Approximating the Value of a Function

- Since the $z$-value in the remainder of an $n$th degree Taylor polynomial lies between $x$ and $c$, use the larger of $\frac{f^{(n+1)}(x)(x-c)^{n+1}}{(n+1)!}$ or $\frac{f^{(n+1)}(c)(x-c)^{n+1}}{(n+1)!}$ as an upper limit.

Finding the error of a Taylor polynomial

The remainder or error of a Taylor polynomial is $R_n(x) = \frac{f^{(n+1)}(z)(x-c)^{n+1}}{(n+1)!}$ for $z$ between $x$ and $c$.

Example: The fourth degree Maclaurin polynomial of $f(x) = e^x$

\[
y = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4
\]

Recall that for Maclaurin polynomials $c = 0$ and $f^{(n)}(0) = e^0 = 1$.

\[
e^{0.1} = 1 + 0.1 + \frac{1}{2}(0.1)^2 + \frac{1}{6}(0.1)^3 + \frac{1}{24}(0.1)^4
\]

\[
e^{0.1} = 1 + 0.1 + 0.005 + 0.00016666 + 0.0000041
\]

\[
e^{0.1} = 1.1051708...
\]

Finding the error of a Taylor polynomial

Q: Can you estimate the error without using a computer?

A: $R_n(x) = \frac{e^2x^2}{5!}$ for $z$ between 0 and $x$.

The largest value that $e^2$ could have is $e^0 = 1$, so

\[
R_n(0.1) = e^{0.01} \leq \frac{e^{0.1}(0.1)^2}{5!} < \frac{9.209 \times 10^{-4}}{5!}
\]

Approximate the error.

The fourth degree Taylor approximation has a very small error.

The remainder term of an $n$th degree Taylor polynomial is based upon the $(n+1)$th term.

The remainder expression contains a $z$-value that must lie between $x$ and $c$. This may seem like a vague definition of $z$, but it is part of an approximation, after all. If you knew the exact value of $z$, that would mean you knew the exact value of the function, and you would not need an approximation.

Here is the value of $e^{0.1}$ that you obtained using the fourth degree Maclaurin polynomial approximation of the exponential function. Since Maclaurin polynomials are special types of Taylor polynomials, you can find their error, too.

You can use the remainder expression to determine what the largest possible error is for your approximation.

When you use the values $n = 4$ and $c = 0$, you arrive at this expression for the remainder.

Since $z$ is a value between $c$ and $x$, it must lie between 0 and 0.1 in this case. Use the larger value when determining an upper limit for the remainder.

Although the remainder may be positive or negative, use the absolute value symbol to make the error positive.

Since 0.1 is close to the $c$-value of zero, this approximation is excellent.