this exercise, teachers help students move from informal or invented strategies to formal mathematical notation. Having students "mathematizing" provides them with opportunities to explain work using purely mathematical language.

• *Use cold calling instead of asking students to raise their hands.* Creating a supportive environment in which "pupils are comfortable with giving a wrong answer" is also important to improve dialogue in the classroom (Black, et al., 2004, p. 12). This technique should be saved for classrooms in which a supportive environment is already established.

• *Use chained questions or student dialogue on using student responses to create new questions* (Wilen, 1991). Here is an example of a conversation using chained questioning effectively:

> *Teacher: Can someone tell me what the x-intercept for the line $y = 4x - 3$ would be?*
>
> *Student: Sure! That would be (3/4, 0).*
>
> *Teacher: How did you find that answer?*
>
> *Student: I set the y equal to zero and then subtracted 3 and divided by 4.*
>
> *Teacher: Why did you set $y = 0$ instead of $x = 0$?*
>
> *Student: Because the $x = 0$ is for the y-intercept.*
>
> *Teacher: Would a similar procedure work for y = 4x^2 - 3? (and so on…)*

Here we can see that the teacher did not stop at just receiving the answer, as he/she wanted to fully understand why the student answered in that way. The teacher also took the opportunity to use further high-order questioning to expand her lesson into another unit on parabolas.

Conclusion

In preparing this article, we were a little surprised that some research on formative assessment focused more on teacher change than student improvement. As such, we had hoped to create a distinction between formative assessment (with a teacher focus) and assessment for learning (with a student focus). Such a distinction, however, is difficult because the terms are rarely used consistently in articles. Despite the lack of clarity, we offer the following final thoughts on formative assessment that appeared to resonate across the articles.

• The research is unclear about how to make instructional changes, likely because such changes depend on many variables (students, curriculum, policies, etc.).

• Teachers can use summative tests in formative ways.

• Students need to internalize the instructional goals.

• Students need clear, actionable feedback for how to improve.

• Once an assessment receives a score, much potential for assessment for learning is lost because students see the score as final, that is, they believe there is no more learning to be done.

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Computer Science Needs Mathematics Teachers

*By Dennis Brylow, Thomas Gendreau, Joe Kmoch, Andrew Kuemmel, Marta Magiera**

> **"The PUMP-CS team is currently recruiting certified high school mathematics teachers interested in teaching computer science."**

Preparing the Upper Midwest for Principles of Computer Science (PUMP-CS) is a National Science Foundation funded project that aims to double the number of certified computer science teachers in Wisconsin and to increase and improve the computer science courses offered in Wisconsin high schools. The PUMP-CS team includes members from Marquette University, the University of Wisconsin-La Crosse, Wisconsin-Dairyland chapter of the Computer Science Teachers Association (CSTA) and DPI.

The PUMP-CS project provides professional development opportunities focused on two courses: Exploring Computer Science (ECS) and AP Computer Science Principles (APCSP). ECS (http://www.exploringcs.org) is a nationally recognized curriculum aimed at 9th and 10th graders. The course is built around 6 units: human computer interaction, problem solving, web design, programming, computing and data analysis, and robotics. ECS was developed as a curriculum intended for all students. Just as most high school students take courses in Algebra or English, ECS is intended as a general introduction to computer science suitable for most high school students.

APCSP (http://apcsprinciples.org) is a new advanced placement course currently under development. The course is built around six computational thinking practices and seven big ideas. The computational thinking practices include: connecting computing, creating computational artifacts, abstracting, analyzing problems and artifacts, communicating and collaborating. The seven big ideas are: creativity, abstraction, data and information, algorithms, programming, the Internet and global impact. From a general education point of view one can see the computational thinking practices are applicable to a wide variety of problem solving situations The seven big ideas make more explicit the range of activities and skills used in the development of computer systems than students frequently see in a traditional introduction to programming course. APCSP is currently offered at 50 pilot sites across the country with three pilot sites in Wisconsin. The first nationwide offering of the course is scheduled for the 2016–17 academic year with the AP test given in May of 2017.

