NCTM Outstanding 2013 Publication Award

Wisconsin TEACHER of MATHEMATICS

Conceptual Understanding, Strategic Competence & Adaptive Reasoning

Understanding Conceptual Understanding

Developing Geometry Knowledge in the Early Grades

WISCONSIN MATHEMATICS COUNCIL, INC.

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Wisconsin Teacher of Mathematics
Spring 2014 Journal

math soup
a recipe for student success

The Spring 2014 *Wisconsin Teacher of Mathematics* journal will focus on the implementation of the Common Core State Standards. The WMC editorial panel would like to showcase instructional activities that are aligned with the CCSSM Content and Practice Standards. We are interested in articles that address the following questions:

- How do you engage students in the Standards for Mathematical Practice?
- What steps has your school district taken to integrate the CCSSM into classroom practice?
- How do you design activities that are aligned to the content domains and/or conceptual categories of the Common Core State Standards for Mathematics?
- How do you support classroom teachers in the integration of the CCSSM into classroom practice?

If you have any questions or ideas for the Spring journal or wish to submit an article or activity for review, please visit the WMC website for more information: www.wismath.org/resources/. The deadline for manuscripts for this Spring's issue is January 31, 2014.

MANUSCRIPT SUBMISSION GUIDELINES

- **Send an electronic copy of your manuscript to the Wisconsin Mathematics Council. Manuscripts may be submitted at any time for review.**
- **Manuscripts should be typed, double-spaced.**
- **Include all figures and photos in .jpg format; submit high resolution copies of figures and student work. Please do not place figures or photos within the document; rather indicate their placement in the document, e.g., Figure 1 here.**
- **All fractions need to be formatted as follows—2/3.** *Do not accept auto formatting of fractions.*
- **All manuscripts are subject to a review process.**
- **Include name, address, telephone, email, work affiliation and position.**

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The deadline for Spring journal submissions is January 31, 2014.

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Wisconsin Teacher of Mathematics

The National Council of Teachers of Mathematics has selected the Wisconsin Teacher of Mathematics to receive the 2013 Outstanding Publication Award. This prestigious award is given annually to recognize the outstanding work of state and local affiliates in producing excellent journals. Judging is based on content, accessibility, and relevance. The WMC editors were recognized at the NCTM Annual meeting in April in Denver.

President's Message

As a teacher, I have always been a proponent for shifting the focus in the mathematics classroom away from a setting that is dominated by computational or procedural work to a more balanced approach where the procedural skills have a meaningful purpose.

I remember the excitement I experienced when I first read about what it means for a student to be mathematically proficient in the 2001 publication by the National Research Council called Adding It Up: Helping Children Learn Mathematics. The writers had reviewed the relevant research related to learning mathematics from pre-kindergarten through eighth grade and they determined there were five major strands that had to occur interdependently if a child is to become mathematically proficient. The five strands include: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition.

The thought that a mathematically proficient student should be on a developmental path in five different strands is sensible, yet overwhelming. When I think about the stages of my personal journey through learning mathematics as a child, a college student, and a math teacher, I know my lack of exposure to all five strands during childhood is something I continually work to overcome. Unfortunately, this is all too common with many teachers. I know my college coursework and my teaching experiences have allowed me to continue on a path toward mathematical proficiency, but the collective work of many dedicated mathematicians and researchers has greatly assisted me along the way. The first people that come to mind are right here in Wisconsin. I have been very fortunate to have access to the mathematical thinking of Dr. Jennifer Kosiak, Dr. DeAnn Huinker, Dr. Kevin McCloud, Dr. Henry Kepner, Dr. Richard Askey, and Dr. Billie E. Sparks, just to name a few.

This is why I always look forward to reading this publication from the Wisconsin Mathematics Council and I hope the contributions shared in this issue provide pathways to greater mathematical proficiency for you and your students.

Doug

Doug Burge

WMC President, 2013-215

Editors' Notes

The Fall 2013 journal focuses on the three of the five Strands of Mathematical Proficiency (National Research Council, 2001). These strands include conceptual understanding, strategic competence, and adaptive reasoning. Conceptual understanding is defined as the "comprehension of mathematical concepts, operations, and relations" (p. 116). Hlas' article further defines conceptual understanding and provides teachers with ways to help build upon this type of understanding at all grade levels. Teaching with a focus on conceptual understanding also provides students with an "integrated and functional grasp of mathematical ideas" (p. 118). McCool and Knudsen's article focuses on the early development of geometric understanding that provides young learners with the opportunities to build upon prior knowledge of shapes to learn new concepts.

Strategic competence is defined as the "ability to formulate, represent, and solve mathematical problems" (p. 116). Often referred to as problem solving, this Strand of Mathematical Proficiency requires students to "formulate the problem so that they can use mathematics to solve it" (p. 124). Askey's article supports students' abilities to solve worthwhile mathematical tasks related to the divisibility tests. Schmid's article also focuses on building students' competency with solving real world problems. In this article, he focuses on how he made connection between physics and mathematics to help students come to a deeper understanding of slope.

Finally, adaptive reasoning is defined as the "capacity for logical thought, reflection, explanation, and justification" (p. 116). Chedister's article related to geometric proofs provides a useful example of how teachers can bring indirect and direct reasoning into the classroom to further support students' abilities to justify their mathematical thinking. Throughout the article, Chedister highlights the importance of students constructing viable arguments and critiquing the reasoning of others (Standard for Mathematical Practice #3).

New to the journal is the Technology Tips section by Dr. Josh Hertel. This new addition to the journal will provide a forum for teachers to discuss the role of technology in the classroom to build conceptual understanding, strategic competence and adaptive reasoning. It will also provide a venue for teachers to share technology activities that they have integrated into their own classrooms.

Jennifer Kosiak
Jenni McCool

WMC Editorial Panel

Reference

National Research Council. (2001). Adding it up: Helping children learn mathematics. J.Kilpatrick, J. Swafford, and B.Findell (Eds.). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Sciences and Education. Washington, DC: National Academy Press.

Understanding Conceptual Understanding

by Christopher S Hlas, Krystal Urness, Kaya Sims, Rosie Ricci, and Lindsey Alger, University of Wisconsin-Eau Claire

Introduction

Consider the following problem:

> *An army bus holds 36 soldiers. If 1128 soldiers are being bused to their training site, how many buses are needed?*

For this problem, Carpenter, Lindquist, Matthews and Silver (1983) reported that, "Approximately 70 percent of the students performed the correct [procedural] calculation, but about 29 percent gave the exact quotient (including the remainder), and another 18 percent ignored the remainder" (p. 656). This result seems to highlight that many of the students have procedural knowledge of division, but are lacking a conceptual understanding of what division means and how to apply it correctly. To investigate this topic further, college students read and analyzed research on conceptual understanding during a directed studies course at University of Wisconsin-Eau Claire. This report details the results and recommendations from their findings.

What is Conceptual Understanding?

The term conceptual knowledge can be traced back to Hiebert and Lefevre, in which conceptual knowledge is defined as "knowledge that is rich in relationships" (1986, p. 3). This is distinct from procedural knowledge, which they defined as a familiarity with the individual symbols, conventions, and procedures for solving mathematical problems. Though definitions have evolved, the idea of relationships or connections is usually central to many definitions of conceptual knowledge.

These past definitions have the problem of confusing knowledge type (conceptual vs. procedural) with knowledge quality (shallow vs. deep) (Baroody, Feil, & Johnson, 2007). To avoid confusion between type and quality, we offer the following distinctions. *Knowledge* exists as a spectrum from isolated bits of information (shallow) to connected ideas (deep). *Concepts* include facts, ideas, and declarative knowledge (similar to Baroody, *et al.*, 2007). Numbers can be seen as an example of conceptual knowledge in mathematics. Procedures involve a sequential order of steps. For example, the addition of two-digit numbers (17 + 15) can be procedural:

1. Add 5 and 7 to get 12.
2. Write 2 in the answer.
3. Carry the 1.
4. Add 1 and 1 and 1 (from the carry) to get 3.
5. Write 3 to the left of the 2 in the answer.
6. Answer is 32.

Here we see that there are some cases "in which procedures cannot function without first accessing conceptual knowledge" (Byrnes & Wasik, 1991, p. 785). Earlier number concepts and procedures are necessary for computing 5 + 7 = 12, which cannot be understood without knowledge about how the natural numbers work.

To highlight the difference between knowledge type and knowledge quality, we introduce a new definition for conceptual understanding. Conceptual understanding is the deeper relationships between conceptual knowledge and procedural knowledge. It is the rich connections between the two types of knowledge that makes students more aware of big ideas in mathematics. Returning to the earlier example of two-digit addition (17 + 15), conceptual understanding might look like the following:

1. Break 17 into 15 + 2
2. 15 + 15 = 30
3. Remember the "2" from the 17: 30 + 2 = 32

In the above example, a student uses a conceptual understanding of what 17 means to make the procedural part easier. Byrnes and Wasik (1991) discuss how conceptual knowledge often leads to the use and understanding of procedure, and then by using these procedures students can better understand the concepts and create conceptual understanding. Building conceptual knowledge helps students build on their procedural knowledge, which in turn helps further perfect students' conceptual knowledge (see Figure 1 for multiplication examples). As depth of knowledge increases, conceptual understanding encompasses deep procedural knowledge (aka. "procedural fluency") because the two types of knowledge become intertwined (e.g., the Strands of Mathematical Proficiency in National Research Council, 2001).

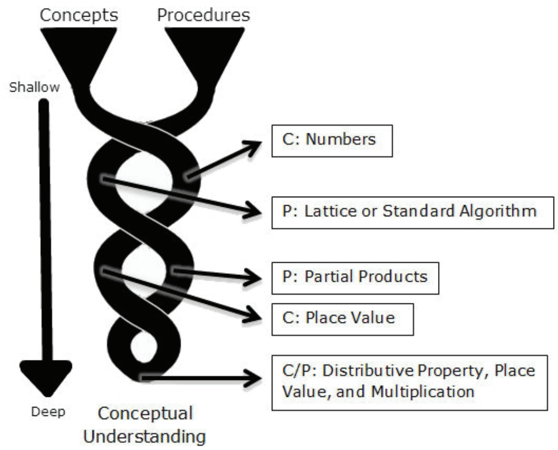

Figure 1. Conceptual Knowledge for Multiplication.

