Wisconsin TEACHER of MATHEMATICS

Focus on Instructional Practices

Structuring Number Knowledge with Anchors to Five and Ten

What, Why and How: A Teacher's Guide to Building on Prior Knowledge

What's in an App?

The Common Core Essential Elements – Wisconsin's New Alternate Achievement Standards

Engaging Students: Look What Can Happen!

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Transforming Assessment, Instruction & Learning

[engaging all students in mathematics]

Wisconsin Teacher of Mathematics Spring 2013 Journal

The Spring issue of the Wisconsin Mathematics Council *Wisconsin Teacher of Mathematics* will focus on high quality assessment practices in the K-16 classroom. With increased attention on balanced assessment practices, the WMC Editorial Panel would like to showcase examples of both formal and informal assessments that guide instructional practices. We are interested in articles that examine the following issues:

- How does assessment guide your teaching practices?
- How does assessment impact the learning of students in your classroom?
- How do you use assessment data to meet the needs of all your students?
- What formative assessment techniques have been successful in your classroom?

If you have any questions or ideas for the Spring Journal or wish to submit an article or activity for review, please visit the WMC website for more information: www.wismath.org/resources/. The deadline for manuscripts for this Spring's issue is January 31, 2013.

MANUSCRIPT SUBMISSION GUIDELINES

- **Send an electronic copy of your manuscript to the Wisconsin Mathematics Council. Manuscripts may be submitted at any time for review.**
- **Manuscripts should be typed, double-spaced.**
- **Include all figures and photos in .jpg format; submit high resolution copies of figures and student work. Please do not place figures or photos within the document; rather indicate their placement in the document, e.g., Figure 1 here.**
- **All fractions need to be formatted as follows—2/3.** *Do not accept auto formatting of fractions.*
- **All manuscripts are subject to a review process.**
- **Include name, address, telephone, email, work affiliation and position.**

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The deadline for Spring journal submissions is January 31, 2013

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Adopting the Common Core State Standards was the Easy Part

In June 2011, the Common Core State Standards were released, and Wisconsin was the first state to adopt the Standards upon release. Forty-four other states have also adopted the CCSS as their official state curriculum for mathematics.

It took Wisconsin less than 24 hours to adopt the CCSS. Adopting the Common Core was the easy part; implementing the Common Core is the difficult part. This is because truly implementing the CCSS involves a shift in our instruction and practice, not simply an adjustment of the content that we teach. There will need to be a change at the district office, at DPI, in our content, and in our classrooms as well. Implementation of the Common Core needs to involve a change in our instruction.

Improving the teaching of mathematics is central to any successful reform initiative. Our intended curriculum is the CCSS, but our implemented curriculum is dependent upon our instruction. Many Common Core implementation efforts have focused on the Standards for Mathematical Content; however the focus needs to be on the Standards for Mathematical Practice. The Standards for Mathematical Practice describe the shift in our instruction that must occur to implement the CCSS.

Steve Leinwand (2009), in the introduction of his book, *Accessible Mathematics: Instructional Shifts that Raise Student Achievement,* writes:

> Fortunately, we all agree that our goal should be to ensure that all students know, like, and are able to apply mathematics as a direct result of their school experiences. Unfortunately, we also agree that as a country, we are currently falling far short of this goal. It's a tough situation for which many solutions have been suggested. However, of all the programs, initiatives, and proposals for addressing this situation, I'm increasingly convinced that the answer is one word: *instruction.* That is, the web of plans, actions, and decisions that constitute what teachers actually do behind closed classroom doors makes all the difference.
>
> There is no question that a coherent and rational curriculum guides and organizes an effective mathematics program. Life is a lot easier when the assessments we use, or are subjected to, are closely aligned with this curriculum and hold us accountable for important mathematics. It certainly helps to have sufficient access to print and non-print materials that support implementation of the program. And it's a relief when we have supportive and gutsy leadership and parental support. But none of these program components has anything like the degree of impact on student achievement as the *quality of instruction.* For many reasons, no component of the K-12 mathematics program seems to get as little attention as instruction – the day in and day out of complex interactions between teachers and students that determine who learns how much mathematics. (p. ix)

Leinwand, speaking at the 2012 WMC Annual Conference, stated that there are essentially four elements of mathematics education: curriculum, assessment, instruction, and professional development and support. With the adoption of the CCSS, the curriculum is set. With the upcoming SMARTER Balanced tests, assessment is set. That leaves the need to focus on instruction and how to improve our instruction through professional development.

One reason the CCSS has been adopted by 45 states across the country is that there is much evidence that shows that having a common curriculum, like many foreign countries, will improve the rankings of United States students on international comparisons. A major reason common standards lead to high student achievement in Korea, Singapore, and Japan is that each of these countries emphasize professional development of teachers and high quality instruction.

One interesting item to note is that the while Common Core State Standards were adopted to standardize our mathematics curriculum nationwide, without a common implementation plan behind the CCSS, any hopes of consistency and standardization are lost. So although we may have a common curriculum, there can be no hope for a common implementation when there are 45 different states and thousands of school districts nationwide creating their own implementation plans. States and districts need to remember that in order for there to be positive change in the learning of school mathematics, the focus on implementation needs to be on instruction.

Reference

Leinwand, S. (2009). *Accessible mathematics: 10 instructional shift that raise student achievement.* Portsmouth, NH: Heinemann.

Common Core State Standards Resources

CCSSM Online:
Wisconsin DPI:
http://www.dpi.wi.gov/standards

Smarter Balanced Assessment Consortium: http://www.k12.wa.us/SMARTER Explore the SBAC site for released items for the new assessment system.

Progressions for the CCSSM: http://ime.math.arizona.edu/progressions/ Examine in-depth analysis of the domain progressions across the grades. Drafts of these progressions are available for K-5 Number and Operation in Base Ten, Counting and Cardinality, Operations and Algebraic Thinking, and Measurement and Data; K-6 Geometry, 3-5 Number and Operations - Fractions, 6-8 Expressions and Equations, Ratios and Proportional Relationships, and Statistics and Probability, and High School Statistics and Probability.

The Illustrative Mathematics Project: http://illustrativemathematics.org This project has a collection of mathematical tasks aligned with the K-8 Domains and High School Conceptional Categories.

CCSSM Learning Trajectory Posters:
http://www.wirelessgeneration.com/posters

Editors' Notes

In this issue, the editors chose to focus on instructional practices as outlined by the Standards for Mathematical Practice. These practices describe "varieties of expertise that mathematics educators at all levels should seek to develop in their students" (CCSSM, 2010). For example, Huinker's article focuses on the Standards for Mathematical Practice (MP) 7 and 8 in which students are looking for structure and repeated reasoning in order to discern patterns and make generalizations. Embedded in Hlas', et al. article on the high-leverage teaching practice of building upon prior knowledge is the mathematical practice of making sense of problems (MP1); as teachers we need to understand what skills and strategies our students are bringing to the classroom. Attending to precision (MP6) is the foundation for Hasenbank's article where communicating precisely involves, not only correct mathematical vocabulary, but also selecting the appropriate units of measure.

As we look forward to our Spring journal, the focus is on using formative and summative assessments to guide instructional practices and promote student learning.

Jennifer Kosiak
Heather Mathison
Jenni McCool
Maggie McHugh

WMC Editorial Panel

Structuring Number Knowledge with Anchors to Five and Ten

DeAnn Huinker, University of Wisconsin-Milwaukee

Sometimes students are stuck in a "counting by ones" paradigm. When these students encounter an addition or subtraction problem situation, they count all, starting at one, or count on by ones as their only strategies. This can be seen with young children using objects or older students who rely heavily on tally marks. These students seem to rarely, if ever, reason with other relationships among numbers. Perhaps these students lack knowledge and experiences with other ways of structuring numbers besides a "one more" or "one less" than relationship. These may be your students who struggle the most in mathematics. Our challenge as teachers is to help all students develop a rich network of number relationships and provide experiences that encourage students to look for and regularly use relationships among numbers in their computation strategies.

This article looks at moving students beyond counting by ones to the use of more sophisticated strategies of reasoning with number relationships. Specifically, it examines the importance of structuring number knowledge with anchors to five and ten through the use of ten frames. It also discusses supporting students to look for and use these anchors in their reasoning. This entails a focus on two of the Standards for Mathematical Practice, "Look for and make use of structure" (MP7) and "Look for and express regularity in repeated reasoning" (MP8). These practices should be embedded and connected to Standards for Mathematical Content in the Operations and Algebraic Thinking (OA) domain and the Number and Operations in Base Ten (NBT) domain. (References to specific standards are included using the established coding system. Refer to the Common Core State Standards document for the specific wording of the standards: www.corestandards.org.)

In the Classroom

A second grade class was growing sunflowers, which provided a context for lots of mathematical work. One Monday, the teacher posed the following problem: The sunflower plant grew 6 inches since we measured it on Friday when it was 28 inches tall. How tall is it now?

Blake solved the problem by counting on from 28, keeping track of his counts by holding up six fingers. He reasoned, "28... 29, 30, 31, 32, 33, 34." Carly made a group of 28 tally marks and a group of 6 tally marks and counted them all starting at one. Daniel drew a group of 2 sticks and 8 dots to represent the 28, and drew a group of 6 dots, and then he counted on from the 28 touching each of the six dots as he counted. Each of these strategies were shared and discussed as a whole class to ensure that all students had access to the mathematics. The teacher began with Blake's strategy, as it was the most common approach used in the class. Next the teacher called on Carly and Daniel to surface and display some visual models. Then she called on Jasmine.