Both courses present computer science in a broader context than traditional entry level programming courses and foster computational thinking skills, which should be part of the education of all students in the 21st century. There is a great deal of overlap between the concepts and dispositions that computational thinking teachers and mathematics teachers seek to develop in students. Concepts such as problem analysis, abstract reasoning, algorithms and search for structure are fundamental to instruction in both areas. Likewise, dispositions such as perseverance, the ability to collaborate, and comfort with complexity and open-ended problems are necessary to solve problems in both mathematics and computation. ECS and APCSP aim to show students the broad range of talents and problem solving skills used in the development of computer systems. Both courses promote a project-based inquiry approach to learning and use student interest and experience as motivation for teaching computing concepts. (http://computinged.wordpress.com/2014/09/14/guest-post-by-joanna-goode-on-cs-for-each/).

As an example of curricular ideas emphasized in these courses, consider an ECS lesson built around the idea of sorting a sequence of elements. The lesson can start by asking students to identify examples where sorting is useful. Since the word sorting is used in a variety of ways and contexts (e.g., sorting a list of names or sorting laundry), an outcome of this part of the lesson is for students to understand the common meaning of sorting in computer science and the wide variety of places sorting is useful.

The next part of the lesson has students learn the steps of two sorting algorithms: selection sort and quicksort. Students learn the algorithms by being given a list of instructions to be followed by a person (i.e., not in any particular programming language) and following the steps with some appropriately created manipulatives. The manipulatives should be designed in such a way that they can only be (easily) compared two at a time to simulate the comparison of two variables. The outcome of this part of the lesson is for students to understand that creating an algorithm requires creating a precise sequence of instructions that can be mechanically followed by any "computer" including a human being.

The last part of the lesson asks the students to compare the two algorithms. Students are asked to consider how two algorithms could be compared and what it means to say one algorithm is better than another. During this process students (possibly with some hints from the teacher) should consider counting the number of times items needed to be compared. Based on this metric students will find that quicksort usually requires significantly less comparisons. The outcome of this lesson is for students to see that evaluating an algorithm not only involves correctness (a necessary feature of any algorithm) but also involves evaluating performance based on some metric.

Although the activities in this lesson address important computational thinking concepts they also engage students in many of the Standards for Mathematical Practice (SMP) in the Common Core State Standards for Mathematics. For example, all three parts of the lesson address SMP 6: attend to precision. In the first part of the lesson, after identifying many ways the word sorting is used in common language the students identify (with help from the teacher) the precise meaning of the term as it is used in computer science. In the second part of the lesson, the students must read and execute the instructions of the sorting algorithm exactly as written in order for the sorting algorithm to work. Finally, while comparing both algorithms, the students must precisely count the number of comparisons each of the two algorithms makes, in order to compare the performance of the two algorithms. This part of the lesson facilitates deeper thinking about precision by encouraging the students to define what it means for one algorithm to be better than another. In this case "better" means the algorithm does fewer comparisons.

Similarly this lesson allows for connections to SMP 1 (make sense of problems and persevere in solving them), SMP 2 (reason abstractly and quantitatively), SMP 3 (construct viable arguments and critique the reasoning of others) and SMP 7 (look for and make use of structure). For example, the quicksort algorithm is an example of a divide-and-conquer algorithm. This is a frequently used technique in algorithm design and requires the designer to identify the underlying structure of a problem to see that this technique can be used to improve the performance of the algorithm.

The PUMP-CS team is currently recruiting certified high school mathematics teachers interested in teaching computer science. The project will summer workshops addressing both ECS and APCSP in 2015 and 2016 and on-going academic year support. The grant provides stipends for participating teachers. Any certified high school teacher is eligible to teach ECS. Teaching APCSP requires a 405 license. We are currently working with DPI to identify a Praxis-like test that certified high school mathematics teachers with previous computing experience could take in order to get a 405 license. The ECS workshops will be held at Marquette University while the APCSP content will be part of the computer science methods workshops held in La Crosse.