Why is Conceptual Understanding Important?

Conceptual understanding is important because it helps students retain and transfer the information they learn. With shallow conceptual or procedural knowledge students can only transfer their knowledge to similar looking situations, compared to deep understanding where students can transfer and apply knowledge to new types of situations (Baroody, *et al.*, 2007). According to Kaminiski, Sloutsky, and Heckler (2009), students learning abstract ideas before seeing concrete examples or applications of the idea were able to build deeper conceptual understanding and therefore transfer the information to new ideas.

When students have shallow conceptual knowledge they tend to only have partial knowledge of both the concepts and procedures involved in a topic (Rittle-Johnson, Siegler & Alibali, 2001). Our investigations found that conceptual understanding and procedural fluency go hand in hand; greater knowledge of one type is associated with greater knowledge of the other. As Rittle-Johnson (2001) stated, "improving children's knowledge of one type can lead to improvements in the other type of knowledge" (p. 347). Because conceptual understanding provides the "how" and "why" to procedures, it gives meaning to procedural information. "One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student's mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from" (National Governors Association Center for Best Practices, 2010, p. 4). Such understanding also makes it easier to learn and retain mathematical information.

The connections between conceptual and procedural knowledge help students see the big ideas of mathematical concepts and "big ideas invite children to view mathematical knowledge as cohesive or structured rather than as a series of isolated procedures, definitions, and so forth" (Baroody, et al., 2007, p. 126). When students see the big ideas and make these connections they solidify their understanding of the concept and can retain and transfer their knowledge more effectively. They come to view mathematics as meaningful, rather than a series of arbitrary procedures.

Porter and Masingila (2000) provide evidence to suggest that students' attitudes and beliefs about mathematics can affect their procedural performance and understanding of mathematical concepts. When students do not believe they are good at mathematics they tend to make more mistakes in their procedures and create less connections between mathematical concepts. Since conceptual understanding must be learned meaningfully, it is crucial to teach conceptual understanding as a way to change students' beliefs about mathematics as only a series of rote memorization tasks. Instead, mathematics is connected so when students understand mathematics as meaningful and connected they gain insight into the structure of mathematics (Ma, 1999).

How Can Teachers Support Development of Conceptual Understanding in Students?

Now that we know what conceptual understanding is and why it is important for student learning, this leads us to discuss how teachers can develop and assess conceptual understanding in their students.

Developing conceptual understanding involves helping students make and apply connections within and between mathematical concepts and procedures. Recommendations for how to create connections include:

• When students are learning a new concept, begin with concrete examples then extend to general examples. One way to do this is to start with concrete manipulatives, then use visual representations, then use abstract mathematical symbols (see Concrete -> Representation -> Abstraction in Miller & Hudson, 2007).

• When students are transferring conceptual knowledge, begin with generic examples then extend to specific examples. For example, start with a mother function like $f(x) = x^2$ and then discuss transformations like $g(x) = x^2 + 5$, $h(x) = -3x^2$, etc. (Kaminiski, *et al.*, 2009).

• Create "knowledge packages" such as categories of functions to establish relationships between other operations and concepts. Concept maps are a typical way to show such knowledge packages (Byrnes & Wasik, 1991; Ma, 1999). See Figure 2 for an example of a concept map.

• Explicitly connect new concept to the categories. For example, introducing division as the inverse of multiplication (Ma, 1999).

• Encourage problem representation, which is an internal depiction of recreation of a problem in working memory during problem solving (Rittle-Johnson, et al., 2001).

• Emphasize that problems can be solved in multiple ways, because the concepts are related to several other concepts (Ma, 1999).

• Ask "why does this make sense?" Exploring the "why" underlying the "how" leads step by step to the basic ideas at the core of mathematics (Ma, 1999).

• Provide examples and non-examples of when concepts and procedures can be applied, these examples/non-examples develop discrimination between concepts (Byrnes & Wasik, 1991).

• Have students make and discuss predictions. Students must use their knowledge of other concepts to rationalize their predictions (Kasmer & Kim, 2011).

• Use concrete examples to connect new concepts to students' prior knowledge, while also using generic examples to help students generalize concepts (Kaminiski, et al., 2009).

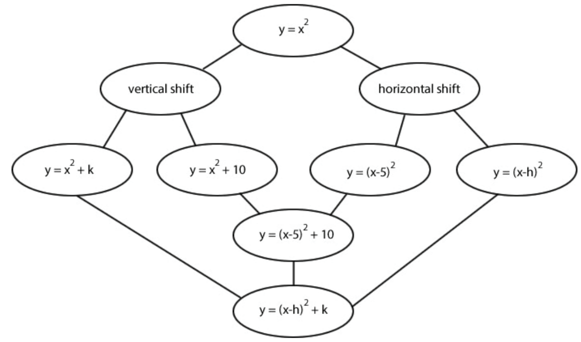

Figure 2. Concept Map of Mother Function (x^2) and Translations

Assessing conceptual understanding involves asking students to use or create multiple representations, often to address new situations because this demonstrates that students have built connections between mathematical ideas. Specific examples for assessment include:

• Conceptual understanding can be assessed through novel tasks, such as counting in nonstandard ways or evaluating unfamiliar procedures (Rittle-Johnson, et al., 2001).

• Use written responses/explanations in assessments. Writing about mathematical concepts cause students to think about mathematical ideas and how to communicate these ideas to others (Porter & Masingila, 2000).

• Students may possess the conceptual understanding, but struggle with explaining the concept verbally. Provide opportunities for students to demonstrate understanding by encouraging students to draw pictures or use mathematical symbols (Rittle-Johnson, Siegler & Alibali, 2001).

• Have students represent mathematical situations in a variety of ways and discuss how these various representations can be useful for different purposes (National Research Council, 2001).

• Analyze students' errors in their work. Analyze how these errors may relate to conceptual misunderstanding (Porter & Masingila, 2000).

• Have students analyze their own errors to address conceptual misunderstandings.

Conclusion

Investigating conceptual understanding was more difficult than we had anticipated. The lack of consistency between knowledge and understanding in the research was problematic so we needed to separate the two ideas. The solution we offered was to use "knowledge" to identify type, and "understanding" to define depth through rich connections between knowledge types. Thus we tried to separate the ideas, but found them difficult to separate as knowledge became more connected. Knowing when to use a procedure, why a procedure works, and how a procedure relates to other mathematical ideas involves conceptual understanding. These connections appear to indicate that an important aspect of mathematics is the "doing" part (procedures) as well as the ideas part (concepts). Knowledge, both procedural and conceptual, is applied to new situations to build mathematical understandings.

"Procedures versus concepts" is one of the many debates in mathematics education. Our research indicates that such debate only exists if we focus on memorization or surface knowledge instead of connected understandings. Teaching for understanding can be done by connecting new ideas and procedures to prior knowledge and by exploring the "why" of mathematics.

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Developing Geometry Knowledge in the Early Grades

by Jenni McCool and Christina Knudsen, University of Wisconsin-La Crosse

According to research by Pierre van Hiele and Dina van Hiele-Geldof, geometry is vital to a student's development (Crowley, 1987). Their research suggests that early geometry instruction should be informal, giving students opportunities to explore geometry concepts through hands-on activities (van Hiele, 1999). Informal activities may include opportunities that provide students with the chance to investigate, create drawings, build and take apart objects, and make observations about the shapes they see in the world around them. Through their work, the van Hieles identified five levels of geometric reasoning: visualization, analysis, informal deduction, deduction, and rigor. The levels are progressive and the accompanying instruction should motivate growth from one level to the next moving from an informal to a formal method of instruction.

The K-6 Geometry domain in the Common Core State Standards for Mathematics (CCSSM) also progresses from informal to formal ways of thinking and is based on three main concepts: (1) geometric shapes, components, and properties; (2) composing and decomposing geometric shapes; and (3) spatial relations and spatial structuring (Common Core Standards Writing Team, 2012). The Geometry domain of the CCSSM expects students to be exposed in Kindergarten to every shape they will see in K-7th grade. What changes over these years is the more refined way we talk about/describe these shapes and how students make more sophisticated connections between the shapes and their attributes. As such, a major emphasis is placed on the vocabulary and the words used to describe the world around us.

In order to gain an extensive understanding of mathematics, students in the early grades need to spend time "playing" with ideas while looking at mathematics from different perspectives and through various experiences (Gavin, et. al, 2013). This article intends to provide examples of ways to engage students in activities about geometric shapes (their components and properties) that are aligned with the K-3 Geometry CCSSM. In order to categorize the activities we used various research studies that summarized similar principles. Research by early childhood mathematics education experts (Clements & Sarama, 2009) suggest successful curriculums for geometry instruction should include four guiding features: (1) varied examples and non-examples; (2) discussions about shapes and their attributes; (3) a wider variety of shape classes; and (4) a broad array of geometric tasks.

Below we will give a description of each of these four guiding features and an example of an activity that creates the intended learning experience. Unless otherwise noted, the activities were taken or modified from *Elementary and Middle School Mathematics: Teaching Developmentally (7th Edition)* by John Van de Walle, Karen S. Karp and Jennifer M. Bay-Williams (Jan 25, 2009).

Varied Examples and Non-Examples

Children need to engage in the exploration of many different examples of types of shapes. More experience helps children so they do not form narrow ideas about any class or type of shape (Clements & Sarama, 2009). Showing non-examples and then comparing them to examples that are similar, help children focus on the attributes of shapes.

Activity 1: Examples and Non-Examples
(CCSSM K.G.A.2, 1.G.A.1, and 2.G.A.1)

In this activity, we asked first and second grade students to think about what makes a triangle a triangle. We then asked students to work together to sort shapes into two piles. Pile one was examples of triangles and pile two were non-examples of triangles. Figure 1 below guided our creation of the shapes we asked students to sort and was adapted from the Progressions document for K-6 geometry, p. 3 (Common Core Writing Team, 2012).

