TRUTH WITHOUT PARALLEL

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by

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There are two young boarders in my house. They condescend to allow me to support them, and occasionally they allow me to help them with their homework. Help them with their homework, indeed -- I am racing to keep up with them. This has led me into some interesting fields of study with which I had had little experience. I speak, of course, of that bane of modern parents -- New Math.

When I was in school, geometry was a straightforward subject. You learned about angles, parallel lines, triangles and the like. You learned to make logical deductions from self-evident propositions, and things were very orderly.

In the New Math, things are orderly (or appear so) but there is a little problem about those self-evident propositions. Consider for example the proposition that "through a given point, one and only one straight line can be drawn which will be parallel to a given straight line," I was taught, and I accepted uncritically, this proposition as self-evident. At least as I was in good company -- Aristotle, Descartes, and Immanuel Kant were on my side. Apparently, we were all wrong, or such is the teaching of the New Math.

Let us look at Euclid, whose "Fifth Postulate" I have just quoted. Euclid's works come to us with all of the textual problems of the other works of Greek antiquity. However, there seems to be general agreement that you can treat Euclid's geometry as an axiomatic system in which all the theorems may be logically deduced from five "common notions" and five "self-evident postulates", the fifth of which I just quoted. It is, again, "through a given point, one and only one straight line can be drawn which will be parallel to a given straight line". There are important consequences that depend on this postulate. For example, it is the basis for the proposition that the sum of the angles of a triangle is 180°. All of this always seemed to me to be in accordance with plain common sense. Anyone can intuitively understand what parallel lines are, even though it is impossible to follow any given set of parallels to infinity. It never seemed to me that there was much to be gained by questioning that the Euclidean system is a representation of some abstract, intuitively comprehended space that we live in. In short, without analyzing the matter in any detail, I understood Euclidean geometry to be as true and as real as the proposition that 2 plus 2 equals 4. That was before my son introduced me to New Math. I now know that I was more than one hundred years behind the times. Now I tentatively accept the proposition that 4 is the symbol for a cardinal number corresponding to a set which is the union of two sets, each of which is represented by the symbol for another cardinal number, namely 2. I am aware that it requires several hundred pages of the Principia Mathematica, the masterpiece of Alfred North Whitehead and Bertrand Russell, to demonstrate rigorously that 1 plus 1 equals 2. (Of course, they don't say it that way.) Even more disconcerting is my newly acquired knowledge that the basic structure of arithmetic, treated as a comprehensive formal, logical system, can be shown to be inconsistent, if it is a complete system, or incomplete, if it is a consistent system.

Who is responsible for these assaults on the sensibilities of the modern parent? Nicholai Ivanovich Lobachevsky is one of the principal culprits. But before we discuss Lobachevsky, let us return to Euclid for a moment. He recognized that his fifth postulate, the parallel postulate, occupied a different position from the other postulates. Its self-evidence was not so plainly self-evident, for the obvious reason that the definition of "parallel" depends on extending two lines into infinity and convincing ourselves that they do not intersect anywhere on their infinite extent. Euclid therefore attempted to derive his parallel postulate as a necessary logical consequence of the other postulates. Euclid failed; Proclus also failed in the fifth century A.D., and the great Arab mathematicians failed. By the 17th Century, there was a large body of so-called proofs of the parallel postulate. All depended on other propositions which were themselves unprovable.

Two 17th Century attempts were more ingenious than the others. The Italian, Saccheri, and the German, Lambert, began to understand more about the parallel postulate problem. Saccheri thought he had solved it by showing that the opposite assertion to the parallel postulate (i. e., that there is more than one line that can be drawn through a point parallel to a given line) leads to unimaginable results. In logic, however, a contradiction to intuitive ideas does not necessarily indicate a logical contradiction. The parallel postulate remained unproved.

A step backward was taken in the 18th Century by Legendre, who reverted to a proof that depended on other unprovable assertions. All the attempts to show that the parallel postulate was a necessary consequence of self-evident propositions failed, because all such attempts were based upon a preconceived notion that the parallel postulate really is self-evident and therefore represents a truth which no sane man can deny.