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Let the Pilot Fly the Plane: Advocating for Our Work as Teachers

By Michael D. Steele, University of Wisconsin-Milwaukee

In a political and policy climate in which teaching and classroom practice are coming under increasing attack, how might teachers situate and advocate for their work? In this article, I articulate aspects of the knowledge base for teaching mathematics to illustrate the ways in which the work of teaching is supported by a professional knowledge base. Three specific suggestions follow for educators regarding ways in which they can use their professional knowledge base to engage educational stakeholders in meaningful conversations about mathematics teaching and learning.

"I can't believe my doctor's office is insisting that my son needs an MRI. Why can't they just use leeches like they used to? Those were standard practice years ago."

"I can't figure out how to work the GPS in my car, so I made it clear to the pilot of this plane that she shouldn't use hers, either."

Statements like these about the fields of medicine and aviation would be patently ridiculous in most circles. Unless you are a doctor or a pilot, one would not presume to tell these professionals how they should be doing their jobs—and even then, the specialized nature of knowledge and expertise within these professions would likely lead for one to defer to the professional (for example, as an amateur pilot, I might compliment my 727 pilot on a nice landing in tough conditions, but would not presume to tell him how he should have handled the deployment of his flaps on descent). So why are similar statements acceptable in the context of teaching mathematics? Like doctors and pilots, electricians and plumbers, engineers and artists, we have specialized preparation and a professional knowledge base that is unique to our work as teachers. In an era in which the work of teaching is increasingly politicized and polarizing, we as a profession need to become stronger, more vocal, and better informed advocates for the work that we do with students.

In this article, I discuss the evolution of the teaching profession and how that evolution has (and has not) shifted public perception of our work. Surveying the research about knowledge for teaching, I articulate the ways in which the knowledge for teaching mathematics is an instantiation of professional knowledge, and make comparisons to the public perception of the nature of knowledge needed to teach. Using the current policy climate as a backdrop, I close with some guidance and challenges for how teachers can advocate for their profession as a whole and for their particular practice in their classrooms, schools, and districts.

Mathematical Knowledge for Teaching: A Brief Retrospective

The work of understanding and describing the mathematical knowledge needed for teaching is marked by a key time point in a teacher's career—when they successfully exit a certification program and begin their classroom teaching career. During the preparation phase, teachers learn broadly about mathematics content, with a focus on the content they will need to teach for their selected certification grade bands; about the pedagogy of teaching, including issues of learning, development, assessment, and management; and the intersections between content and pedagogy, including perspectives on student thinking and misconceptions, and ways in which to support learning in specific topics. Pre-service teachers learn a few things along the way from student teaching that are not necessarily found in research articles, books, or university classrooms about how students learn and how we might teach them well. Following this phase, teacher learning is dominated by practical experience in the classroom, dotted with more formal learning through professional development experiences large and small. These learning experiences lead to the development of different aspects of mathematical knowledge for teaching: subject-matter knowledge, pedagogical content knowledge, and pedagogical knowledge, and two types of knowledge development: theoretical knowledge and craft knowledge.

Aspects of Mathematical Knowledge for Teaching

From nearly the moment that a pre-service teacher begins his/her student teaching, it becomes abundantly clear that simply knowing how to do mathematics problems is not sufficient to know how to teach mathematics effectively to students. Shulman (1986) in a presidential address to the American Educational Research Association nearly 30 years ago, identified three aspects of knowledge that teachers need to teach effectively: subject-matter knowledge, general pedagogical knowledge, and pedagogical content knowledge. The notion of pedagogical content knowledge was relatively unheralded, and characterized the ways in which content and teaching interact. For example, understanding three different ways in which one can find the solution to a system of equations and good (and bad!) examples for each one, is a prime example of pedagogical content knowledge. Pedagogical content knowledge is important, as it is uniquely useful in the profession of teaching.

A number of research studies have found that aspects of this unique knowledge base matters with respect to student learning. Research has shown that aspects of pedagogical content knowledge are measurable and distinct from other kinds of mathematical and pedagogical knowledge (Hill, Rowan, & Ball, 2005). More importantly, differences in mathematical knowledge for teaching are linked both to the quality of mathematics instruction that teachers enact and the substance of student learning that results (Baumert et al. 2010; Charalambous, 2010; Hill, Rowan, & Ball, 2005; Tchoshanov, 2011). This particular sort of knowledge of the multiple ways in which we might think about mathematics content, the misconceptions that students might have about that content, and the range of examples and pedagogical practices that support the teaching and learning of that content, is the quantifiable and unique professional knowledge base for teaching mathematics not found or needed in other mathematics-intensive professions.