Jasmine solved the problem by reasoning with ten as an anchor. She explained, "I broke apart the 6 inches into 2 and 4. Then I started at 28, jumped to 30, and jumped to 34." The teacher pressed Jasmine to say more about her strategy to make her reasoning decisions visible to the class. The teacher recorded her thinking, as shown in Figure 1.

Jasmine: First I broke apart the 6 inches into 2 and 4. (The teacher recorded the number relationship.)

Teacher: Why did you decide to break it apart this way and not use some other number pair, such as 3 and 3?

Jasmine: Well, 3 and 3 would have been harder. It's easier for me to use tens, so I knew I needed to add 2 to get to 30, and then added 4 to get 34. (The teacher recorded the two equations.)

Teacher: How did you know that you needed to add 2 to get to 30?

Jasmine: I think about the ten frame in my head and I just see those two empty spaces.

The teacher displayed the ten frames shown in Figure 2, and the class re-examined Jasmine's strategy using her mental model of the ten frames. The teacher then had the class use ten frames and counters to practice Jasmine's strategy to solve 8 + 6, 18 + 6, and 28 + 6, and then challenged them to solve 78 + 6 by visualizing the ten frames in their minds. Some students were able to visualize

2 empty spaces and jump to the next ten, and also able to add 4 more, whereas others still needed to count on by ones.

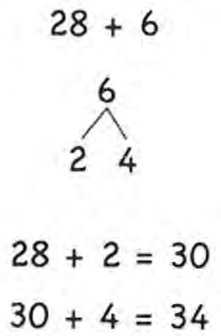

Figure 1. Jasmine's Strategy

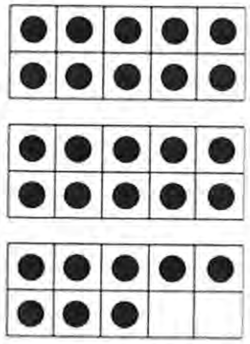

Figure 2. Jasmine's Mental Model

This scenario raises several questions. What enabled Jasmine to use ten as an anchor? Why did so many students persist in counting by ones? What was the teacher trying to achieve by focusing on Jasmine's strategy? How much experience did the students have using ten frames, and how much more do they need? Why did the teacher use the sequence of tasks involving 8 + 6, 18 + 6, 28 + 6, and 78 + 6?

Jasmine began using five and ten frames in kindergarten and then used ten frames and double ten frames in first grade (see Figure 3). She came to second grade with a strong visual image of the ten frame and knowledge of how numbers relate to the anchors of five and ten. For many of the students in this class, the ten frame was a new visual model. In addition, many students were not fluent in decomposing ten into pairs and still relied heavily on counting by ones to solve addition and subtraction problems.

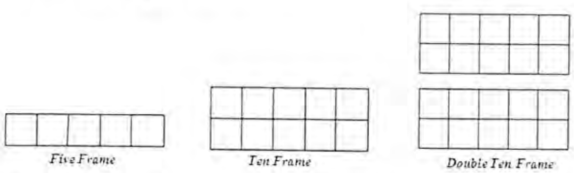

Figure 3. Frames for Structuring Number Knowledge

Why Five and Ten?

Before students can recognize the structure inherent in our numeration system, they must see the consistency and organization of numbers related to ten. Ten frames, as well as five frames, provide students with a "rich, visual tool for developing understanding of number, place value, and computation" (Losq, 2005, p. 310). As Jasmine showed us, ten frames support numerical strategies beyond counting by ones each time students solve addition or subtraction problems. The frames encourage students to consider how numbers are related (e.g., 8 is three more than 5 or 2 less than 10) and to use these relationships as a foundation for more complex mental reasoning strategies.

A five frame is a 1-by-5 array and is filled from left to right with one counter in each box. Young children can begin using a five frame in preschool and then shift to a ten frame when they have established meaning for five. A ten frame is a 2-by-5 array. The top row is filled first and then the second row, again filling from left to right. This provides a consistent way to show numbers and reinforces visual images of fives and tens as anchors.

The ten frame should be a common tool in PK-2 classrooms. It is especially helpful in meeting the Common Core State Standards for Mathematics expectations for standards K.OA.3 and K.OA.4 (see Figure 4). These standards expect students to decompose numbers within 10 into pairs and to find the number that makes ten for any number from 1 to 9. This foundation is needed for students to develop the expected reasoning strategies for single-digit computation in standard 1.OA.6, as well as standard 2.OA.2.

Figure 4. Selected Common Core Standards

K.OA.3.	Decompose numbers less than or equal to 10 into pairs in more than one way, e.g., by using objects or drawings, and record each decomposition by a drawing or equation (e.g., 5 = 2 + 3 and 5 = 4 + 1).
K.OA.4.	For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation.
1.OA.6	Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten (e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 – 4 = 13 – 3 – 1 = 10 – 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 – 8 = 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13).
2..OA.2	Fluently add and subtract within 20 using mental strategies.2 By end of Grade 2, know from memory all sums of two one-digit numbers

Figure 4. Selected Common Core Standards

Starting with a Five Frame

The first transition is moving from counting by ones to a focus on knowledge of five including decomposing five into pairs and relationships to five as an anchor. Given that we have two hands with five fingers on each for a total of ten fingers, presents itself with an obvious starting point for discussing number relationships with children. Finger patterns play "an important role in early numerical strategies, and their use and development is to be encouraged" (Wright, Martland, & Stafford, 2006, p. 27).

Throughout children's work with five frames, connections should be made to finger patterns. You might also want to establish a context, such as a row of five cubbyholes or mailboxes or a box that holds five chocolates. Here are some activities for getting started with five frames.

Fill Five Frames. Have students practice showing quantities from 0 to 5 using five frames and counters. Once the students have displayed a number, ask questions that emphasize the relationship to five: How many spaces are filled? How many spaces are empty? If all the spaces were filled, how many counters would you have? Initially, students should count by ones to figure out or to check their answers.

Build It. Display or flash a five frame that contains from 0 to 5 dots for about five seconds. It works well to have a set of five frames pre-filled with dot stickers or to use an overhead projector or document camera. Students should put the image in their minds and then build it on their own frames using counters. Flash the image again for students to check and adjust if needed. Then show the image and have students compare it to what they built.

Hold Up. Display or flash a five frame that contains from 0 to 5 dots. Have students hold up the number of fingers that matches the number of dots.

Five And Some More. As a next step, call out numbers from six to nine. Students place five counters in the frame and place the additional counters outside the frame. Discuss how each number is anchored to five. For example, 7 is two away from five because there are two extra counters not in the five frame.

Using a Ten Frame

The next transition is using ten frames to reinforce how numbers are anchored to both five and ten. For example, 8 is three more than five and is two less than ten. Ten frames also reinforce the relationship of subtraction to addition as an unknown addend (standard 1.OA.4). For example, to subtract 10 – 8 students visualize the number of counters to add to 8 to make 10. As students develop visual images of number combinations that make ten, they gain fluency in identifying these complements to ten, which serves as an important building block for single-digit computation strategies.

The activities for five frames can now be extended to ten frames. First, students need experiences representing quantities from 0 to 10 on ten frames using counters and discussing the organization of those quantities in relation to the anchors of five and ten. How many spaces are filled in the top row? How many spaces are filled in the bottom row? How many total spaces are filled? How many more counters are needed to have ten? (i.e., How many spaces are empty?) How many counters would you need to add or remove to have five?

The following activities provide ideas for ways to use ten frames. Continue asking questions that encourage students to visualize and describe the quantities and their relationships to five and ten. It is helpful to have a set of prefilled ten frame cards, as shown in Figure 5, for some of the activities.

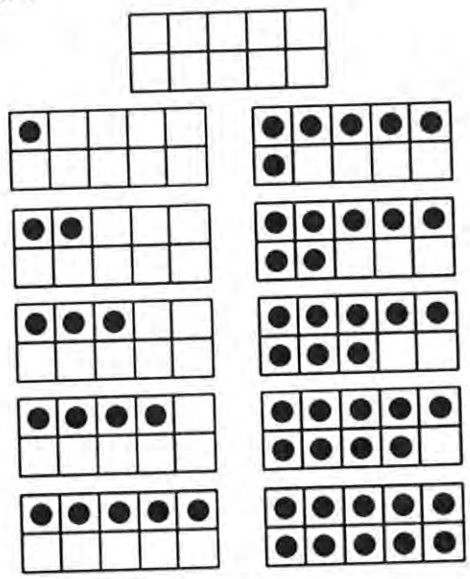

Figure 5. A set of Ten Frame Cards

Build It. Display a ten-frame card to the class for a few seconds. Students visualize the quantity and its relationship to ten. Have students build it with counters on their ten frames. Provide a quick second or third look for students to check their visual reasoning. For each card, ask, "How many dots did you see? How did you see it?" Eventually, students should be able to tell the number without having to build it.

Ten-Frame Pairs. Flash a ten-frame card for a few seconds. Allow time for students to visualize it. Ask students to state the pair of numbers that make ten. For example if seven dots are shown, they state, "Seven and three." For a variation, ask students to write the expression (e.g., 7 + 3).

Make Ten. Display an empty ten-frame on the board as a visual reference. Call out a number from zero to nine. Students determine the number needed to make ten. Have students turn and talk with a partner to compare answers and reasoning.

I Wish I Had. Hold up a ten frame card and say, "I wish I had 5 (or 10). Then students determine and write down how many more (or fewer) counters are needed to make that number. Initially, they can use ten frames and counters to figure out the answers. Eventually, they should use their mental images.