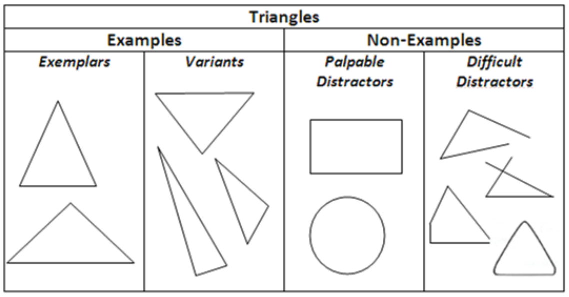


Figure 1. Examples and Non-Examples of Triangles.

Note the difference between variants, palpable distractors and difficult distractors. The most commonly misplaced variant was the long "skinny" acute triangle. Students even used the words "too skinny" when explaining why this shape was a non-example. Ensuring that we remember the variants when we draw and display triangles in our classroom will help those students who think the only triangles are those that have all three sides the same length and one side parallel to the ground. On the pre-assessment, many second grade students did not think the equilateral triangle in Figure 2 was a triangle because it was "turned".

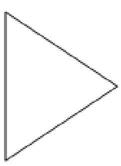


Figure 2. Rotated Equilateral Triangle.

The palpable distractors are helpful when students are first learning to distinguish between shapes at a very basic level. For example, students are focused on straight or curved sides and the number of sides a shape has. Difficult distractors add the complexity of there being three "sides" but the shape not being closed. This notion of a shape needing to be "closed" is important for students in early grades to learn. One of the most frequently misplaced shapes was the difficult distractor that had three round corners. Students would say, "three sides and three corners, so it is a triangle." When we discuss sides and "corners" with our students, placing emphasis on *straight sides* and *pointed corners* may help eliminate this common misconception.

Discussions about Shapes and Their Attributes

If children are given opportunities to discuss shapes using geometric vocabulary, they will develop their mathematical language (Clements & Sarama, 2009). Using mathematical terms in the classroom can highlight early geometric skills of recognizing, describing, and naming shapes (Cooke & Buchholz, 2005). Allowing students to share with the whole class introduces the formalization of mathematical concepts and processes.

Activity 2: Shape Sorting (CCSSM K.G.A.1 and 3.G.A.1)

The shape sorting activity (see Figure 3 below) encourages students to use descriptive words to describe attributes of a shape and to compare two shapes. As you will see, each of the four stages of the activity increases the cognitive demand of the task and thus allows for differentiation to meet students' needs. From simply naming characteristics of a single shape, to comparing two shapes to determine similarities and differences, and finally making comparisons between all of the shapes, this activity motivates students to explain and justify their thoughts to the group.

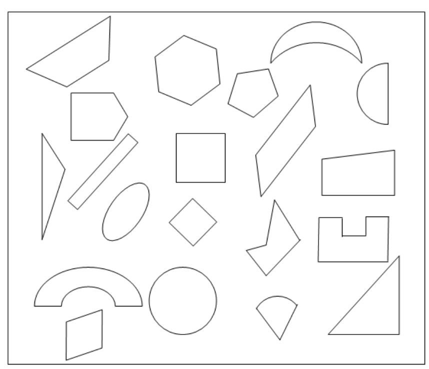


Figure 3. Shape Sorting Activity Set of Shapes

Shape Sorting Activity (Groups of 3-4) Prompts and example set of shapes (see Figure 3 above):

1. One student selects a shape. Group members take turns telling two things they found interesting about that shape.

2. One student randomly selects two shapes and tells one thing that is alike and one thing that is different about the two shapes.

3. One student selects one shape and identifies an attribute(s). The rest of the group must find all other shapes that have the same attribute(s).

4. One student creates a small collection of shapes that fit a *secret rule*. By comparing the pile of shapes that fit the rule and those that do not fit the rule, the other group members try to guess the rule.

First and second grade students actively engaged in all four of these activities. Many vocabulary words were used to describe shapes, and students were able to evaluate whether a particular shape belonged in a particular group based on group member's stated rules. This activity allowed the teacher to observe how often informal language was used by students in regards to shapes and build on this language, moving to more formal language when appropriate. For example, when the teacher observed students say the semi-circle was a "shape with two sides" the teacher then brought this shape and characteristic to the whole class for discussion.

A Wider Variety of Shape Classes

Early curriculums often begin with four distinct shape classes: square, circle, triangle, and rectangle. The best way to move forward with shape classes is to present various examples, all-varying in size, orientation, etc. (Clements & Sarama, 2009). Clements and Sarama (2009) also suggest teachers can develop skills by encouraging students to describe why a figure belongs or does not belong to a specific shape category. If young children never see a square as an example of a rectangle for instance, then they may have a difficult time later in fifth grade when it is imperative that students see the connections between these quadrilaterals.

The previous shape sorting activity can be a precursor to the more sophisticated activity described below. The property lists for quadrilaterals activity gives students an opportunity to begin to formalize their thinking about one class of shapes, quadrilaterals. This activity motivates students to focus their attention on the properties within a class of shapes (quadrilaterals) that distinguish one quadrilateral group from another so students can eventually create a quadrilateral hierarchy in fifth grade.

Activity 3: Property Lists for Quadrilaterals
(CCSSM 3.G.A.1 and 5.G.B.3)

In groups of 4, each person selects a different type of quadrilateral (see Figure 4 below). Students should list as many properties as they can for their shape. Students then take turns stating a property and everyone decides if their shape also has that property. This activity requires students to explain and justify why their shape does or does not have a stated property. This activity can be used in the lower grades if students are just focused on such concepts as side lengths and parallel sides, for example, or can be used in higher grades when students investigate other properties such as angles and diagonals.

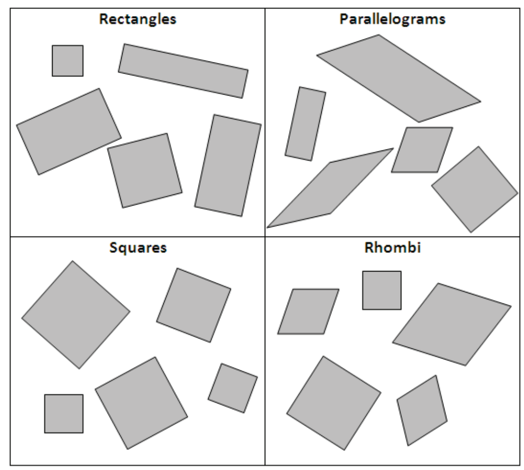
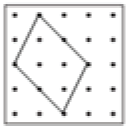


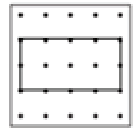
Figure 4. Four Types of Quadrilaterals.

Broad Array of Geometric Tasks

Research suggests using computer environments and manipulatives to engage children in geometry activities (Clements & Sarama, 2009). These activities should give students opportunities to reflect on the task, not just perform the task. Clements and Sarama (2009) recommend exploring, identifying, matching, and even making shapes can be a great way for students to learn. The task in Figure 5 (adapted from K-5 Math Teaching Resources, 2010) is an open-ended task that gives students the opportunity to create their own shapes as they apply their understanding of shape attributes. This *Geoboard Quadrilaterals* activity (see Figure 5) allows a student to use his/her own knowledge to describe attributes for quadrilaterals as well as compare and contrast the different types of quadrilaterals.



Geoboard Quadrilaterals



Word Bank: square, rectangle, trapezoid, parallelogram, rhombus, right angle, parallel sides

1. Create a quadrilateral on your geoboard. Copy your quadrilateral onto geoboard paper.
2. Compare your quadrilateral with a partner. How is it alike? How is it different?
3. Make as many different quadrilaterals as you can. How are the quadrilaterals you made alike and how are they different.

Figure 5. Geoboard Quadrilaterals

Geometry seems to be the lost language of children and is often perceived by adults as the high school class where they "had to write a lot of proofs." It is more than proofs. The reasoning that goes into writing these proofs begins in the primary grades and earlier. The development of the geometry vocabulary moves from informal to a more formal language as students move through the grades. Students are motivated to learn through language and situations that are familiar to them. Engaging students in activities such as the ones described above, are a step toward ensuring students are prepared to meet the Geometry CCSSM as they move through the grades.

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Tests for Divisibility

by Richard Askey, Professor Emeritus, University of Wisconsin

When tests for divisibility are given in school, the only reasons why they work is that a pattern seems to hold true. While this is adequate for divisibility by 2, 5, and 10, it is not as good for 4 and is inadequate for divisibility by 3 and 9. We will give a motivated treatment of divisibility conditions for these numbers, and then mention a problem where they can be used.

Divisibility tests for 5 and 10 are easy. Ten works because of place value, and more than a test for divisibility is easy to determine. For example, 546 = 5*100 + 4*10 + 6, and 100 and 10 are divisible by 10, so the remainder when 546 is divided by 10 is 6. The remainder is 0 when the unit digit is 0. For divisibility by 2, the same argument works. The remainder is 0 when the unit digit is divisible by 2, and it is 1 when the unit digit is not divisible by 2. For 5, the remainder is 0 if the unit digit is 5 or 0, and the remainder is k, k = 1,2,3,4, if the unit digit has a remainder of k when divided by 5.

We will be using divisibility by 4 and 8, and it is easy to show that a whole number is divisible by 4 if the last two digits are divisible by 4. Why? 100 = 25*4, so it and 1,000, etc. are also divisible by 4. Likewise, a whole number is divisible by 8 if the last three digits are divisible by 8 because 1000 = 125*8.

Tests for divisibility by 3 and by 9 can be found using a slight change in the argument above. The simple argument used above will not work since 10 and 100 are not divisible by 3 or 9. However, we can look for the closest number to 10, 100, etc. which is divisible by 3 and by 9. These numbers are 9, 99, and so on. The general case is treated in the same way a three digit number is treated. Each whole number up to 999 can be written as a*100 + b*10 + c, where a, b, c are digits. Write 10 = 9 + 1, 100 = 99 + 1, and use the distributive property to get a*100+b*10+c = a*(99+1)+b*(9+1)+c = a*99+b*9+a+b+c. As above, a*99 is divisible by 3 and by 9 since 99 is, and so is 9*b. Thus the three digit number given by abc is divisible by 3 or by 9 exactly when a + b + c is divisible by 3 or by 9.