Theoretical Knowledge and Craft Knowledge

Mathematical knowledge for teaching evolves in two primary forms. University teacher preparation and in-service professional development cultivate knowledge that we would generally refer to as *theoretical knowledge*. These experiences might build subject-matter, pedagogical content, or general pedagogical knowledge, and they do so in ways that aren't immediately connected to the work of teaching in classrooms. The important work of translating that knowledge into practice is left to the teacher as they plan, teach, and reflect on their practice with students. That is not to say that theoretical learning is impractical—the contemporary press towards practice-based professional development (Smith, 2001) that makes use of student work, narrative and video cases, lesson plans, and other artifacts of teaching, has been shown to have significant potential to more easily connect theoretical knowledge to classroom practice (Steele & Hillen, 2012).

Craft knowledge evolves from the practice of teaching and is described as, "deep, sensitive, location-specific knowledge of teaching [including] fragmentary, superstitious, and often inaccurate opinions" (Leinhardt, 1990, p. 18). The development of craft knowledge is essential to growth in teaching—it is the primary context in which teachers learn new and important ideas about their practice that can develop into mathematical knowledge for teaching. Capturing and generalizing craft knowledge can be a challenge—by its nature, it is highly contextualized, and teachers have limited opportunities to codify that knowledge in writing, reflect on it, and discuss it with other teaching professionals. This can lead to the notion that what works in one classroom may not, or does not, work in other classrooms, and the fragmentation of the knowledge base.

A Profession Suggests a Professional Knowledge Base

The nature of mathematical knowledge for teaching and the types of knowledge that develops (theoretical and craft) is important to consider when we think about the ways in which the public views the work of teaching. The notion of a profession brings with it a sense of a professional knowledge base that is unique to that line of work, such as the biological and medical knowledge of a veterinarian, the mechanical engineering and biomechanics knowledge of a bicycle designer, or the understanding of human cognition and therapeutic techniques of a clinical psychologist. Note that for these professions, one assumes a strong set of theoretical knowledge, alongside craft knowledge gained through experience. Experience is significantly valued in these professions—we tend to trust practitioners who have the sorts of established track records that come with experience working in the field.

In contrast, teachers tend to be associated with careers that are more heavily reliant on craft knowledge. "Alternative" certification programs like Teach for America rush teachers into the classroom with little or no pedagogical training and no attention to the particulars of pedagogical content knowledge, relying on the individual's disciplinary knowledge gained through a college degree to be sufficient to get started with teaching. The rise in popularity of these programs reflects a cultural norm that the work of teaching is learned on the job, and there is little or no benefit to the development of a theoretical knowledge base to prepare for the work. Compare this view to how pilots are trained and certified to fly. A typical training program for aviators begins with ground school, a rigorous study of the theoretical aspects of flight, combined with logging hours of practical experience that has a specific and identifiable curriculum to the work. When a pilot is certified through both the practical and written assessment, regular logged hours are required, and recertification is needed if the pilot chooses to fly a different make or model of aircraft or class of planes. This combination of theoretical and practical work, with assessments in both areas, is not unlike a university-based teacher preparation program. So why is the public's perception so very different?

A New View of Outreach: Advocating for Our Professional Knowledge

There are myriad reasons as to why teaching finds itself at the nexus of a political crisis at present. The involvement of children, the broad impact of the work of teaching on every community, and the complex policy and funding structures of education are prime reasons for turning teaching into a political issue. These facts notwithstanding, one would find it unlikely that politicians and legislators would have the same level of involvement in regulating such aspects of professions that are seen as having a respected and codified professional knowledge base. To truly silence continued unprincipled attacks on teachers and overreach into our professional practice, we must work to change the perception of the profession of teaching. Three specific suggestions for teachers to work towards that goal are provided below.