Crazy Mixed-Up Numbers. Prepare a list of mixed-up numbers. Call out the numbers one at a time. Students build each number as it is called, but do not clear their frames between numbers. They must figure out what changes are needed to make each new number. If the frame contains 4 counters and 8 is called, students respond, "Add four." If 5 is then called, they respond, "Subtract three." Students can prepare their own list of about 15 numbers and take turns calling them out to others in small groups.

Collect Ten. Students work in pairs. Each has a ten frame and ten counters. They take turns rolling a dot cube that has 1-3 dots on a side. The goal is to fill the ten frame. The exact amount must be rolled.

Collect 30 Together. Place three ten frames and a bowl containing counters in the middle of a small group of students. They take turns rolling a dot cube and placing counters on a ten frame. If the entire number does not fit on a ten frame, a new ten frame must be started. If this is not possible, the player loses a turn. The goal is to fill all the ten frames by working together.

Using a Double Ten Frame

Once students are skilled at reasoning with a single ten frame, have them work with a double ten frame to develop mental strategies of making ten or using fives for single-digit addition and related subtraction (standards 1.OA.6 and 2.OA.2). Consider how students might solve this *Add To Result Unknown* word problem: Alaina had 8 tennis balls. Today at the court she found 5 more tennis balls. How many tennis balls does she have now? Students would put 8 counters on the top frame and 5 counters on the bottom frame to represent the situation (see Figure 6). Then a student might move 2 counters to the top frame to make ten and then see that they have 13 total. Another student might say, "I see 2 rows of five. That's ten, and I see 3 more and that makes 13."

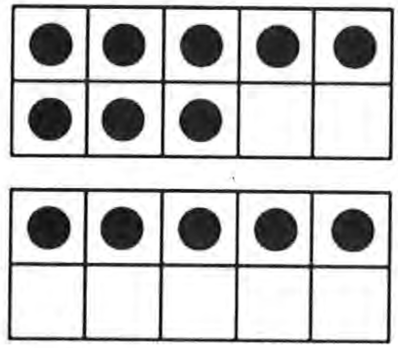

Figure 6. Solving 8 + 5

An activity that supports the making ten strategy is "Nine and Some More." Display a double ten frame with 9 counters on it. Have students visualize adding 6 more counters (or any number from 3 through 9). Then ask, "What do the frames look like now?" Discuss how they visualized the changes. For example, to add 4 a student might say, "I imagined putting one counter in that empty space to make ten and then put three on the other frame. That gave me 13." It is helpful to record, as shown earlier with Jasmine, how the four was decomposed into one and three to make it easy to add. Then extend this reasoning by displaying 8 counters or 7 counters and have students visualize and discuss adding some more.

Embedded in this computation work is also an opportunity to develop place-value knowledge of teen numbers. As students instinctively look for ways to complete a ten to make the computation easier, they are beginning to visualize and use place-value ideas. For example, with a double ten frame students can see that 14 is composed of 10 ones and 4 ones (standard K.NBT.1) or as 1 ten and 4 ones (standard 1.NBT.2).

Math Practices: Use Structure and Regularity in Reasoning

The use of ten frames provides a visual tool that supports students in the mathematical practice to "look for and make use of structure" (MP7). The structure is knowledge of ten. This includes developing part-whole understanding of ten through its decompositions and its relationships to other numbers. It also includes supporting students' cognitive shift in unitizing, knowing "ten" simultaneously as both one group of ten (the size of the unit) and as a group of ten individual items. This shift supports students in counting "groups of ten" with meaning (e.g., seeing with ten frames that 3 tens is the same as 30 individual items), a major idea required to understand place value and use place-value based strategies for computation.

Furthermore, the use of ten frames engages students in the mathematical practice to "look for and express regularity in repeated reasoning" (MP8) as students look for and use anchors of five and ten to solve single-digit addition and subtraction. This can then be extended, as Jasmine demonstrated, to larger numbers. Consider using problem sequences and multiple ten frames to have students extend their make ten strategy using these problem sequences:

- 9 + 4, 29 + 4, 39 + 4, 69 + 4, and then perhaps, to 99 + 4 and 99 + 34.
- 7 + 6, 27 + 6, 37 + 6, 77 + 6, and then perhaps, to 97 + 6 and 97 + 26.

The development of number knowledge is an expectation of the *Common Core State Standards for Mathematics.* The use of five and ten frames supports students in visualizing and structuring number knowledge anchored to five and ten and, just as importantly, supports them in the mathematical practices of using this structure regularly in their mathematical reasoning and problem solving.

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Resources

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Conklin, M. (2010). *It makes sense! Using ten-frames to build number sense, grades K-2.* Sausalito, CA: Math Solutions.

National Council of Teachers of Mathematics (NCTM) Illuminations Site. Ten Frame applet. Website: http://illuminations.nctm.org/ActivityDetail.aspx?ID=75.

What, Why, and How: A Teacher's Guide to Building on Prior Knowledge

Christopher S Hlas, Lindsey Alger, Deana Petersen, Julia Baranek, Krystal Urness, Leah Grancorvitz and Mitchell Hahn

Overview

A *high-leverage teaching practice* is defined as a practice that is "most likely to affect student learning" (Ball, D.L. & Forzani, 2010, p. 43). Identifying such practices could provide focus for teacher education programs and novice teachers, while serving as reminders for expert teachers. Unfortunately the conceptualization of high-leverage teaching practices is relatively new so there is limited evidence and agreement regarding what these practices might be (e.g., see Teaching Works).

Despite this lack of agreement, it is still a worthwhile task to speculate possible practices that may qualify as "high-leverage" then determine if there is support for the high-leverage classification. To accomplish this, college students investigated the teaching practice of *building on prior knowledge* during a directed studies course at University of Wisconsin-Eau Claire. This article details the results of their findings. Please note that this is not a comprehensive meta-analysis of the research, but is instead a multi-faceted review of the practice of building on prior knowledge.

What is *Prior Knowledge?*

Students start building prior knowledge in mathematics during infancy. Seigler (2003) states, "Children's learning of mathematics in school contexts builds on a substantial base of understanding that they acquire *before* they begin their formal education" (p. 219, emphasis added). Therefore we define prior knowledge as any previous knowledge that a student brings to a new learning experience. Such knowledge may be affected by non-academic or academic factors.

Socioeconomic status (SES) is an example of a non-academic factor that can affect students' prior knowledge. Prior to a child's formal education, some children learn less mathematics than others. Specifically, "most middle-income children know the relative magnitudes of all of the single-digit numbers when they enter school"(Siegler, 2003, p. 221), however, low-income children typically do not. These gaps in prior knowledge prove as hindrances once these students start their formal education.

An academic factor that affects prior knowledge is the extent to which the teacher provides the students with an opportunity to learn through specific learning experiences. For example, tasks that demand high-level skills provide an opportunity for high-level learning (Henningsen & Stein, 1997). Such tasks are often nonroutine problem-solving tasks that require students to explain their mathematical thinking. For example:

> *Shade 6 small squares in a 4x10 rectangle. Using the rectangle, explain how to determine each of the following: a) the percent of area that is shaded, b) the decimal part of area that is shaded, and c) the fractional part of area that is shaded.* (Henningsen & Stein, 1997, p. 539)

A good way to build on prior knowledge in mathematics is by having instruction full of problem-solving situations.

Why is Building on Prior Knowledge Important?

"Some researchers have pointed out that if cognitively demanding tasks are appropriate with respect to students' levels and kinds of prior knowledge, students' cognitive processing during task implementation stands a better chance of remaining at a high level" (Henningsen & Stein, 1997, p. 527). This indicates a task building on a student's prior knowledge has a better chance of being a high-leverage teaching practice. Our analysis of these articles indicate that building on prior knowledge is important for increasing comprehension of the material, student motivation, and collaborative performance.

For example, Tobias (1994) finds that students can better apply their prior knowledge and make connections when they engage in problem solving questions and topics of interest. Hailikari's study (2008) finds that students' self-beliefs and previous success, as well as their prior knowledge, affect their future success. Rittle-Johnson, Durkin, and Star (2009) discuss benefits for students building up their base of prior knowledge about the topic before they view different problem solving methods. The importance of building a base of prior knowledge is also a factor when it comes to participating in cooperative learning practices (Oortwijn, Boekaerts, & Vedder, 2008). This research states that building on prior knowledge helps motivate students to participate in activities and feel more comfortable in a small group atmosphere by using "high quality helping behaviors" like asking, providing, and applying explanations (Oortwign, Boekaerts & Vedder, 2008, p. 252).

It should also be noted that mathematics is a subject in which new material builds on previous material that requires students to transfer knowledge within and between different contexts (National Council of Teachers of Mathematics, 2000; National Research Council, 2001). Teachers must not only activate students' prior knowledge, but also connect it to new concepts and learning environments.

From this information, we gather that building on prior knowledge is a fundamental concept of learning mathematics. Building on prior knowledge is important for increasing comprehension, motivation, and collaboration among students.

How Can Teachers *Build on Prior Knowledge?*

For organizational purposes, we classify recommendations from the research into three broad categories. What follows is a discussion of each in more detail.

Recognizing prior knowledge involves a formal or informal assessment to determine a student's prior knowledge of a specific subject. Recommendations include:

- Provide frequent and early assessments to gain insight to a student's prior knowledge and foresee possible misconceptions (Hailikari, Nevgi and Komulaninen, 2008; Siegler, 2003).
- Account for a student's linguistic proficiencies, which may mask a student's actual prior knowledge within a subject (Oortwign, Boekaerts & Vedder, 2008).
- Identify a student's interests to identify areas of (likely) higher prior knowledge (Tobias, 1994).
- Possible measures of prior knowledge might include student grade point average, pretest scores, standardized test scores, observations from in-class discussions, and student collaboration (e.g., Rittle-Johnson, Star, & Durkin, 2009).