If you want to approach these two cases more slowly, here is a suggestion of an exercise described by a Japanese teacher . He made cards with one digit marked on each. He passed out three cards to groups of students. They were to make as many different three digit numbers with the three cards as possible and check to see which were divisible by 3. For example, one set of three cards might be 2,4,5, and a second set might be 4,6,8. It may come as a surprise to our students that the first set can make 6 different three digit numbers, and none of them is divisible by 3, while the second set gives 6 different three digit numbers, and all of them are divisible by 3. I have not done this with students so I do not know how easy it will be for them to come up with the sum of the digits as the test for divisibility. If this is done, this is just a start toward the mathematical reasoning given above which shows why this test works.

A test for divisibility by 6 is a combination of the conditions for divisibility by 2 and by 3. There is a test for divisibility by 7, but it is easier to just divide by 7 to check divisibility.

Here is a problem which can be approached using these tests for divisibility. Take the digits 1,2,3,4,5,6,7,8,9 and use each digit once to form a nine digit number that has the following properties. The first digit on the left is divisible by 1. Don't laugh just because this tells nothing. The useful conditions start with the next one. The first two digits on the left as a two digit number is divisible by 2. The first three digits as a three digit number is divisible by 3. Here is an example where these three conditions are satisfied.

123 satisfies these conditions since 12 is divisible by 2 and 123 is divisible by 3. However, 1234 does not satisfy the fourth condition, which is that the first four digits have to be divisible by 4 when considered as a four digit number. Continuing, the first five digits are divisible by 5, and so on up to 9. Is there such a number, and if so, is there just one or is there more than one?

A solution to this problem will appear elsewhere in this issue, but before reading it, try it yourself.

At the End of the Hallway: Collaborating with Physics Teachers

by Scott Schmid, Talawanda High School, Oxford, Ohio

Introduction

Many times math teachers attempt to draw from other subjects when creating a lesson. This allows students to take knowledge in one context and apply it to another. In many of these cross-curricular lessons, math teachers fail to use a great resource. These resources are in every high school. They generally tell bad jokes, are friendly, and are usually stuck at the end of a random hallway. Of course I am talking about physics teachers!

Recently, I took a class with a number of secondary school mathematics teachers. The goal of the class was to present a lesson, review literature on the topic and revise that lesson. Being a high school physics teacher, I had assumed I would have a similar point of view as my fellow mathematics teachers when it came to improving lessons. I discovered that science and math teachers could learn a lot from each other on how to improve teaching in our respective classrooms. In general, I found that the mathematics teachers were exceptional at thinking of numerous ways to approach a problem. They were also strong in using technology in a number of ways to display a problem. The physics teacher perspective seemed helpful in certain types of problem solving skills, graphing and areas of mathematics that are frequently used in science class. In this article, I will magnify the physics teacher perspective, with an example from the class.

The Initial Lesson

The initial lesson taught was through Cedar Point's math and science day program. Before students went to Cedar Point Amusement Park, they got a feel for linear and angular speed, by having four students link arms and having one person act as a pivot point while the other move in circles with varying radii. The students are to notice that the people on the outside move much more quickly than those on the inside. At Cedar Point Amusement Park, students ride a carousel two times. One time they are near the center of the carousel and the other trial they are toward the edge of the carousel. While riding, the students measure the time it takes to make three revolutions, and later calculate the distance traveled in three revolutions. Using this data, the students then calculate their average linear speed. They also plot a point on a position time graph and calculate the slope, which the students then discover has the same value as the average linear speed. The students were to complete a worksheet (a selected sample is shown in Figure 1) taken from the *Cedar Point Math and Science Student Handbook* (pp. 26-29) available at https://www.cedarpoint.com/images/uploads/file/2013_MandS_HS_Worksheet.pdf

The concepts of linear and angular speed are used in numerous real world concepts including sports, manufacturing, transportation, and astronomy. Mathematics standards covered in this activity include:

CCSS.Math.Content.HSF-IF.B.6: Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

CCSS.Math.Content.HSN-Q.A.1: Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and the origin in graphs and data displays.

Wisconsin science standards covered in this lesson include all of the grade 8 forces in motion standards.

Review of the Literature

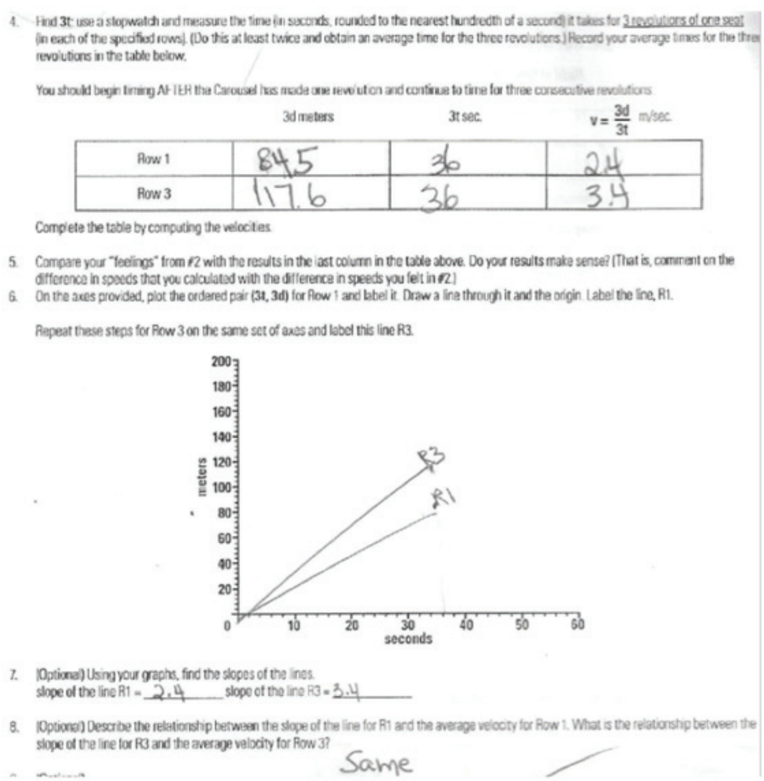
Students and teachers have many misconceptions when it comes to rate of change. For example, when given a graph with points that are not linear students struggled to find the rate of change at a particular point (Orton, 1984). Students tend to struggle with the terms average, constant, variable and instantaneous when it comes to speed (Orton, 1984). In a study done with math teachers, the majority of those teachers did not relate the concept of slope to the rate of change between two variables (Stump, 1999). Students also seem to struggle when applying mathematics concepts to physics. For instance some students thought it inappropriate to put units on slope, because slope is a mathematical idea (Woolnough, 2000). In another study, students were given similar problems involving slope, one in a physics context and one in a math context. Students did far worse on the physics problems (Planinic, Milin-Sipus, Katic, Susac, & Ivanjek, 2012). It is evident in these articles that a different approach is needed to promote a better understanding of slope as the rate of change.

Student Work

The students generally completed the above work accurately. If fact, all the student work provided had the same answers, but were just rounded to different decimal places. All of the students' worksheets just had answers, no work was provided to show how they solved the problems. Some difficulties did occur. For example, some students mistook the formula for circumference to be $C=\pi r^2$ instead of $C=2\pi r$. Students frequently left units off answers. Figure 1 below is a good representation of the class work, as it shows the student correctly finding velocity from the equation v=3d/3t. At the bottom of the sheet it is clear the student did not calculate slope, but merely wrote down the same numbers from above. This student, like most students in the class, left the units off of his/her answers. I am fairly certain the goal of this worksheet was to show velocity is the slope of a distance time graph, but the worksheet did not meet the intended goal.

Lesson Revision

Looking at the student work and the review of the literature, our class of teachers worked on improving the lesson. The lesson seemed to be too direct for the level of students in the class as it focused on calculation rather than learning the concept. The students certainly did not seem challenged with the lesson. The goal of the lesson was working with linear and angular speed, but it focused solely on linear speed. Even if the lesson had used angular speed, the values for the angular speeds would all be equal, which may lead to more misconceptions. Therefore, as a team, we revised the lesson to include finding both linear and angular speed on a running track as shown below in Figures 2a and 2b. In part one (Figure 2a), each student runs around the curve of a track, three times, in various lanes, timing how long it takes to cover the distance. Students must then calculate the distance of the curve, most likely using the circumference formula. Students should notice they have covered a half of a circle, or 180 degrees. From this, students can calculate linear and angular speed. In part two (Figure 2b), a group of students spread out across a number of lanes while holding a pole and cover the same part


4. Find 3t: use a stopwatch and measure the time (in seconds, rounded to the nearest hundredth of a second) it takes for 3 revolutions of one seat (in each of the specified rows). (Do this at least twice and obtain an average time for the three revolutions.) Record your average times for the three revolutions in the table below.

You should begin timing AFTER the Carousel has made one revolution and continue to time for three consecutive revolutions.

	3d meters	3t sec.	$v=\frac{3d}{3t}$ m/sec.
Row 1	84.5	36	2.4
Row 3	117.6	36	3.4

Complete the table by computing the velocities.

5. Compare your "feelings" from #2 with the results in the last column in the table above. Do your results make sense? (That is, comment on the difference in speeds that you calculated with the difference in speeds you felt in #2.)
6. On the axes provided, plot the ordered pair (3t, 3d) for Row 1 and label it. Draw a line through it and the origin. Label the line, R1.

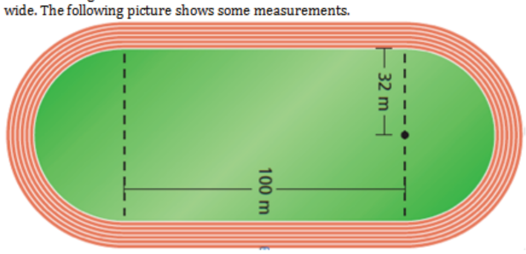
Repeat these steps for Row 3 on the same set of axes and label this line R3.

7. (Optional) Using your graphs, find the slopes of the lines.
slope of the line R1 = 2.4 slope of the line R3 = 3.4

8. (Optional) Describe the relationship between the slope of the line for R1 and the average velocity for Row 1. What is the relationship between the slope of the line for R3 and the average velocity for Row 3?
Same

Figure 1. Student Work Sample

A standard high school is about 400 meters around and has 8 lanes. Each lane is one meter wide. The following picture shows some measurements.



1. Run or walk around the curve of the track in lane 1. Repeat the process in lanes 4 and 8 while attempting to maintain the same speed each time. Time each of your trips and record your results in the table below.