Communicating to Parents and Administrators the What and the Why

In the past twenty years, communicating efficiently and effectively with parents and community members has become increasingly easy for schools and districts. Web presences, email, physical and electronic newsletters, and stronger school records have allowed parents, teachers, administrators, and community stakeholders to exchange more information about what is happening in schools. Initiatives such as District Math Nights have given parents a stronger knowledge of what teachers are doing in their classrooms. Often times, however, these communications focus more strongly on what is going to happen than why it is taking place.

For example, a meeting I recently attended related to the adoption of a new district curriculum focused almost exclusively on describing the curriculum adoption process, the particular features of the new materials, and the timing of the rollout. Little time was spent with parents and community members helping them understand why a new curriculum was needed, how teachers had learned and would continue to learn about new mathematical strategies and content that would be taught, and why these changes were important for students' short- and long-term educational futures. In addition to communicating to parents why we are making changes to our teaching practice that may be different from their own experiences, we as teachers should make efforts to explain why we believe these changes will be successful for students. For example, helping parents understand the alternative algorithms for multiplication and division that their students will see is important. Beyond this, a strong parental outreach should also describe the ways in which these processes support the development of number sense, and set students up for success with arithmetic and algebraic concepts that are traditionally sticking points for students.

Administrators serve two important roles in this work—as the evaluators of teaching practice, and as one of the primary points of contact for parents with concerns and questions. As such, we also have a responsibility to provide our own school administrators with outreach to help them understand our work in the classroom. Stein & Nelson (2003) describe the notion of leadership content knowledge, which they cite as a critical component to the success of instructional reform efforts. Leadership content knowledge includes knowledge at the system level, the instructional (classroom) level, and knowledge of the content. Supporting administrators in understanding examples of the content to be learned and the ways in which new pedagogical approaches support that content will help them be better and stronger advocates for teachers' work in the classroom.

Prominently and Publicly Displaying Professional Learning

Displaying artifacts of student learning is a common practice that emphasizes the norms and values of the classroom community and communicates about what it means to know and do mathematics in the classroom. Yet as teachers, we rarely create similar displays that represent our own learning and establish the teacher as a growing and learning member of the classroom community. Displaying artifacts of teacher learning invites engagement and conversation about how teachers are evolving their own teaching practice. For example, a teacher might choose to prominently display that they are members of the National Council for Teachers of Mathematics and the Wisconsin Mathematics Council, their artifacts from professional development experiences, or their work towards National Board certification. Educational credentials, seminar participation certificates, and awards can be prominently displayed in the classroom for all to see, in the same ways that we might see a mechanic's certification or a doctor's diplomas and professional memberships displayed in the public areas of their offices and businesses.

Professional learning frequently occurs out of the view of students, parents, and sometimes colleagues. Providing public and prominent artifacts of that professional learning and making the work the topic of discussion and inquiry can lead to stronger understandings of what teachers learn outside of the classroom and why that learning is important. For example, I have suggested to my pre-service teachers who are entering the first year of solo classroom teaching that they create signs to encourage student, parent, and administrator dialogue about their practice, such as, "Ask me why our desks are in groups," or "Ask me why I require multiple solution methods for some problems." Displays and invitations to engage such as these communicate to students, parents, and administrators that teaching is not an independent, idiosyncratic exercise—that it is thoughtful, principled work that builds on both theoretical and practical knowledge.

Connecting to Educational Research

A final suggestion to strengthen outreach efforts with all education stakeholders is for teachers to remain connected to contemporary work in educational research. This is an area in which local professional learning communities can support the work of reading research and engaging teachers in meaningful discussions about what current findings might mean for policy and practice. Community beliefs that teaching is something that one learns on the job and that knowing the content is the only important preparation can be combatted by making use of data that demonstrates how and why students learn mathematics content in particular ways, and why we might take specific instructional approaches in mathematics.

It is also important that school and district professional learning communities continue to find ways to collect and make use of data in systematic ways that constitute action research. By being able to describe local trends in addition to national and international research, a district can demonstrate the ways in which policies and practices are influencing student learning. Too often, this work does not leave the confines of a department meeting conversation around the development of a new assessment, or a school-level analysis of standardized test scores. Making public, where appropriate, the scholarly work that teachers and departments engage in to better understand student learning would go a long way towards conveying a professional knowledge base for teaching.