Activating prior knowledge relies on making connections between previously learned material and the current topic. Recommendations include:

- Draw explicit connections between current and previous tasks by encouraging a student to recall prior experiences (Henningsen & Stein, 1997).
- Model the use of prior knowledge for a student by recalling relevant past experiences in a variety of ways (Henningsen & Stein, 1997).
- Incorporate a student's interests into a lesson to better activate his or her prior knowledge (Tobias, 1994).
- Compare different mathematical procedures only after the student has learned at least one procedure well (see the expertise reversal effect or Rittle-Johnson, Star, & Durkin, 2009).
- Emphasize links between a procedure and the concepts involved (Siegler, 2003).

Taking prior knowledge into consideration identifies appropriate learning experiences given a student's level of prior knowledge in a subject. Recommendations include:

- Help a student find procedures and strategies that work for him or her because it is important to "utilize a student's individual thought process" (Siegler, 2003, p. 233).
- Use individual experiences more for students with higher prior knowledge. Use collaborative experiences more for students with lower prior knowledge (Buschang, Chung & Kim, 2011; Serafino & Cicchelli, 2003).
- Change collaborative groups often to allow students to interact with other students of varying levels of prior knowledge (Buschang, Chung & Kim, 2011).
- When using problem-based curricula, use more student-centered and student-paced instruction with low prior knowledge students (Serafino & Cicchelli, 2003).

Conclusion

Building on prior knowledge is a natural teaching strategy, so we were comforted to find so much support for it within the research literature. More surprising, however, was the amount of variability regarding how prior knowledge is defined, the factors that can influence prior knowledge, and the numerous suggestions from research regarding prior knowledge. These results show that building on prior knowledge is an important, if not an essential or "high-leverage" teaching practice, and that there is no "one way" to implement that practice.

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What's in an App?

Kimberly A. Markworth & Molly Kutsick, Western Washington University

Currently, there are few resources for reviews and evaluations of mathematics applications (apps) for the iPad. Several educators' blogs discuss the use of the iPad in general within classrooms; however, information specifically related to mathematics apps is lacking. Despite the prohibitive expense of the device (approximately $500 for a single unit), iPads are making their way into classrooms through grant funding and school- or district-level initiatives. Many teachers are also willing to utilize their personal iPads for student use. Of course, iPads and other devices that support apps (i.e., iPods and iPhones) are becoming ubiquitous in students' homes. The ability of students of all ages to access and engage with math apps makes their educational potential impressive.

Both fortunately and unfortunately, there are thousands of math apps from which to choose. A simple app store search of "math" yields 4,330 hits. This is fortunate in that these apps can address many content domains in a variety of contexts. Is a student interested in penguins playing soccer? Yes, there's a math app for that! Unfortunately, this vast selection of math-related apps makes selecting a strong mathematical app a challenging and time-consuming task. With so many apps to choose from, how can one find interesting, engaging, and accurate apps that address the intended content or skill?

The purpose of our ongoing research is to develop a scoring rubric for math apps that will provide useful information for math educators and parents. In our development of this rubric, we have attempted to address areas of mathematical importance, student interest and maneuverability, and teacher factors for classroom application. Our long-term goal is to develop an online, searchable database for teachers and parents, so that teachers and parents can be directed to relevant and quality apps for classroom or individual student use.

In the next sections of the article, we will discuss the five categories of the app scoring rubric. A current draft of the rubric can be found at http://faculty.wwu.edu/markwok/mathapps.shtml. Following this discussion, we will consider how several of these scoring categories apply to a free app, Coop Fractions. You may wish to download and play the app before reading further!

The App Scoring Rubric

Section 1 – Description

Section 1 of the scoring rubric focuses on general description. In this section, the app is given a two to three sentence description, and cost and availability (i.e., iPhone, iPod, iPad) are indicated. In addition, the relevant content standards of the Common Core State Standards for Mathematics (CCSSM) (Common Core State Standards Initiative, 2010) are identified. This last aspect may prove especially useful in an online database in that it will enable the user to search by content standard, yielding apps that are related to the targeted content areas.

Section 2 – Developing Mathematical Understanding

For the authors, this second section has been the most important and the most difficult in its development. Developing Mathematical Understanding consists of three criteria, each of which is scored on a scale from 1-4.

1. Conceptual Connection. The Conceptual Connection criterion refers to the ability of the app to provide support for conceptual understanding. The highest rating (4) reads, "There are two or more conceptual connections/representations for procedural tasks," indicating that there is meaningful, conceptual support, even when tasks may be presented procedurally. Representations that support conceptual understanding may include number lines, chip models, or various fraction models. An app that is purely drill – and that does not provide support for making sense of the operation that is being drilled – would score poorly in this category.

2. Contextual Connection. This criterion relates to the connection that is made between the mathematical content of the app and its context. Apps in which there is no connection between the context and the mathematical ideas (e.g., penguins playing soccer and addition) are unfortunately common; these score poorly in this category. To score a 4, an app must make a contextual connection that is both "meaningful and relevant to the mathematical ideas."

3. **Feedback for User.** In order to develop mathematical understanding, students should have immediate feedback for their mistakes and be expected to correct their work. Ideally, this feedback should be accompanied with support, such that students not only know that they are incorrect, but they have tools or representations to assist them in correcting their mistakes and misconceptions. An app would score a 4 in this category if it provides such feedback.

Section 3 – Ease of Use

The Ease of Use section encompasses three criteria that relate to how easy the app would be to use at the student level, although there are implications for a classroom teacher or parent.

1. **Directions.** Directions refers to the availability and quality of the directions that are provided by the app. The ratings range from a score of 1, which indicates that no directions are available, to a score of 4, which indicates that clear, age-appropriate directions are provided.

2. **Support for Use.** This criterion has presented difficulties with scoring, in part because it requires us to take the perspective and skill of students into account. To score a 4, "Students would be able to engage effectively with the app with no instructional support." Since younger generations are more confident in their use of technology, a lot of children would be able to 'figure out' how to work an app with little to no adult assistance.

3. **Settings.** Have you ever wished you could control the sound on an app, but haven't been able to? In this section, Settings refers to the ability of the user to control various settings (e.g., sound, display, number choices) and access these settings from all areas of the app, not just from the main menu. To score a 4, the user must be able to control multiple settings and access these with ease from any area of the app.

Section 4 – Motivation

Ideally, apps ought to be engaging for the students who explore them, and their engagement factors should not detract from the mathematical concepts to be learned or practiced. This section ranks two categories that may influence a student's willingness and enthusiasm to spend some time with the app.

1. **Entertainment.** This criterion identifies the factors contributing to how entertaining an app might be to the user. The ranking progresses through 1) lacking any engagement factors, 2) engagement due to sound or display functions, 3) engagement due to contextual factors unrelated to the mathematical ideas, to 4) engagement due to the mathematical ideas being addressed.

2. **Stimulation.** App sounds and displays can sometimes distract from the mathematical ideas or overwhelm the user. To score a 4 in Stimulation, the auditory, visual, and/or kinesthetic stimulation must be appropriate and not distract from successful engagement with the app.

Section 5 – Other Criteria

Several other criteria emerged in our evaluation of apps that we thought would be of interest to mathematics educators. However, these criteria were not appropriate for a ranking system or score. Instead, questions around levels, speed, assessment opportunities, and settings were included as an additional section.

1. **Levels.** Four dichotomous questions (yes/no) indicate if progressive levels are locked, if levels increase in difficulty, if students can select levels, and if additional levels or content skills are available for purchase.

2. **Speed.** One primary question indicates if speed is a factor in the app. If so, three secondary questions indicate if speed is used punitively, if speed is used for self-competition, and if speed can be controlled by the user or administrator (e.g., setting a time limit).

3. Assessment Opportunities. This criterion contributes to a teacher's ability to track students' progress with app work. One question indicates if the app can negotiate different users and if there is a limit to the number of programmable users. A second question indicates if the app has the ability to keep track of scores and if those scores are public to all users.

4. Settings. Although settings have already been addressed in the Ease of Use section, this category further indicates if all app sounds can be controlled and if app settings are password protected.

5. Additional Comments. Finally, there have been issues of interest or concern that have emerged in relation to specific math apps. A section for additional comments provides a place for these issues to be stated.

Coop Fractions

Let's consider how this rubric applies to Coop Fractions in a few areas. Coop Fractions is a multi-level app activity in which children estimate the value of a fraction on a number line marked in whole number or decimal increments. A nest is moved to the location of their estimate, and a chicken (with substantial discomfort and drama!) projects an egg to the actual value of the fraction on the number line. If the estimate is too far from the actual value, the egg misses the nest and breaks!

This app scores fairly well in the *Developing Mathematical Understanding* section. A conceptual connection for the procedural task is provided (score 3) in that it utilizes a number line representation marked with whole number or decimal increments. The context for the app is superficially connected to the mathematical ideas; the size of the nest allows for a range in the fraction estimate, and a higher score results from a closer estimate (score 3). Feedback for incorrect answers is immediate (the egg breaks!), but there is no opportunity for correction (score 2).