Notwithstanding the mathematicians' difficulties with the parallel postulate, Euclidean geometry has played a significant role in other areas of human thought. Most importantly, to Immanuel Kant, Euclidean geometry was a prime example of synthetic judgments *a priori*.

In the *Critique of Pure Reason*, he stated, "geometric principles are always apodictic, i. e., united with the consciousness of their necessity...; theorems of this kind cannot be empirical judgments or conclusions from them." To Kant, space was an *a priori* form, and the axioms of geometry (including the parallel postulate) must be true and must be self-evident, since they are properties of thought itself. Thus, if the student of mathematics is to think about space at all, he must think about space that conforms to Euclid's geometry.

This seems eminently sensible to me, and as nearly as I can recall, this philosophy was part and parcel of the geometry that I studied in school. They don't teach it that way any more. After nearly two centuries, the geometry of Nicholai Ivanovich Lobachevsky is now taking over.

Lobachevsky is generally given credit for being the founder of non-Euclidean geometry. As we shall see later, the time was ripe in the early 1800's for the new approach to geometry. Several other mathematicians arrived at the same conclusions at about the same time. Still, Lobachevsky was first, when, on February 23, 1826, he delivered a lecture at the University of Kazan, where he was a young professor of mathematics. In this lecture, he announced the solution to the by-now two thousand year old problem of the parallel postulate. The solution is simply that the parallel postulate is an arbitrary assumption. Perfectly valid and consistent geometries may be constructed on the assumption that there is more than one line, parallel to another (in the same plane), that can be drawn through a given point.

Lobachevsky, like many great creative scientists before and after his time, did not speak to a world ready and waiting for his revolutionary discoveries. The magnitude of his contributions to mathematics, to logic and to philosophy was not properly recognized or rewarded in his lifetime.

Lobachevsky was born in 1792 in Nijny-Novgorod. His father, of Polish descent, was employed by the Russian government as a surveyor. Lobachevsky grew up in poverty but was able to secure a scholarship at the gymnasium in Kazan, a trading post on a Volga tributary with a population of about 25,000. While he was there, Czar Alexander I promulgated a law changing the gymnasium to a university. There was some substance to the change. A number of German professors came to the new university and brought with them the latest works of mathematicians such as Gauss and Legendre. Lobachevsky was fortunate enough to study under one professor who had been Gauss's teacher. These influences led him to abandon a previous interest in medicine and to turn to pure science.

His work both at the gymnasium and the university levels was brilliant. He did not spend all his time on studies, however. He was known as a party-goer, and he once constructed an illegal rocket which was set off in the university courtyard. These activities, together with a personal antipathy towards one of the university officials nearly cost him his degree. Only a confession before the school's council and the intervention of his German professors enabled him to receive his master's degree at the age of 19, in 1811.

Lobachevsky stayed on at Kazan as an instructor, and graduate student. His lowly academic rank required him to teach only the most elementary courses, but this was a blessing in disguise, since his attention was focused directly on Euclidean geometry. In 1815, he attempted a proof of the parallel postulate, but soon realized that his demonstration was as erroneous as those of all his predecessors.

In 1816, he was promoted to associate professor, but the triumph was short-lived. Alexander I embarked on a program of repression which included stamping out atheism in the universities. Lobachevsky's teachers returned to Germany; and he was forced to assume a number of dull and distasteful administrative duties, including the preparation of reports on subversive faculty members. He wrote a book but was unable to get it published because it used the new-fangled metric system. The political authorities in Moscow could not approve the publication of any work having such flagrant connections with the French Revolution.