	Distance ( )	Time ( )	Average Linear Speed ( )
Lane 1			
Lane 4			
Lane 8			

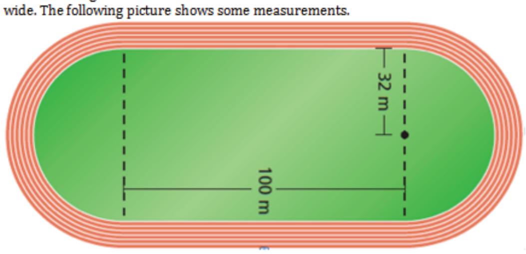
2. Why were your times different for each lane?
3. Fill in the following table

	Degrees Covered ( )	Time ( )	Average Angular Speed ( )
Lane 1			
Lane 4			
Lane 8			

4. Make Time vs. Lane graph
5. What was your slope? What does that slope mean?
6. What was your y-intercept? What does the y-intercept mean?
7. Predict what your time, distance, linear speed and angular speed would be if you were in lane 6.

Figure 2a. Revised Lesson Plan Part 1

A standard high school is about 400 meters around and has 8 lanes. Each lane is one meter wide. The following picture shows some measurements.



1. Walk around the curve of the track while holding a pole. One person should be walking in lane 1, lane 4 and lane 8. Time your trip and record in the table below.

	Distance ( )	Time ( )	Average Linear Speed ( )
Lane 1			
Lane 4			
Lane 8			

2. Why were your speeds different for each lane?
3. Fill in the following table

	Degrees Covered ( )	Time ( )	Average Angular Speed ( )
Lane 1			
Lane 4			
Lane 8			

4. Make Distance vs. Lane graph
5. What was your slope? What does that slope mean?
6. What was your y-intercept? What does the y-intercept mean?
7. Predict what your time, distance, linear speed and angular speed would be if you were in lane 6.

Figure 2b. Revised Lesson Plan Part 2.

To download a copy of the revised lesson plan please visit www.wismath.org/resources.

of the track as before, and once again calculate linear and angular speed. Students will find in the first part that their linear speeds will be relatively similar, because they are attempting to run at the same rate in all three trials. In part one, the angular speed will be different in all three trials, because each trial takes a different amount of time, but the same change in angle occurs. In the second part, since the time is constant, the angular speed will be the same, but the linear speed will be different because each student is walking a different distance in the same amount of time. In each part, students are asked to graph data, explain what their slope and y-intercept mean and make predictions about linear and angular speed if they were in another lane of the track.

Strengths of the Revision and Conclusion

During the revision process, I was surprised at how my approach to revisions as a physics teacher was different than the mathematics teachers. For instance, amongst the math teachers there was no discussion of the confusion between the equations for area and circumference of a circle. When the lesson was taught, if a student confused the equations, another student or the teacher would correct him/her by just telling the correct formula. If students know the definition of area and circumference they should be able to figure out which formula is correct, on their own, by unit analysis. For example $2\pi r$ has to have distance units, while πr^2 has to have the units of distance squared, only the first one can be a circumference. Unit analysis is a common approach in physics classes to see if an equation is even possible. The units on the original worksheet were usually given to a student, which generally means students will overlook them. By including empty parenthesis on the revised worksheet, students will remember they need to include units and more likely understand their importance. This is one component of the Common Core Standards for Mathematical Practice: #6 Attend to Precision.

Physics teachers also seem to take a different approach to graphs compared to mathematics teachers. The graphing section of the new lesson will enforce a variety of topics. By asking what the slope and y-intercept mean, students now can see graphing as a mathematical concept that also has applications. Suppose the slope in part 1 is 8. The units would be seconds per lane, as those are the y and x units respectively. The answer for #5 on the worksheet could read: As the lane number increases by one, the amount of time it takes to run increases by 8 seconds. The correct answer to the y-intercept question might read: If there were a lane 0, it would take y-intercept (seconds) to run the distance of lane 0. Asking what the slope means also stresses the importance of units.

The final and most obvious use of the physics teacher in this lesson is to discuss the physics involved. During the revision of the lesson, math teachers occasionally had misused physics vocabulary. This is a common mistake, as many words have similar meanings. For instance, the term centrifugal force was used when explaining the feeling students had on the carousel, when they felt like they were flying out of the circle. This feeling of flying out of the circle is actually called inertia. There is no force pushing the student out of the circle, they are merely trying to go in a straight line which takes them out of the circle. Speed and velocity were also interchanged. Velocity is speed with a direction. For example, your speed could be 45 miles per hour, and your velocity could be 45 miles per hour North. The point is, physics teachers discuss these concepts constantly, and can assist math teachers and students in better understanding these concepts. So if you are doing a lesson that involves motion or some other physics topic, take a walk down the long hallway, laugh at the bad jokes and collaborate with your friendly physics teacher.

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Pentagrams and More – An Introduction to Reasoning and Proof

by Matthew Chedister, University of Wisconsin-La Crosse

Within the *Common Core State Standards* (CCSS), specifically the S*tandards for Mathematical Practice*, there is an emphasis on students constructing and critiquing arguments and proofs. However, many students struggle with reasoning and proof. The concept of proof is difficult for many students because they have not been formally introduced to proof prior to their high school geometry class. More problematic is that once they are introduced to proof in high school geometry, they are taught proof as a topic of study rather than an overarching mathematical language. Problems involving proofs often focus only on the form of the proof and ask students to prove things that are already obviously true.

One way to generate students' understanding of proof is to have them work on proof tasks that require both inductive and deductive reasoning and whose results are not immediately obvious. Further, types of problems that are especially effective are problems whose entry points are manageable. One such problem that I have used in my geometry classes is called *Pentagrams and More* (see Figure 1). In this problem (adapted from Craven 2010), students must experiment to discover that the sum of the measures of the angles at the points in any pentagram is always constant. Furthermore, to prove this, students need only to have prior knowledge of basic angle and polygon facts and theorems.

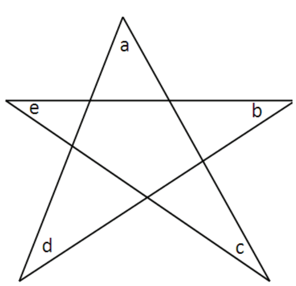

Figure 1. The Pentagrams and More Problem. What is the sum of the measures of the five labeled angles that make up the points of the pentagram? (Craven, 2010)

Beginning the Problem with Experimentation

Inductive reasoning and deductive reasoning are two important types of reasoning that students should use in communicating their understanding of mathematics. Unfortunately, these two types of reasoning are only briefly discussed in many classes. Inductive reasoning is defined as the process of observing data, recognizing patterns, and making generalizations about those patterns (Serra, 2008). Deductive reasoning is defined as the process of showing that certain statements follow logically from agreed-upon assumptions and proven facts (Serra, 2008). *The Pentagrams and More Problem* can be used to give students an opportunity to work with both types of reasoning.

Because the result about the sums of the measures of the points is not immediately obvious, it is best to begin this problem by having students construct several example pentagrams and then use inductive reasoning to determine that the sum of the measures of the angles that make up the points is always 180 degrees. This can be done using paper, pencil, and protractor or using a dynamic software like Geometer's Sketchpad or Geogebra. The paper, pencil, and protractor method is sufficient if there is limited technology easily accessible. The disadvantage of this method is that accuracy is difficult and students may get a variety of sums for the angle measures. If using paper and pencil, having students draw pentagrams at least 2 inches by 2 inches is more effective because smaller ones result in small angles that are harder for many students to measure with accuracy. The use of dynamic software is an effective way to show students that even when the vertices of their figures change sizes, the sum of the angle measures does not change. It also allows students to see that as one angle grows or shrinks, the other angles change correspondingly to compensate.

After students have made their generalization, they are ready to begin writing deductive proofs of their findings. I will describe four proofs and their extensions that were developed by one of my geometry classes.

Solution Method #1 – The Big Triangle Method

One group in the class came up with a solution that we chose to call *The Big Triangle Method* as shown in Figure 2. In the method, the students drew a line from vertex labeled d to the vertex labeled c and labeled the two new angles *1* and *2*. The students argued that the sum of the measures of angles *1* and *2* was equal to the sum of the measures of the angles labeled e and b. They justified this argument by observing that the angles labeled *3* and *4* were vertical angles and therefore equal in measure. This meant that since all triangles' interior angles sum to 180 degrees, the remaining angles' measurements must have the same sum.

The students then outlined one large triangle (see Figure 2) and noted that all triangles have interior angles whose measures sum to 180 degrees. The angles in this big triangle were angle *a* and an angle made of angle d and 1 and an angle made of angle *c* and *2*. Therefore, the sum of the measures of angles *a*, *d, c*, *1*, and *2* equals 180 degrees. Further since the sum of the measures of *1* and *2* is equal to the sum of the measures of *e* and *b,* the sum of the measures of the original five angles is 180 degrees.

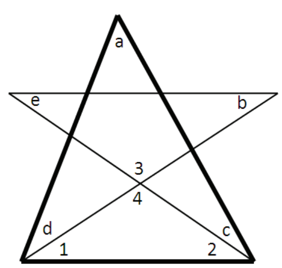

Figure 2. The Big Triangle Method.

Solution Method #2 – The Exterior Angles and Remote Interior Method

Another group came up with a method using two of the exterior angles of two triangles found in the pentagram as shown in Figure 3. The class decided to call this method *The Exterior Angles and Remote Interior Angles Method*. In this method, the group labeled two angles of one the triangles. The students stated that they knew that the exterior angle of a triangle is equal to the sum of the remote interior angles and therefore angle 1 had a measure equal to the sum of the measures of angles *b* and *d* (see Figure 3). By a similar argument, angle *2* is equal to the measures of angles *e* and *c*. The group then said that they knew that the sum of the interior angle measures in a triangle is 180 degrees so the measures of angles a, *1*, and *2* sum to 180 degrees. By substitution, the sum of the angle measures a through e must also be 180 degrees.

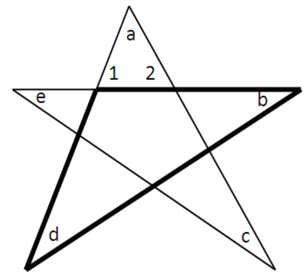

Figure 3. The Exterior Angles and Remote Interior Angles Method.