For *Entertainment*, the app's engagement is reliant on contextual factors that are unrelated to the mathematical ideas (score 3). Specifically, the laying of the eggs – including the faces and sounds that accompany this process – is quite entertaining. This auditory and visual stimulation is distracting in a way that might moderately affect performance (score 2).

The egg-laying aspect of this app is something that is attended to in the *Additional Comments* area. The mathematical ideas addressed by Coop Fractions are important, and the contextual connection, although limited, is unique and engaging. However, the appropriateness of the context is a potential limitation for its use in a classroom. Although students would surely get beyond its distracting qualities, this would be a substantial hurdle to overcome!

Initial Findings

Perhaps unsurprisingly, our initial exploration of almost 40 apps has revealed that there are many drill apps related to particular computational procedures. Unlike Coop Fractions, approximately half of these apps lack any conceptual connection for procedural tasks. Also, 75% of these apps utilize contexts that have no connection to the mathematical ideas addressed by the app. This is not to say that there are not quality apps that address mathematical concepts in an effective way. Adding Apples, Lobster Diver, and Candy Factory are apps that score well, especially in the *Developing Mathematical Understanding* category. Of course, Coop Fractions develops some important concepts around fraction representation and estimation, despite its distracting features!

Although our research and development is still in its initial phase, we are excited about what our current version of the rubric has been able to differentiate and identify. Currently, we are drafting a rubric scoring guide to accompany the scoring rubric. We invite your comments, suggestions, and questions, whether you have used math apps in the classroom or if you are considering their use. What are your thoughts? Have we neglected any important criteria that you would include? Are there any particular math apps that you would like to see evaluated? Would you find a searchable database an important resource? We welcome your feedback!

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The Common Core Essential Elements – Wisconsin's New Alternate Achievement Standards

Kristen Burton and Erin Faasuamalie, Wisconsin Department of Public Instruction

In June 2012, Wisconsin adopted the Common Core Essential Elements (CCEE) in English language arts and mathematics. These alternate academic achievement standards describe challenging, grade-level expectations for students with significant cognitive disabilities and increase access to the general education curricula based upon the Common Core State Standards (CCSS). The CCEE were developed by a consortium of 13 states called Dynamic Learning Maps (DLM). These new standards will be used as the basis for the new Dynamic Learning Maps assessment, to be implemented in 2014-15. The CCEE can be found at: http://dpi.wi.gov/sped/assmt-ccee.html.

All students, including students with significant cognitive disabilities, deserve and have a right to a quality educational experience. This right includes, to the maximum extent possible, the opportunity to be involved in and meet the same challenging expectations that have been established for *all* students.

The general education mathematics curriculum for all students includes five domains: number and operations, measurement, data and probability, geometry, and algebra. Historically, students with significant cognitive disabilities have received focused instruction on the functional skill activities—numbers, operations with basic computations, telling time, and measurement with money—while the opportunity to learn other aspects of the mathematics curriculum have generally been overlooked (Browder, Spooner, Ahlgrim-Delzell, Harris, & Wakeman, 2008). Low academic expectations for students with significant cognitive disabilities have resulted in a lack of access to the mathematical skills found in all five domains. In fact, research, although limited, already demonstrates that when provided with supports, students with significant cognitive disabilities are capable of learning other mathematical skills such as computation, measurement, graphing, shapes, and money (Browder et al., 2008). When students are provided with academic opportunities and held to high expectations, the potential for increased opportunities in life after graduation, including post-secondary education and sustainable paid employment, are more likely to occur.

The Common Core Essential Elements reflect the rigor of the CCSS and introduce skills that have not been taught previously in the classrooms that include students with significant cognitive disabilities. This shift in academic expectations will present challenges for schools. Teachers may need a deeper understanding in the foundations of mathematics. This challenge brings an opportunity for collaboration between the special education teacher and the general education teacher. As mentioned by Agnello, Ahlgrim-Delzell, Knight, and Jimenez, 2009:

> The Dynamic Learning Maps system is designed to provide formative tools, periodic assessments and professional development modules to be used throughout the year as well as an end-of-year summative assessment. Educators will be able to use this data and information to make adjustments in their instructional strategies in order to meet the needs of their students. The learning maps help guide the day-to-day instruction for students with significant cognitive disabilities, thereby hoping to improve access to the general education curriculum.

While the special educator is skilled at adapting the general education lesson for his/her students to ensure participation in a meaningful activity, a general education teacher can help the special education teacher with concerns of alignment [to the standards] and the big ideas of the lesson.

Collaboration between educators of students with significant cognitive disabilities and their colleagues will not only help special education teachers strengthen their knowledge of content stan-

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While the special educator is skilled at adapting the general education lesson for his/her students to ensure participation in a meaningful activity, a general education teacher can help the special education teacher with concerns of alignment [to the standards] and the big ideas of the lesson.

Collaboration between educators of students with significant cognitive disabilities and their colleagues will not only help special education teachers strengthen their knowledge of content standards, but will also help to ensure that the fidelity of the standards are maintained in the classroom. Likewise, general educators can learn new methods and strategies for instructional differentiation from their special education colleagues.

Since the adoption of the Common Core Essential Elements, the Wisconsin Department of Public Instruction has been working closely with stakeholders to develop a three-year professional development plan to implement the CCEE and help students prepare for the spring administration of the DLM assessment. Trainings in the first year will enable participants to:

- Understand the Common Core State Standards for English language arts and mathematics (CCSS) are for all learners and provide the foundation for the CCEE.
- Understand the purpose of the CCEE is to provide access to academic content and instruction.
- Explore the CCEE for English language arts and mathematics, and understand the progression of skills across grade levels and content.
- Interpret the implications for instruction embedded in the knowledge, skills and understandings in grade level CCEE.

Additional information about the work of the Dynamic Learning Maps Consortium can be found at: http://dynamiclearningmaps.org/.

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Engaging Students: Look What Can Happen!

Jodean E. Grunow, Department of Mathematics, University of Wisconsin-Platteville
Jason Thrun, Department of Mathematics, University of Wisconsin-Platteville
Patrick Wengelewski, Mathematics Methods Student, University of Wisconsin-Platteville

"I couldn't miss math class; I'd miss too much; I'd never be able to make it up."

"I love the mathematical discussions we have in class; I keep thinking about them when I leave."

"It's sure a lot of work, but, boy, did I learn!"

"I definitely was not a fan of math, but I think now I kinda like it."

"I really feel prepared to go forth and do math…"

These are comments taken from students leaving mathematics methods classes. They seem to indicate: 1) there is substance in the mathematics classes; 2) students are enthused; 3) students are trying; 4) students are gaining in confidence; and 5) students feel that they can apply what they have learned. These comments also would seem to indicate engagement at various levels.

Following is a discussion of a problem solving experience that took place last semester that, as we consider engagement, is worthy of consideration.

Both Grunow and Thrun use the Pizza Problem shown in Figure 1 near the beginning of their respective classes. It is a problem of interest to students and is an excellent problem for inductive and deductive analysis.

The Pizza Problem

What is the greatest number of pieces of pizza you can get if you cut a pizza from edge to edge using 5 straight cuts?

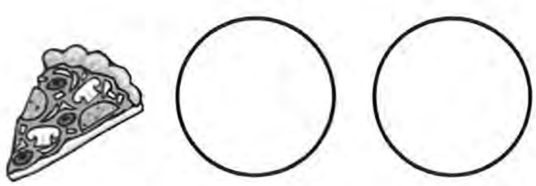


Figure 1.

What would happen if you made 8 cuts? What would happen if you made n cuts?

Figure 1. The Pizza Problem.

After exploration, discussion of pieces ensues and results are analyzed in terms of equal areas or, if equal areas is not necessary, and pieces of different sizes are possible, in consideration of maximization (see Figure 2). One way to mathematically analyze the situations and develop generalizations is to use the method of finite differences. Readers unfamiliar with finite differences may wish to view the comments at the end of this article. Figure 3 shows the patterns that emerge when using the finite-difference method. These patterns can be used to find the coefficients of linear, quadratic, and other higher order relationships. Figure 4 shows the finite-difference table and the generalizations for the Pizza Problem. An alternate strategy to the Pizza Problem is shown at the end of this article.

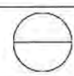
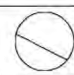
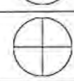
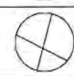
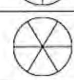
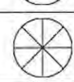
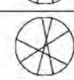
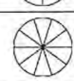
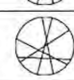
Number of Cuts	Equal Area Model		Creative-Cuts Model	
	Diagram	Number of Pieces of Pizza	Diagram	Maximum Number of Pieces of Pizza
1		2		2
2		4		4
3		6		7
4		8		11
5		10		16

Figure 2. Dividing the pizza using the equal-area model and the creative-cuts model.

Linear		
n	$an+b$	First Difference
1	$a+b$	
		a
2	$2a+b$	
		a
3	$3a+b$	
		a
4	$4a+b$	
		a
5	$5a+b$	

Quadratic			
n	an^2+bn+c	First Difference	Second Difference
1	$a+b+c$		
		$3a+b$	
2	$4a+2b+c$		$2a$
		$5a+b$	
3	$9a+3b+c$		$2a$
		$7a+b$	
4	$16a+4b+c$		$2a$
		$9a+b$	
5	$25a+5b+c$		

Figure 3. The relationship between the coefficients of a linear and quadratic polynomial, as realized through the finite-difference method.