Advocacy: Teaching Beyond the Classroom Walls

In the current era of political turmoil in Wisconsin and nationally, we are witnessing unprecedented attacks and intrusions into our practice from politicians and special interest groups. Current rhetoric regarding the Common Core State Standards—that it is indoctrination, that it takes away local control, that it represents a lowering of academic standards, and that it is inherently detrimental to students—represents uninformed views and misconceptions about standards, curriculum, teaching, and assessment. If we as educators do not speak back to these voices with informed, research-grounded advocacy, we risk the continued erosion of the professional standing of teaching. Educators must take on advocacy as a form of teaching that extends beyond the classroom walls, and that not only defends our profession from uninformed attack, but also positions educators as professionals with expertise that use knowledge to make informed decisions. Educators should be able to describe to the community the ways in which the Common Core State Standards provides curricular coherence and reflects similar standards to other high-achieving countries (Schmidt & Houang, 2012), the specific cognitive strategies that teaching with the Standards can support (Conley, 2011), and the ways in which their district is continually monitoring progress and modifying teaching practice. As educators, we must convey that we have the professional knowledge and expertise to fly our planes to new heights of student achievement.

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Notes

Notes

WMC PUZZLE PAGE

Search-A-Word: Mathematicians

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E G N A R G A L O I S A B E L
O E L I L A G O N A D R A C U
Y K S V E L A V O K R C T M I
R I E M A N N N O R E H L S C
M G L Y M E L O T P H I O N C
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L Y H S I P M R N I E E E A N
L J T B E A O A E R O D L M O
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K R M O E P F R L S N R T A S
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ABEL	LAGRANGE
AGNESI	LEIBNIZ
ARCHIMEDES	LOVELACE
BANNEKER	MANDELBROT
CARDANO	NEWTON
DESCARTES	NOETHER
EINSTEIN	PASCAL
FIBONACCI	POINCARE
GALILEO	POLYA
GALOIS	PTOLEMY
GERMAIN	PYTHAGORAS
HERON	RAMANUJAN
HYPATIA	RIEMANN
KHAYYAM	SOMERVILLE
KOVALEVSKY	THALES

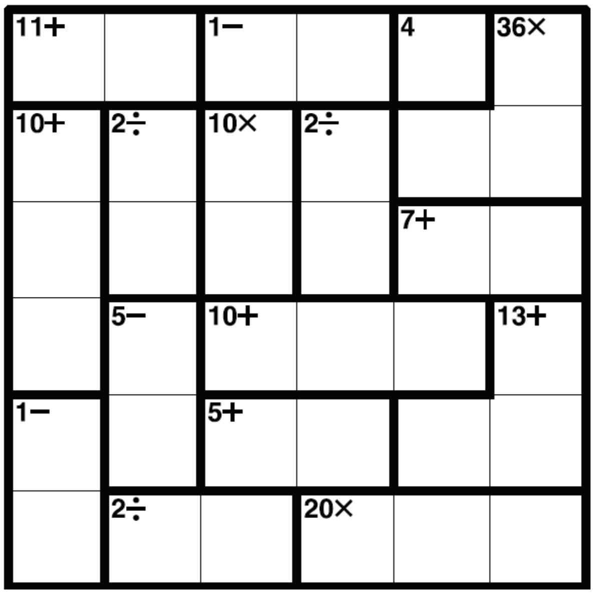
State Mathematics Competition

The following problem is from the 2010 High School State Mathematics Contest. For additional questions and solutions, visit www.wismath.org/resources/math-contests .

I am a 3-digit number that is divisible by 25. If you were to subtract 1 from me, I would then be divisible by 26. What number am I?

Ken Ken

Fill in the blank squares so that each row and each column contain all of the digits 1 through 4. The heavy lines indicate areas that contain groups of numbers that can be combined (in any order) to produce the result shown with the indicated math operation.



Sudoku

Fill in the blank squares so that each row, each column and each 3-by-3 block contain all of the digits 1 through 9.

			5			6	7	2
	6				3			9
			2		7	8		
			9	5		3		
7	9						8	6
		6		2	8			
		5	1		2			
1			8				6	
3	7	8			5			

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