Solution Method #3 – The Many Triangles Method

A third group came up with a method that they called *The Many Triangles Method*. This method required a large amount of "bookkeeping." These students noted that they could make triangles using two points of the pentagram and one angle of the interior pentagon. One such triangle is shown in Figure 4 using angles *a* and *d*. Four more of these triangles can be made – one for each of the angles of the interior pentagon. Within these five triangles, each angle at the point of the pentagram appears in two triangles. For example, angle *a* appears in the triangle with angle *d* as shown, but also in a triangle with angle *c*.

The students noted that they had five triangles and that the sum of the angle measures in these five triangles was 5 x 180 or 900 degrees. The students then noted that they included five angles which were not part of the points of the pentagram (the five interior angles of the pentagon which sum to 540 degrees) so they subtracted 540 from 900 to get a sum of 360. They finally noted that each angle at the points had been counted twice so they dived 360 by 2 to get a solution of 180 degrees for the sum of the measure of the five angles that make up the points of the pentagram.

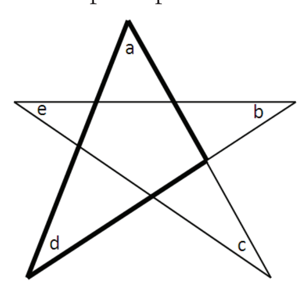

Figure 4. One of the Triangles in the Many Triangles Method.

Solution Method # 4 – The Exterior Angles Twice Method

The final method presented by a group used an idea we had recently discussed in class – that the measures of the exterior angles of a polygon sum to 360 degrees. This group noted that within the triangles around the interior pentagon, there were two complete sets of exterior angles – labeled with 1's and 2's (see Figure 5). They then noted that there were five triangles and the sum of the measures of all the angles in these triangles was 900 degrees. The students then used the fact that each complete set of exterior angles adds to 360 degrees and they subtracted 720 degrees from their total to arrive at the answer that the measures of the five angles that make up the points of the pentagram sum to 180 degrees.

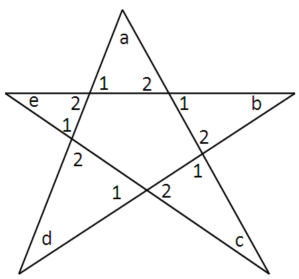

Figure 5. Exterior Angles Twice Method.

Extendibility

As a class, we concluded the lesson by discussing which of the methods developed extended to hexagrams, heptagrams, or even n-grams. After further experimentation and discussion the class decided that each time a side was added to the n-gram, the sum of the measures of the angles at the points increased by 180 degrees. They also decided that this could be proven using two of the four methods: *The Exterior Angles Twice Method* and *The Many Triangles Method*.

The argument for the *Exterior Angles Twice Method* was more straight forward. The students noted that every time a side was added to the n-gram, there was new triangle added increasing the sum by 180 degrees. The sum of measures of the exterior angles did not change so therefore the total sum of the measures of the angles at the points also increased by 180 degrees.

The Many Triangles Methods was more complicated but still seemed to generalize but in two different cases. When n was even, the figure could be broken into two n/2 polygons whose vertices were all points of the n-gram and the sum of the angle of measures of these two polygons added to the appropriate total. As shown in Figure 6, the hexagram can be broken into two triangles whose vertices make up the points of the hexagram. The sum of these angle measures is 360 degrees.

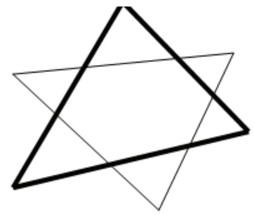

Figure 6. Hexagram Example

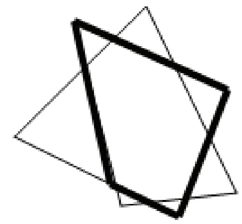

Figure 7. Heptagram Example

When n was odd, the student found that they could make *(n+1)/2* sided polygons that had one vertex that was part of the interior *n*-gon. Making these polygons, they used each point of the n-gram *(n-1)/2* times. They found that they could make n of these polygons. They then subtracted the sum of the measures of the interior angles of the interior polygon and divided by *(n-1)/2* to obtain their final answer. See Figure 7 for example using a heptagram. Within the heptagram, we can make quadrilaterals that contain three vertices of the heptagram and one vertex of the interior heptagon. Seven of these heptagrams can be created. The sum of the measures of these seven quadrilaterals is 2520 degrees. To find the sum of the measures of the points, we need to subtract the sum of the measures of the interior heptagon (2520 – 900 = 1620). Each point of the heptagram is included in 3 different quadrilaterals so we have triple counted so we divide 1620 by 3 to get our answer of 540 degrees.

Conclusion

The use of a problem like *Pentagrams and More* exposes students to a number of key ideas involving proof such as inductive reasoning and deductive reasoning. More importantly, the implementation of a problem like this can motivate students' understanding of proof. Since, the problem is built around basic angle and polygon facts and theorems, it can serve as a positive springboard into other proving tasks throughout the school year.

References

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Problem Using Divisibility Tests

by Richard Askey, Professor Emeritus, University of Wisconsin

Here is the problem that was stated in the earlier article: Use the digits 1,2,3,4,5,6,7,8,9 once each to make a nine digit number with the properties that the first k digits as a k digit number is divisible by k, k=1,2,3,…,9.

The first five digits as a five digit number is divisible by 5, so the fifth digit has to be 5. The first two digits as a two digit number is divisible by 2, so the second digit has to be even. Similarly the fourth, sixth and eighth digits have to be even, which uses all the even numbers, so the remaining digits have to be odd.

The first four have to be divisible by 4, and so by the test for divisibility by 4 the third and fourth digits as a two digit number have to be divisible by 4. The numbers 14, 34, 74, 94, 18, 38, 78, 98 are all not divisible by 4 so the fourth digit has to be either 2 or 6. Similarly, the first eight digits as an eight digit number has to be divisible by 8, and so divisible by 4, so the eighth digit has to be either 2 or 6. The first three and the first six each have to be divisible by 3, so the sum of the first three digits has to be divisible by 3, as does the sum of the first six digits. This implies that the sum of the fourth, fifth, and sixth digits has to be divisible by 3. The only possibilities for this are 258 and 654. The test for divisibility by 8 is that the last three digits has to be divisible by 8. The possibilities for the digits in the sixth, seventh and eighth places are 432, 472, 816, 896. The sum of the seventh, eighth, and ninth digits has to be divisible by 3, and this is not possible for the numbers which end in 16 and an odd number, since the only odd digit which works is 5 and 5 has already been used. This leads to ten possible numbers:

789654321, 987654321, 981654327, 189654327

189654723, 981654729, 381654729, 183654729

247258963, 741258963

One and only one of these numbers has the first seven digits as a seven digit number which is divisible by 7. Let me illustrate how what was called "short division" was done when I was in elementary school to determine if a number was divisible by 7. Consider the first number, and just the first seven digits. When dividing by a single digit number, here 7, the quotient was written above the bar if one wants to do the division, but here we do not care about anything except to see if the division is exact or there is a remainder. This would have been done mentally after a little practice, or the partial remainders could have been recorded. Here in words is what to do (see also Figure 1).

7)7896543

First, there is no remainder when 7 is divided by 7, there is a remainder of 1 when 8 is divided by 7 so the next division is of 19. The remainder is 5 since 19-14=5, so the next division is of 56. There is no remainder when this is divided by 7, a remainder of 5 when 5 is divided by 7, and of 5 when 54 is divided by 7, 54-49=5. Finally, 53 is not divisible by 7 so 7896543 is not divisible by 7.

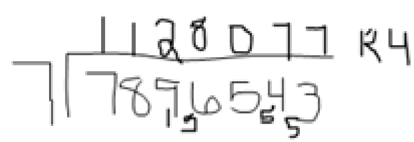

Figure 1. Short Division for 7.

The rest of the calculations are left to you.

Technology Tips: GeoGebra Tablet App

by Josh Hertel, University of Wisconsin-La Crosse

Welcome to a new section in the Wisconsin Teacher of Mathematics called Technology Tips. The goal of this section is to provide a forum to discuss the use of technology in the classroom, share projects or activities that incorporate technology, review new or emerging technologies, and promote an open dialogue about the place of technology in mathematics education. If you have comments, questions, or an idea for an article, please send me an email at jhertel@uwlax.edu.

An Overview of the GeoGebra Tablet App

This article will present a brief review of the new GeoGebra app for android and iOS tablets. If you have never used GeoGebra before, it is a dynamic mathematics software that is freely available for Windows, Mac, and Linux systems (to download visit http://www.geogebra.org). Interest from the community of users for a tablet version of GeoGebra led to a Kickstarter campaign in the Fall of 2012. The campaign was fully funded and as of September 2013, both Android and iPad apps have been released and are free to download from the Google Play and Apple App stores. Experienced GeoGebra users might wonder whether there are major differences between the desktop edition and the tablet app. In fact, there are a number of differences in terms of features and interface. The tablet app is a streamlined version of the desktop edition that includes a simplified user interface and a limited tool set. These limitations aside, the app incorporates several features that are prominent on the desktop edition and easily work with the tablet medium. I will discuss several of these features briefly and then present an overall assessment of the app. Note that this review was written primarily from an iPad app; however, I have also tried the Android App on a Nexus 7 and the features appear identical. This review also assumes basic knowledge of the desktop edition.

Main Features

A main feature of the desktop edition of GeoGebra is its integration of several "views" that allow the user to manipulate different representations of a mathematical object. The desktop edition includes a dynamic geometry environment (graphics view), an algebraic representation view (algebra view), a computer algebra system (CAS view), and a spreadsheet (spreadsheet view). These views are connected together so that changing a mathematical object in one view will result in changes to the representation of the object in the other views. The current tablet app has the graphics view, algebra view, and computer algebra system commands (the CAS and spreadsheet views may be incorporated in the future). For example, Figure 1 shows a quadratic function that has been defined in the algebra view (right panel) and displayed in the graphics view (left panel). If the user changes the definition of the function in the algebra view, these changes will immediately be updated in the graphics view. Likewise, if a user interacts with the function in the graphics view by dragging it around the screen, the shifts are applied to the equation in the algebra view.