Fair Share Model		
n	Pieces of Pizza	First Difference
1	2	
		2
2	4	
		2
3	6	
		2
4	8	
		2
5	10	
n	$2n$	

Creative-Cuts Model			
n	Pieces of Pizza	First Difference	Second Difference
1	2		
		2	
2	4		1
		3	
3	7		1
		4	
4	11		1
		5	
5	16		
n	$\frac{1}{2}n^2 + \frac{1}{2}n + 1$		

Figure 4. Finite-difference table and generalization for the Pizza Problem.

Later in both courses, in a discussion of inductive and deductive reasoning, the Circle Problem (see Figure 5) is posed. This problem, appropriately entitled the "What? Why? Problem," appears in *Planning for Curriculum* (Grunow, 2001, p. 95), the Wisconsin State guide, because it is an example of a pattern that does not do what is expected. Grunow and Thrun use the problem in that context.

The Circle Problem

Notice the four circles shown below. The first has two points highlighted, the second has three points highlighted, the third has four points highlighted, and the final circle has five points highlighted. The points are chosen at random, so there is no obvious symmetry. Chords are drawn from one point to another for each circle. In each case, the circle is divided up into a number of regions.

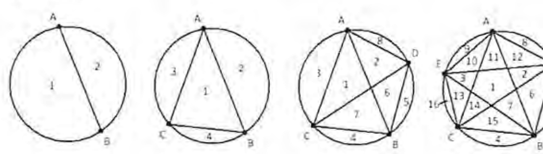


Figure 5

Predict the number of regions in the fifth circle if it had 6 points highlighted. Test your prediction by drawing a circle and finding the number of regions.

Figure 5. The Circle Problem.

This past semester, a student common to both classes was intrigued by the Circle Problem, the significance of its use in both classes, and the several questions it seemed to raise. Exhibiting all of the traits of the engaged student described in the retrospective included in this issue, this student investigated the problem in depth. Figure 6 shows the circle Patrick used to develop his conjecture and Figure 7 shows how Patrick organized his work.

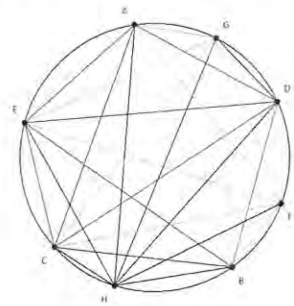


Figure 6. Patrick's circle with 8 points highlighted.

Points	New Chords	Total Number of Chords	Structure of Regions	Total Number of Regions
1	0	0	1	1
2	1	1	1+(1)	2
3	2	3	2+(1+1)	4
4	3	6	4+(1+2+1)	8
5	4	10	8+(1+3+3+1)	16
6	5	15	16+(1+4+5+4+1)	31
7	6	21	31+(1+5+7+7+5+1)	57
8	7	28	57+(1+6+9+10+9+6+1)	99
9	8	36	99+(1+7+11+13+13+11+7+1)	163
10	9	45	163+(1+8+13+16+17+16+13+8+1)	256

Figure 7. The relationships between the points, chords, and regions, as realized by Patrick.

Patrick's organizational scheme is revealing. Each time a new point is added, new chords are added. In fact, when point n is added, n - 1 new chords are added. The sequence representing the total number of chords is the sequence of triangular numbers. Of particular interest is the structure of the regions. Patrick noticed that the number of new regions generated forms a Pascal-like triangle. This is shown in the "Structure of Regions" column in Figure 7. The entries for the Pascal-like triangle are shown in parentheses. Notice that the entries along any main diagonal form an arithmetic sequence.

The following excerpt, in Patrick's words, describes how he thought about generating the Pascal-like triangle.

> The way in which the rows for the Pascal-like triangle are developed can be seen when looking at the Circle Problem in Figure 5. It is important to note that each number in the Pascal-like triangle represents the number of new regions that are created with each *new* chord. This Pascal-

like triangle starts when a second point is added to the circle. When a chord is drawn connecting points A and B, one new region is formed. This is represented in the first row of the Pascal-like triangle. The (1) represents the new region created by chord BA. When point C is added to this circle, two new chords are created to connect point C with points A and B. This is why there are two terms in the second row of the Pascal-like triangle, one term to represent each chord. When looking at the chord CA, it divides one region of the circle. This means one new region is created. The same thing occurs when looking at the chord CB. One new region is created when this chord divides a previous region. On the Pascal-like triangle, this is represented in the second row as (1+1). The first 1 represents the new region created by CA. The second 1 represents the new region created by chord CB. When point D is added to the circle, three new chords are created to connect point D to points A, B, and C. As noted before when point C was introduced to the circle, the three new chords means that there are three terms in the third row of the Pascal-like triangle. Moving clockwise around the circle from point D, the first chord that is created is chord DB. Since this chord is cutting through one region, one new region is formed. An interesting thing happens when looking at creation of the chord DC. It cuts through chord AB. This means that chord DC will cut through two regions. This means that two new regions are formed from this chord. The last chord that is created is chord DA. This chord only cuts through one region so only one new region is created. The new regions formed by these three new chords are represented in the third row of the Pascal-like triangle. This row shows (1+2+1). The first 1 represents the new region created by the chord DB. The 2 represents the two new regions created by chord DC. The last 1 represents the new region created by chord DA This process continues for each new point that is added to the circle.

To summarize, the creation of the Pascal-like triangle involves the number of points on the circle and the number of new regions created with each chord. The row number plus one is the number of points on the circle. This is because that the Pascal-like triangle starts when there are two points since two points are needed to create a chord. The number of terms in each row represents the number of new chords that are added when a new point is added to the circle. The number of new chords should be one less than the number of points on the circle. Finally, each term represents the number of new regions that are created by each chord. This can be seen when moving in a clockwise (or counterclockwise) order in the creation of each new chord from the new point on the circle, starting with the point closest to the new point. The number of new regions that each of these new chords create is determined the by the number of previous chords that it goes through. Each new chord is going to create at least one new region because it is cutting through one piece. When each new chord passes through a previous chord, it begins to divide another region. This means that the number of new regions each new chord creates is one plus the number of previous chords it cuts through. This is how the Pascal-like triangle is developed for the Circle Problem.

Patrick concluded his analysis by using the finite-difference method to find the quartic relationship between the number of points on the circle and the number of regions created. Figure 8 shows the relationship between the coefficients of a quartic polynomial, as realized through the finite-difference method. Figure 9 shows the finite difference table and the generalization for the Circle Problem. Comparing corresponding entries from Figure 8 and Figure 9,

$24a = 1$, $60a + 6b = 1$, $50a + 12b + 2c = 1$, $15a + 7b + 3c + d = 1$, and $a + b + c + d + e = 1$

Solving and back substituting, we find

$a = 1/24$, $b = -1/4$, $c = 23/24$, $d = -3/4$, and $e = 1$.

Both Grunow and Thrun were, needless to say, overjoyed with this analysis. It is this kind of *engagement* that delights educators. It IS the *mathematical practices* in action!

		Quartic			
n	$an^4 + bn^3 + cn^2 + dn + e$	First Difference	Second Difference	Third Difference	Fourth Difference
1	$a+b+c+d+e$				
		$15a+7b+3c+d$			
2	$16a+8b+4c+2d+e$		$50a+12b+2c$		
		$65a+19b+5c+d$		$60a+6b$	
3	$81a+27b+9c+3d+e$		$110a+18b+2c$		$24a$
		$175a+37b+7c+d$		$84a+6b$	
4	$256a+64b+16c+4d+e$		$194a+24b+2c$		$24a$
		$369a+61b+9c+d$		$108a+6b$	
5	$625a+125b+25c+5d+e$		$302a+30b+2c$		
		$671a+91b+11c+d$			
6	$1296a+216b+36c+6d+e$				

Figure 8. The relationship between the coefficients of a quartic polynomial, as realized through the finite-difference method.

n	Pieces of Pizza	First Difference	Second Difference	Third Difference	Fourth Difference
1	1				
		1			
2	2		1		
		2		1	
3	4		2		1
		4		2	
4	8		4		1
		8		3	
5	16		7		1
		15		4	
6	31		11		1
		26		5	
7	57		16		
		42			
8	99				
n	$\frac{1}{24}n^4 - \frac{1}{4}n^3 + \frac{23}{24}n^2 - \frac{3}{4}n + 1$				

Figure 9. Finite-difference table and generalization for the Circle Problem.

Note: *Dr. Grunow has done a webinar discussion of this problem. It can be found on the CESA 3 website under the Gifted and Talented Grant. Go to the CESA 3 website (http://www.cesa3.k12.wi.us). On the site, go to Instructional Services on the top menu bar. Under Instructional Services, find Gifted and Talented on the left hand menu bar. Click on it and Gifted and Talented Math Grant 2012 comes up. The particular webinar that is referenced in these materials is Mathematical Practice 7: Look for and make use of structure. There are three discussions: Food for Thought: The Pizza Problem; What? Why?: A Look at Patterning; and Four Lines: Connections?*

Notes on the Finite Difference Method

In this article, the authors used a finite-difference method to find a polynomial expression that generalizes the relationship between two sets of values. In the Pizza Problem, our input values were consecutive whole numbers representing the number of cuts on the pizza, and the outputs were whole numbers corresponding to the maximum number of pieces of pizza. In the Circle Problem, the inputs were consecutive whole numbers representing the number of points selected on the circle, and the outputs were whole numbers corresponding to the number of regions generated by connecting all possible chords between these points. Because these relationships were polynomial in nature, the finite-difference method provided a way to find the coefficients of the polynomial.

In problems like the ones shown in this article, students frequently generate a table for small values of n and look for a pattern from these numbers. This is a very natural, inductive method. Of course, this can lead to problems if there are competing patterns. For example, students frequently generalize the total number of regions in the Circle Problem as 2^{n-1}, where n is the number of points selected on the circle. If students do not extend their tables far enough, they may never see a reason to consider another pattern. For this reason, it is best to combine this inductive approach with a deductive approach. If students can explain why a pattern continues, then the finite-difference method can be a useful tool.

In our examples, the finite-difference tables began with input and output values. Since the input values were consecutive whole numbers, the output values can be represented by {f(1), f(2), f(3), …f(k), …}.

The first-order finite difference, f_1, is given by $f_1(k) = f(k+1) - f(k)$. A second-order finite difference, f_2, is given by $f_2(k) = f_1(k+1) - f_1(k)$ or equivalently, $f_2(k) = f(k+2) - 2f(k+1) + f(k)$. A third-order finite difference, f_3, is given by $f_3(k) = f_2(k+1) - f_2(k)$ or equivalently, $f_3(k) = f(k+3) - 3f(k+2) + 3f(k+1) - f(k)$. In general, the ith finite difference, f_i, is given by $f_i(k) = f_{i-1}(k+1) - f_{i-1}(k)$.

The following chart shows a general finite difference table with first-, second-, and third-order differences.

Input Values	Output Values	First-Order Difference	Second-Order Difference	Third-Order Difference
1	$f(1)$	$f_1(1) = f(2) - f(1)$	$f_2(1) = f_1(2) - f_1(1)$	$f_3(1) = f_2(2) - f_2(1)$
2	$f(2)$	$f_1(2) = f(3) - f(2)$	$f_2(2) = f_1(3) - f_1(2)$	$f_3(2) = f_2(3) - f_2(2)$
3	$f(3)$	$f_1(3) = f(4) - f(3)$	$f_2(3) = f_1(4) - f_1(3)$	$f_3(3) = f_2(4) - f_2(3)$
4	$f(4)$	$f_1(4) = f(5) - f(4)$	$f_2(4) = f_1(5) - f_1(4)$	$f_3(4) = f_2(5) - f_2(4)$
⋮	⋮	⋮	⋮	⋮
n	$f(n)$	$f_1(n) = f(n+1) - f(n)$	$f_2(n) = f_1(n+1) - f_1(n)$	$f_3(n) = f_2(n+1) - f_2(n$
$n+1$	$f(n+1)$	$f_1(n+1) = f(n+2) - f(n+1)$	$f_2(n+1) = f_1(n+2) - f_1(n+1)$	
$n+2$	$f(n+2)$	$f_1(n+2) = f(n+3) - f(n+2)$		
$n+3$	$f(n+3)$			

Figure 10. A general finite-difference table.

Suppose the set of values, $\{f(1), f(2), f(3), \ldots, f(k), \ldots\}$, is governed by a linear relationship $f(n) = an + b$, , for some constants a and b, then each first-order finite difference is given by

$f_1(k) = f(k+1) - f(k) = a(k+1) + b - (ak+b) = a.$

In this case, the first-order finite difference is constant. If the set of values, $\{f(1), f(2), f(3), \ldots, f(k), \ldots\}$, , is governed by a quadratic relationship, $f(n) = an^2 + bn + c$, for constants a, b, and c, then each first-order finite difference is given by

$f_1(k) = f(k+1) - f(k) = a(k+1)^2 + b(k+1) + c - (ak^2 + bk + c) = 2ak + (a+b).$

Since the difference is linear in k, the second-order difference will be constant.

In general if the set of values, , $\{f(1), f(2), f(3), \ldots, f(k), \ldots\}$, is governed by a polynomial of degree *m*,

$f(n) = a_m n^m + a_{m-1} n^{m-1} + a_{m-2} n^{m-2} + \ldots + a_1 n + a_0,$

for constants $a_0, a_1, a_2, \ldots, a_m$, then each first-order finite difference is given by $f_1(k) = f(k+1) - f(k)$ or equivalently

$a_m(k+1)^m + a_{m-1}(k+1)^{m-1} + \ldots + a_1(k+1) + a_0 - (a_m k^m + a_{m-1} k^{m-1} + \ldots + a_1 k + a_0).$

Since the coefficient of the k^m term in $(k+1)^m$ is 1, this difference reduces to a polynomial of degree m – 1. Similarly, the second-order finite differences will be polynomial of degree m - 2. Continuing in the way, the m^{th} finite difference will be constant.

An Alternate View of the Pizza Problem and the Circle Problem

For readers unfamiliar with the method of finite differences, an alternate method of the Pizza Problem and the Circle Problem is provided.

The Pizza Problem begins with an uncut pizza, so zero cuts corresponds to one region or piece of pizza. One cut divides the pizza into 1+1=4 pieces. To get the maximum number of pieces, the second cut must pass through the first cut. Imagine the cut beginning on an edge of the pizza. As the cut passes though the previous cut, the old region is divided into two so the number of pieces increases by one. When the cut is continued to the outer edge of the pizza, again the number of pieces increases by one. In all, there are 1+1=2 pieces of pizza corresponding to two cuts. Figure 11 shows the maximum number of pieces corresponding to zero, one, two, three, and four cuts.

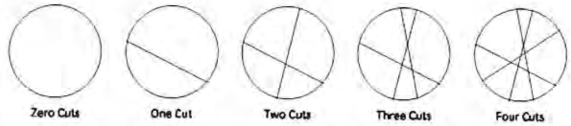


Figure 11. Diagrams for maximizing the number of pieces of pizza for the Pizza Problem.

This idea can be generalized. In order to maximize the number of pieces created, each new cut must pass through all the previous cuts. As the cut passes from edge to cut, cut to cut, or cut to edge, the number of pieces increases by one. On the n^{th} cut, the number of pieces increases by n. The following chart suggests a generalization for the total number of pieces of pizza with n linear cuts.

Cut	Increase in Number of Pieces of Pizza	Total Number of Pieces of Pizza
0		1
1	1	1 + 1 = 2
2	2	1 + 1 + 2 = 4
3	3	1 + 1 + 2 + 3 = 7
4	4	1 + 1 + 2 + 3 + 4 = 11
⋮	⋮	⋮
n	*n*	???

Figure 12. A generalization for the maximum number of pieces of pizza for the pizza problem.

Specifically, the total number of pieces of pizza is given by

$$1+(1+2+3+4+\cdots+n)=1+\frac{n(n+1)}{2}$$

This same reasoning can be used to generate the Pascal-like triangle for the Circle Problem. Suppose we begin with a circle with seven points highlighted and all the chords drawn. To this circle, we add one new point, H. See Figure 13.

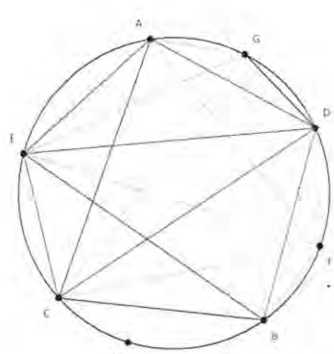


Figure 13. Generating new regions when adding an additional point for the Circle Problem.

Remember how the number of new regions increases. If we begin at point H, each time we reach a chord or an endpoint, we increase the number of regions by one. If we begin at point H and arrive at point C, we have added one new region. From point H to point E, we cross five chords and arrive at point E, so we have increased the number of regions by six. From point H to point A, we cross eight chords and arrive at point A, so we have increased the number of regions by nine. This idea is easily generalized. When creating chord HA, we cross each line that begins at point C or E and ends at point B, F, D, or G. This means there are a total of 2x4=8 chords. Similarly, the chord from H to G must pass through 3x3=9 chords. The following chart generalizes this idea.

Number of Points on the Circle	Structure of the Increase in Number of Regions
2	$0\cdot1+1$
3	$0\cdot1+1,\ 1\cdot0+1$
4	$0\cdot2+1,\ 1\cdot1+1,\ 2\cdot0+1$
5	$0\cdot3+1,\ 1\cdot2+1,\ 2\cdot1+1,\ 3\cdot0+1$
6	$0\cdot4+1,\ 1\cdot3+1,\ 2\cdot2+1,\ 3\cdot1+1,\ 4\cdot0+1$
7	$0\cdot5+1,\ 1\cdot4+1,\ 2\cdot3+1,\ 3\cdot2+1,\ 4\cdot1+1,\ 5\cdot0+1$
8	$0\cdot6+1,\ 1\cdot5+1,\ 2\cdot4+1,\ 3\cdot3+1,\ 4\cdot2+1,\ 5\cdot1+1,\ 6\cdot0+1$
⋮	⋮
n	$0(n-2)+1,\ 1(n-3)+1,\ 2(n-4)+1,\ \dots,\ k(n-2-k)+1,\ \dots,\ (n-2)0+1$

Figure 14. Generalization for the n^{th} row of the Pascal-like triangle.

From the chart in Figure 14, we can see the increase in the number of regions when the nth point is added is given by

$$\sum_{k=0}^{n-2}\left(k(n-2-k)+1\right)=\frac{n^3-6n^2+17n-12}{6}$$

Since this cubic polynomial represents the first difference in the difference table, the total number of regions must come from a 4th degree polynomial. Alternatively, the total number of regions is the sum of the first region and all the subsequent increases. If there are m-1 increases, the total number of regions is given by

$$1+\sum_{n=2}^{m}\frac{n^3-6n^2+17n-12}{6}=\frac{m^4-6m^3+23m^2-18m+24}{24}$$

Of course, this agrees with Patrick's formula.

Does $a^2 + b^2$ Really Equal c^2? The Case of the *Mystery Triangle*!