The toolbar at the bottom includes a majority of the typical tools (e.g., New Point, Perpendicular Line, Polygon, Reflect Object About a Line, Create a Slider). As with the desktop edition, touching one of the visible tools brings up a sub-menu of tools that can be selected. The graphics view can be shifted horizontally/vertically by dragging a finger across the screen. Zooming can be done using a pinch motion.

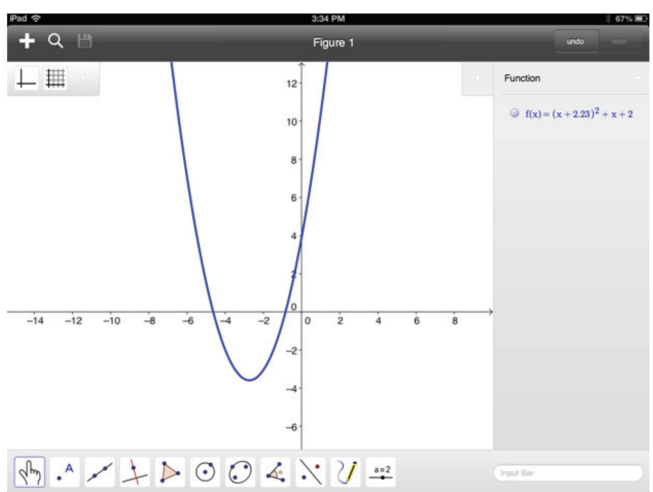

Figure 1. A parabola show in the graphics and algebra views.

Objects can be created in the graphics view by selecting a tool and then touching the screen. Once created, an object can be selected by tapping on it. Selecting an object brings up a properties menu in the upper left of the graphics view, which can be used to change the color of the object and change the text that is displayed (e.g., name, label). Objects can also be moved around the screen by tapping and dragging (note that the hand tool must be selected from the toolbar). To delete objects, select the eraser tool (found in the hand tool sub-menu) and then tap on the object.

Algebra View and Input Bar

The algebra view can be opened by tapping on the arrow in the right hand corner of the graphics view. Any object that has been created in the graphics view is automatically created within the algebra view. Similar to the desktop edition, the algebra view allows the user to hide/show an object by tapping on the radio buttons next to the name. Likewise, the algebra view allows the user to redefine an object by double-tapping on its definition. Double-tapping on a definition brings up an input box showing the current definition of the object, which can be edited using the on-screen keyboard.

Objects can be created in the algebra view using the input bar. This bar is at the bottom right in landscape mode and at the bottom of the screen when in portrait mode. The input bar disappears when the tablet is rotated into portrait mode, but can be revealed by tapping on the arrow in the lower right-hand corner. As with the desktop edition, the input bar can be used to define functions, redefine objects, and execute predefined commands. I tested several commands (e.g., derivative, extremum) and everything appeared to be present. However, unlike the desktop edition, entering the first few letters of a command does not bring up a list of possibilities. Additionally, using the on-screen keyboard to enter commands can be cumbersome because a large portion of the screen is covered by the keyboard itself (Figure 2).

Figure 2. The on-screen keyboard covers a large portion of the screen.

New Sketch, Search, and Save

One other prominent feature, which is unique to the app, is the set of three buttons in the top left of the screen (see Figure 1). These buttons are: a new sketch symbol (plus sign), a search symbol (magnifying glass), and a save symbol (floppy disk). The first and last are self-explanatory, but the second is not. The search symbol provides the means to access previously saved sketches. You can open the sketch for viewing, open the sketch for editing, or delete the sketch from the tablet. Additionally, the search tool allows the user to search for sketches on GeoGebraTube (an online sharing site for GeoGebra sketches). Sketches can then be downloaded directly from GeoGebraTube for display or editing. At this time it is not possible to upload sketches to GeoGebraTube, but according to the developers this feature is a top priority. On an Android tablet it is possible to email a sketch, but this is not possible on the iPad at the moment.

Final Thoughts

The GeoGebra tablet app presents several possibilities for classroom use. First, the app allows students to build geometric constructions within the graphics view. These constructions can be manipulated to form/test conjectures about specific mathematical ideas or relationships. In a more structured approach, a teacher could create a sketch using the desktop edition, upload it to GeoGebraTube, and then have students download the sketch onto classroom iPads. Thus, the app itself provides some of the same opportunities for mathematical reasoning and sense-making found in the desktop edition but places these opportunities within a more user-friendly interface.

Second, I can imagine the app serving as a useful tool for teachers who use iPads regularly in their own teaching. The flexibility of the application to create graphs or construct mathematical objects on demand may make it useful to illustrate concepts or ideas on a regular basis. For example, I can imagine a classroom of students divided into small groups with each group working on graphing functions that have undergone some type of transformation (e.g., shift, compression, stretch). As the teacher moves from group to group, they might use the GeoGebra app to demonstrate different ideas or allow students to experiment with an existing sketch. The inclusion of sliders in the app make it relatively easy to change the parameters of a particular function.

Overall, I believe that there are two barriers to the implementation of the GeoGebra tablet app: (a) the price and availability of tablets and (b) the technical knowledge necessary to use the application. There is no question that these are real barriers that will prevent the app from being used within some classrooms. However, the last several years have seen a general trend of more tablet use both within and outside of classroom. Likewise, both students and teachers are gaining increased familiarity with tablet based applications. Thus, I believe the development of the GeoGebra tablet is timely given the shifting focus on mobile technology. Moreover, the ease of use of the app allows the user to forget the interface and instead focus on the mathematical concepts at hand. This puts the GeoGebra tablet app into a strong position for incorporation into the mathematics classroom.

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We are accepting articles for future issues; please refer to the guidelines for submission that are printed on the inside cover of this journal. Articles should be sent to kosiak.jenn@uwlax.edu.

Wisconsin Mathematics, Engineering, and Science Talent Search Celebrates its 50th Anniversary

This year the Wisconsin Mathematics, Engineering, and Science Talent Search celebrates its 50th anniversary of bringing challenging, intriguing mathematics problems to Wisconsin students, and recognizing some of the state's most talented young mathematicians. It all began in 1963, when University of Wisconsin-Madison Professor Laurence C. Young wrote the first competition. It started with the following problem.

Given n billiard balls numbered 1, 2, 3...n, for some positive whole number n, show that there are exactly two ways to arrange them in a circle so that the difference of the numbers on two adjacent balls is ± 1 or ± 2. [Show that] further of the two arrangements, each is a mirror reflection of the other.

This is a great example of a talent search problem. It is not easily classified as being solved by some technique. It doesn't require students to know a particular fact or formula, and such knowledge will not particularly bring students closer to a solution. Yet in the solution to this problem (see "Solving the problems" below) we can see important themes of mathematical problem solving.

The problem above and four other problems were distributed to Wisconsin students, and students mailed in their solutions to be graded at the Mathematics Department at the University of Wisconsin-Madison. Thirty-one students sent in a correct answer to the problem above. This set of five problems was the first of four problem sets mailed out in the 1963-64 school year. On the second set, one problem was solved correctly by 170 students.

Now, the Talent Search consists of five problem sets, each of five problems. The problem sets are mailed to all of the middle and high school math teachers in Wisconsin, and the teachers are asked to distribute the problems and information about the competition to interested students. The five problem sets are available online as well. Students have approximately a month to work on each problem set. They may send their solutions by email or postal mail, where they are still graded in the Mathematics Department at the University of Wisconsin-Madison.

These days, there are a lot of different math competitions that students and schools may participate in. However, the Talent Search is different from most of these competitions in several respects. It requires that students provide reasoning and justification, that is, "proofs," for their answers and the solutions are graded on the quality of these proofs. The problems are more challenging than many in math competitions because the students have a month to wrestle with the problems, which allows for problems that require more ideas and insights than anyone can be expected to have in a timed format. Any student may participate in the Talent Search. While the Talent Search relies on teachers to help spread the word about the competition, participation does not require any further commitment from the school. A school that is too small for a math team or without the resources to travel to competitions may still have one or two students who are interested in mathematical challenges and who could participate in the Talent Search.

In 1991, the Talent Search began offering a four year scholarship to UW-Madison as the top prize. At that time the scholarship was \$4,000 a year for four years. Currently, the Van Vleck Scholarship, the top prize in the Talent Search, is \$6,000 a year for four years. After the five rounds of the mail-in problem sets, approximately 25 of the top scorers from Wisconsin are invited to take a proctored and timed scholarship exam. The winner or winners of this scholarship exam are awarded the Van Vleck Scholarship to UW-Madison. The top scorers and their parents are also invited to UW-Madison for an Honors Day. There the students talk to professors, attend some mathematics and science lectures, and visit a research facility on campus.

Where the Problems Come From

The Talent Search has also been run by the Mathematics Department at UW-Madison. Since Professor Laurence Young started the Talent Search, many other faculty members in the department have helped run the program, most notably Professor I. Martin Isaacs, who was Talent Search Director for 32 years from 1980 to 2012. Professor Donald S. Passman was Assistant Director for 21 years from 1991 to

2012. Since the retirement of these long standing Talent Search organizers, a new team consisting of Professors Benedek Valko, Melanie Matchett Wood, and Jonathan Kane has been writing the problems and organizing the competition. They have been helped immensely by Sharon Paulson, who has been running the day-to-day operations of the competition since 1991.

For each problem set, the organizers each write a list of proposed problems. They then get to enjoy the challenge of working on the others' problems before they come together to decide which five will be best for the competition. The organizers look for problems that don't require too much specialized knowledge, but give students the opportunity to think creatively. Ideally the problems are inviting, and give students something to play around with and puzzle out as they examine small cases and try different strategies. Some problems in each set should be more accessible to allow more students to get involved, and some should be challenging enough to give even the best students something to think about over the course of the month they have to work on them.

Solving the Problems

Each problem in the talent search gives students a chance to explore a small piece of mathematics. Often, there is some surprising or interesting structure in the mathematics. The problems are designed to give students a chance to stretch their minds and develop their problem solving skills. They give students a chance to develop their mathematical skills without just going further in the school curriculum.