Jon Hasenbank, Assistant Professor of Mathematics Education, Grand Valley State University and Ben Sturomski, Teacher Candidate, University of Wisconsin-La Crosse

The Common Core Math Practices outline a vision of what it looks like to do mathematics across all grade bands. The Standards for Mathematical Practice describe the mathematical habits of mind that every student should develop if they are to become proficient with both the art and science of mathematics. Below, we outline a lesson we taught to a high school geometry class in an effort to help them better appreciate the importance of being attentive to precision in measurement and computation (Math Practice #6). The activity can be adapted to work at a variety of levels, with appropriate adjustments depending on the prior knowledge of the students. For instance, the activity would align nicely with an eighth grade lesson related to standard 8.G.6 "Explain a proof of the Pythagorean Theorem and its converse" or 8.G.7 "Apply the Pythagorean Theorem to determine unknown side lengths in right triangles in real-world and mathematical problems," and extensions can lead to discussions related to the high school standard A-APR.4 "Prove polynomial identities and use them to describe numerical relationships. *For example, the polynomial identity* $(x^2 + y^2)^2 = (x^2 - y^2) + 2(xy)^2$ *can be used to generate Pythagorean triples.*"

"Attending to Precision" encompasses a wide variety of habits and skills. In part, it involves communicating precisely: using correct mathematical vocabulary, writing precise definitions, carefully stating the meaning of symbols used – including specifying the unit of measure – and using correct mathematical syntax. It also encompasses precision in measurement, which not only means selecting an appropriate unit of measurement but also reporting answers with an appropriate degree of precision for the problem context. It is this latter aspect we intended to address with our lesson.

The lesson is ostensibly an exploration of the Pythagorean Theorem. In Euclid's *Elements*, the Pythagorean Theorem is stated as follows (Proposition 47): "In right-angled triangles, the square on the side opposite the right angle is equal to the squares on the sides containing the right angle." A close reading reveals an important insight — to Euclid and his contemporaries, "the square on the side" was taken literally as the square constructed on the given side. The theorem is therefore not an *algebraic* statement about the relationship between the side lengths a, b, and c, but rather a *geometric* statement about the areas of the literal squares constructed on the sides of a right triangle (Dunham, 1990, p. 48). This observation served as a launching point for our activity.

We divided the students into groups and gave each group copies of two right triangles: one with legs 3 in and 4 in, and another with legs 3 in and 5 in. Students were not told the lengths of the sides; instead, they are asked to use 1 inch x 1 inch plastic square tiles (physical manipulatives found in many elementary and middle school classrooms) to measure the side-lengths and to form the literal squares on the sides, as shown in Figure 1. Can you spot the apparent error in the diagram associated with the 3-5 right triangle?

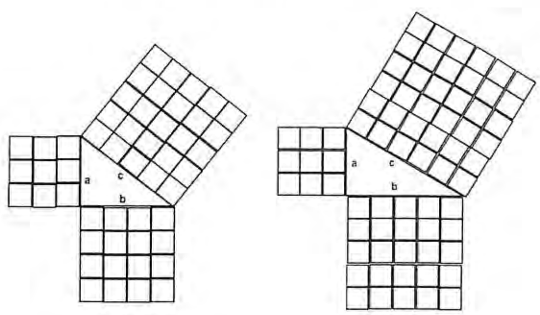


Figure 1. Literal Squares

Students were directed to test whether or not the Pythagorean relationship holds for each triangle. By inviting students to use the 1 inch x 1 inch tiles to measure the lengths and construct the squares on each side, we accomplished two goals. First, we reinforced the geometric meaning of the Pythagorean Theorem. Second, we retained some control over the level of precision students used for their measurements by surreptitiously forcing them to measure to the nearest whole inch.

Students discovered that when the tile squares were counted and areas calculated, the squares of the legs did not in fact equal the square on the hypotenuse: i.e., $9 + 25 \neq 36$! We asked them: Could it be that we have discovered a right triangle that does not adhere to the results outlined by the Pythagorean Theorem? Could the Pythagoreans and Euclid have been wrong about their famous conjecture?

At this point, some students express concerns that perhaps the right angle was not truly "right." The class investigated, some by overlaying the angle in question with square corners of paper, others by using a protractor to measure its size. We also presented students with a dynamic construction built with Geogebra, where we fixed two sides of a triangle at 3 in and 5 in and allowed the included angle to vary. We learned that it would take an angle of nearly 94° to yield an opposite side of length 6 inches. The students agree: such an angle is plainly obtuse (see Figure 2). In the end, all are convinced that a slightly inaccurate right angle would not be enough to explain the discrepancy.

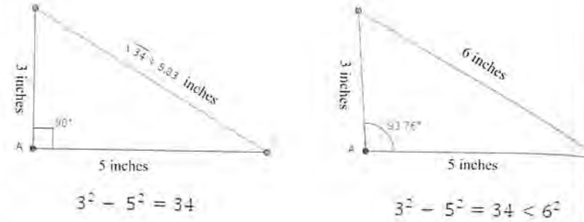


Figure 2. An Extension of Pythagorean Theorem

Numerical error analysis is concerned with quantifying the errors that may be introduced in numerical computations. The Pythagorean Theorem and the 3-5-$\sqrt{34}$ right triangle considered in our activity can lead to a nifty introduction to the topic.

Let $a = 3$ in, $b = 5$ in, and $c = \sqrt{34}$ in. In the student activity, we have induced a measurement error for the length of the hypotenuse. Measuring to the nearest whole inch, we obtain an approximately correct length of $c^* = 6$ in. This means we have a measurement error of

$e = c^* \quad c \approx 6 - 5.83 = 0.17$ in.

So instead of the correct equation $a^2 + b^2 = c^2$, we have the approximate equation $a^2 + b^2 \approx (c^*)^2$. Substituting $c + e$ in for c^*, we obtain the following:

$$a^2 + b^2 = (c^*)^2$$

$$a^2 + b^2 = (c + e)^2$$

$$a^2 + b^2 = c^2 + 2ce + e^2$$

Here, the $2ce + e^2$ represents the divergence from the correct equation $a^2 + b^2 = c^2$; this is the error induced by our inaccurate measurement of the hypotenuse.

At this point, numerical analysts would argue that if the measurement error e is small, then e^2 will be insignificant, and so the magnitude of our induced error is dominated by the middle term, 2ce. A more thorough analysis would consider the worst case scenario for this error term, but we may simply substitute the known values $c = \sqrt{34}$ and $e = 0.17$. We find $2ce \approx 1.98$, which helps illuminate the source of those extra two square tiles from a numerical analysis perspective.

If the angle has been accurately constructed, our students reasoned, then perhaps the source of the error was our measurement of the side lengths. After all, 1 inch tiles are not an especially precise measurement tool. If the Pythagorean relationship were to hold, we should have found that the hypotenuse had a length of $\sqrt{34}$, or about 5.83 inches. Once again, the class investigated. As students reconsidered their measurements, they confirmed the length of the hypotenuse was not quite 6 full tiles. The measurement error of just over 1/8 of an inch (actually about 0.17 inches) represents a deviance of less than 3% from the correct value: it is hardly perceptible in the physical model, and yet it is enough to yield two extra square tiles in the square on the hypotenuse. (If the reader wishes to take the error analysis a step further to discover analytically the source of those two extra tiles, please refer to the sidebar.)

We have found that many students have a difficult time anticipating the impact that early rounding may have on their final results. So we asked them: Is it surprising that a measurement error of just 3% should produce such a significant final error of 2 full square units? The Common Core State Standards and common sense suggest students should be able to provide a thoughtful response to such a prompt. Activities like this one are important scaffolds for helping our students learn to attend to precision in measurement, both in school mathematics and in practical situations that will crop up throughout their adult lives.

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WMC Puzzle Page

Search-A-Word: Trigonometry

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G	A	S	E	N	I	S	F	O	W	A	L	S	O	A
L	I	S	T	A	O	T	H	A	C	H	O	S	C	N
E	H	Y	P	O	T	E	N	U	S	E	J	X	A	G
O	P	P	O	S	I	T	E	D	E	I	W	C	L	L
E	L	G	N	A	I	R	T	C	C	O	S	I	N	E

adjacent
amplitude
angle
cosecant
cosine
cotangent
degrees
hypotenuse
identities
inverse
law of cosines
law of sines
opposite
period
ratio
reciprocal
right angle
sine
sohcahtoa
tangent
triangle
trigonometry
unit circle

State Mathematics Competition

The following problem is from the 2011 High School State Mathematics Contest. For additional questions and solutions, visit www.wismath.org/resources/math-contests .

Schroeder High School has 2011 lockers in it, each one starting closed. The students decide to play a prank on April Fool's Day. The first student to enter the building opens every locker. The second student to enter closes every second locker. The third student changes the status of every third locker. The fourth student changes the status of every fourth locker, and so on until the 2011th student changes the status of the final locker. How many lockers end up open?

Ken Ken

Fill in the blank squares so that each row and each column contain all of the digits 1 through 6. The heavy lines indicate areas that contain groups of numbers that can be combined (in any order) to produce the result shown with the indicated math operation.

6×		2	2÷	11+	
1−	4	9+		3÷	
			1−		2−
11+	3÷			2÷	
	2−	1−	2÷		6
				5−	

Sudoku

Fill in the blank squares so that each row, each column and each 3-by-3 block contain all of the digits 1 through 9.

	5		3					4
2								
			7	6	4		1	5
	8		9	4			7	
		9		7		8		
	6			3	8		5	
5	3		4	2	1			
								3
7					3		8	

Notes

Notes

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