For example, how would a student solve the problem above, which is on the easier end of talent search problems? First, one can consider small cases. For example, if there are three balls, some playing around will quickly convince you that there are only 2 ways to put three balls in a circle--such that they are 1,2,3 in order clockwise or counterclockwise--which both satisfy the conditions of the problem and are mirror reflection of one another. However, when there are four balls, you can see that there are now more circular arrangements--6 of them in fact--but by looking at each one you can again observe that the statement of the problem holds for four balls. Though for any particular n one could in theory list the arrangements and then check each one to see if it satisfies the difference condition of the problem, such an approach will never solve the problem for all n. For this, one needs to think systematically about how to build such arrangements.

Here one can use another important problem solving strategy by trying to make partial progress on the problem. How about before trying to see there are exactly two allowed arrangements (where an allowed arrangement is one where the difference of the numbers on two adjacent balls is ±1 or ±2), you could just try to see if you can construct any allowed arrangements at all? To construct one in an organized fashion, it makes sense to start with the placement of ball 1. What can be next to 1? Only 2 or 3. Yet 1 has to have two balls adjacent to it, so they must be it in some order. If $n>3$, what can be next to 2? Only 1,3,or 4, and 1 and 3 are already placed, so 4 must be on the other side of 2 from 1. Then, if $n>4$, what can be next to 3? Only 1,2,4,5, but 1,2,4 are already placed, so 5 must be on the other side of 3 from 1. We can proceed like this. We see that at the first step, we had two choices for which side of the 1 held the 2 and which held the 3. But after this, every ball position is fixed and there are no more possible choices. What started out as an attempt to build any allowed arrangement at all has turned into a proof that there are at most two allowed arrangements!

It still remains to be seen that there are indeed these two allowed arrangements. We see that the balls are being placed in order: 1,2,3,4..., alternating on the left and then the right of a line. At each point, every ball is difference ±1 or ±2 from all adjacent balls. When we run out of balls, the two balls on the ends will become adjacent to form a circle. Since the last two balls placed are consecutively numbered, we see that the difference of those adjacent balls is 1. Thus, we have constructed (when $n>2$) two different allowed arrangements which are mirrored reflections of each other. This completes the problem.

Past Winners

Over the years, the winners have come from many different schools across the state of Wisconsin, and they have gone on to many different paths after the Talent Search. However, they commonly recognize the important contribution the Talent Search had in the development of their mathematical skills. David Boduch, a winner in 1981 and 1982 from Waukesha Memorial High School in Waukesha said, "The problems were harder than any others I encountered in the classroom or at math meets, and I looked forward to their challenge and to competing against some of the region's most talented math students. Each set contained at least a couple problems that did not succumb to an hour, a day, or sometimes even a week of pondering. I was often spurred to leaf through books on geometry, combinatorics, or number theory for helpful results. Occasionally I tested hypotheses or looked for patterns by writing BASIC programs on a 16K RAM microcomputer." After studying mathematics at Harvard and MIT, David pursued a career in finance and created one of the largest and most successful fixed-income hedge funds. David reports that "The autographed copy Professor Laurence Young gave me on Honors Day of his "Mathematicians and Their Times" still rests on my bookshelf."

Matt Wage from Appleton East High School was a Van Vleck Scholarship winner in 2007. He said that, "The Wisconsin Talent Search was great for me because it was my first experience with really difficult math problems." After studying math and philosophy at Princeton, he has also gone into a career in finance.

Some Talent Search winners have gone on into academic mathematics. Daniel Kane was a Van Vleck Scholarship recipient in 2000. He said "The Talent Search was actually a really significant learning experience for me, as it was my first real exposure to proof-based problem solving. I also found the problems generally interesting and continued to participate for years after I had already won." Daniel majored in math and physics at MIT, got his PhD in mathematics at Harvard, and is currently a postdoctoral scholar at Stanford.

The Loh family famously had three Van Vleck Scholarship winners: Po-Shen in 1997, Po-Ru in 1999, and Po-Ling in 2002. Po-Shen said, "I thought that the Wisconsin Talent Search was a great experience, and indeed, it was my first encounter with formal proof writing. All of my previous mathematical activities had focused on numerical answers, and I found it interesting and challenging to shift the focus from calculation to proof." Po-Shen is now an Assistant Professor of mathematics at Carnegie Mellon University and is deputy leader of the US International Mathematical Olympiad team. Po-Ru is a postdoctoral scholar in the Harvard School of Public Health working in statistical medical genetics. Po-Ling said, "The Wisconsin Math Talent Search played an integral role in my development as a young mathematician. Although I was never one to enjoy competition, I always looked forward to the monthly problem sets, with carefully-crafted problems that I appreciated even more as I learned more mathematics in later years." She is pursuing a PhD in statistics at Berkeley.

The most recent Van Vleck Scholarship winner is Thomas Ulrich who is from Appleton, where he was homeschooled. Thomas will use his scholarship to attend the University of Wisconsin-Madison starting this fall.

How to Get Students Involved

When teachers receive the Talent Search problem sets in the mail, it is wonderful if they can distribute copies of them to students they think might be interested. However, any student can get involved by getting the problems from the website https://www.math.wisc.edu/talent/.

The problem sets will be available online at the start of October, November, December, January, and February and are each due a month later.

WMC PUZZLE PAGE

Search-A-Word: Common Core

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F N O I S I C E R P S V U V Z
U E U N A Y V F L C X N K T F
N R R Q I T D Y I B D O X J W
C U E D S I T T R E T S U L C
T T A K Q L S D R A D N A T S
I C S N O I S S E R G O R P E
O U O Q T B T D M T Z Y G G P
N R N A Z A P R A C T I C E S
S T T X N B D P J I R O R O A
A S T D S O I B T E N S I M R
O L I E M R R N B T E K T E B
J N D A Z P A M E V R I I T E
G D I U K U U N E P E L Q R G
G N L M Q N T R O D T L U Y L
X R G M O D E L I N G S E O A
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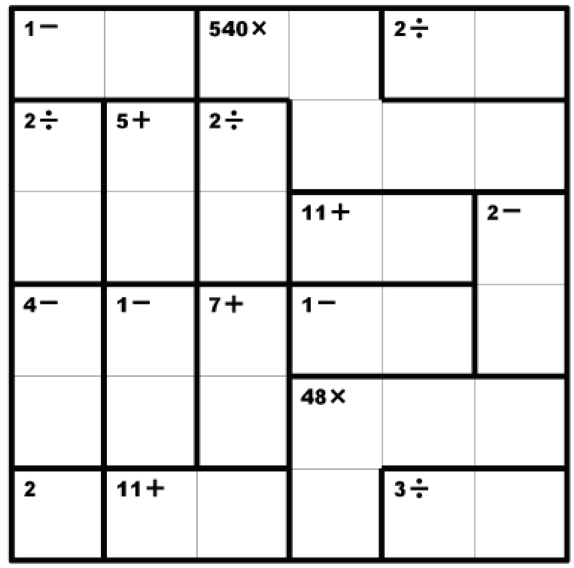
State Mathematics Competition

The following problem is from the Team Portion of the 2009 High School State Mathematics Contest. For additional questions and solutions, visit www.wismath.org/resources/math-contests.

> Use the digits 1 through 8, once each, to create the largest integer you can that is divisible by 11.

Ken Ken

Fill in the blank squares so that each row and each column contain all of the digits 1 through 6. The heavy lines indicate areas that contain groups of numbers that can be combined (in any order) to produce the result shown with the indicated math operation.



Sudoku

Fill in the blank squares so that each row, each column and each 3-by-3 block contain all of the digits 1 through 9.

		7			9			
	9		1					8
			4	7	3	9		
					4	3	8	
	3	1		2		7	9	
	8	5	9					
		8	3	9	7			
1					2		5	
			5			8		

Notes

The Wisconsin Mathematics Council in collaboration with the Wisconsin Department of Public Instruction present

WI Administrators Mathematics Initiative: Effective Leaders, Effective Educators

Attend 4 sessions, at the location(s) of your choice, for $150 or attend individual sessions for $75/session!

The Wisconsin Mathematics Council, in collaboration with the Wisconsin Department of Public Instruction, is offering a year-long initiative to support school and district administrators to improve teaching and learning of mathematics, PK-12. **WMC recognizes that mathematics is a gateway to our children's future and values the importance of *leadership* in implementing the necessary changes in teaching and learning in order for *all* Wisconsin students to be fully prepared for college and career.**

Open to **district administrators, school principals, and curriculum directors**, this four part mathematics professional learning initiative, features nationally and internationally respected mathematics education thought leader, Dr. Timothy Kanold, to lead you through highly motivational and energizing vision implementation planning sessions. Dr. Kanold will focus on strategies for sustaining your leadership in Common Core, Smarter Balanced, and Educator Effectiveness.

Timothy D. Kanold, PhD, *is an award-winning educator, author, and consultant. He is former director of mathematics and science and served as superintendent of Adlai E. Stevenson High School District 125, a model professional learning community district in Lincolnshire, Illinois. Dr. Kanold is committed to equity and excellence for students, faculty, and school administrators. He conducts highly motivational professional development leadership seminars worldwide with a focus on turning school vision into realized action that creates greater equity for students through the effective delivery of professional learning communities for faculty and administrators.*

He is a past president of the National Council of Supervisors of Mathematics (NCSM) and coauthor of several best-selling mathematics textbooks over several decades. He has authored articles and chapters on school leadership and development for education publications. Dr. Kanold earned a bachelor's degree in education and a master's degree in mathematics from Illinois State University. He completed a master's in educational administration at the University of Illinois and received a doctorate in educational leadership and counseling psychology from Loyola University Chicago. You can follow Dr. Kanold's blog at http://tkanold.blogspot.com.

2014 Dates (Parts 1, 3 and 4 offer two location options for each one-day session; times vary)

Part 1: February 12 (Kalahari Resort & Conference Center, WI Dells in conjunction with AWSA Secondary Principals Conference) or February 13 (Olympia Conference Center, Oconomowoc)	**Part 3:** April 23 (Jefferson Street Inn, Wausau) or April 24 (Country Springs Hotel, Pewaukee)
Part 2: March 13 Webinar (4:00-5:00 p.m.)	**Part 4:** June 24 (Jefferson Street Inn, Wausau) or June 25 (Country Springs Hotel, Pewaukee) (School/district teams encouraged to join administrators for Part 4)

For more information, please contact:

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