Nonetheless, Lobachevsky continued his work in geometry, and on February 23, 1826, gave a report in French to the physico-mathematical section of the University entitled, A Brief Statement of the Principles of Geometry with a Rigorous Demonstration of the Theorem of Parallels. No one could understand it, and certainly no one in his audience was aware that they had witnessed a historic turning point in mathematics. Lobachevsky's subsequent publications on the subject were greeted with no great interest in Russia. Although he became rector of the university, he was forced to resign in 1846 and became an "adjunct superintendent" of a small school district. His later years were marred by the loss of a son by tuberculosis, by financial difficulties and finally by arteriosclerosis and blindness. He died in 1856. As a final blow, the professor who delivered his eulogy was charged with being a freethinker and was fired from a position as professor of mathematics at Kazan.

The Russians do not think so little of Lobachevsky now. A current Russian treatise on mathematics states: "Lobachevsky displayed the true grandeur of a genius who defends his convictions without wavering and does not hide them from public opinion for fear of misunderstanding and criticism."

This is a point well taken. As I said earlier, the times were ripe for the development of new geometries. The great German mathematician Gauss, for example, had come, independently, to the same conclusions as Lobachevsky at about the same time, but he declined to publish anything for fear of criticism by the "Boethians", or scholastic philosophers.

Bolyai, a Hungarian geometer, who also arrived at a non-Euclidean geometry independently of Lobachevsky, did publish, but his system was not so fully developed as Lobachevsky's.

Even Lobachevsky was not fully aware of the consequences of his discovery. In 1835, Lobachevsky wrote about his revolutionary conception as follows:

> "It is well known that in geometry the theory of parallels has so far remained incomplete. The futile efforts from Euclid's time on throughout two thousand years have compelled me to suspect that the concepts themselves do not contain the truth which we have wished to prove, but that it can only be verified like other physical laws by experiments, such as astronomical observations. Convinced, at last, of the truth of my conjecture and regarding the difficult problem as completely solved, I put down my arguments in 1826."

Lobachevsky did not construct his geometry as an arbitrary exercise in logic. As he stated in 1835, he intended to see whether his new geometry could be shown, by astronomical observations, to be a more accurate representation of space. To this end, he conducted experiments in astronomy, but they were inconclusive. Within the limits of accuracy of the instruments then available, it appeared that cosmic space corresponds to Euclidean geometry. It is interesting to note that Gauss also conducted such experiments. He had observers on three mountain peaks in Germany make measurements of the sum of the angles formed by the three peaks, which were widely separated but still visible to each other. These results also were inconclusive.

It was only much later that the physical meaning of non-Euclidean geometries became known. Lobachevsky worked out a nearly complete set of theorems based on his hypothesis that through a single point in a plane outside a line in the plane, more than one parallel can be drawn. Based on this groundwork, later geometers discovered that Lobachevskian geometry is the simplest way to represent the properties of figures on a pseudosphere. What is a pseudosphere? It is formed by rotating a tractrix on its axis, and it looks like the cone-like instrument that an ear-nose-throat specialist uses to look in your ear. Lobachevskian geometry is also a specific account of the geometry of figures in a circle or on a sphere.

To all of this, you may well say that it is interesting, but not very. Its real importance lies first in the fact that the theory of relativity employs non-Euclidean geometry to represent space. Lobachevsky's discoveries formed the basis for the work of a German geometer, Riemann, and there is now general agreement that Riemannian geometry best describes cosmic space as it is known to us by the measurement of light from heavenly bodies. Euclidean geometry remains perfectly valid and useful, of course, provided that it is applied on a non-cosmic scale. It also can be thought of as a special case in a more general concept of geometry.

There are still other practical consequences of Lobachevsky's discoveries. These include the development of geometries in more than three dimensions. If you are like me, you thought of the fourth dimension as strictly a science fiction concept. In the New Math, however, this is not so. Four dimensional (and, for that matter, n-dimensional) geometries turn out to be very useful in describing the reaction of the human eye to the colors that make up light and in describing phase relationships in physical chemistry.

We see that there can be truth without parallel, that is, without the Euclidean concept of parallel lines extended to infinity. For this reason alone, the New Math is the only math that can be taught by the modern school. But we spoke earlier of the part that Euclidean geometry has played in the development of other areas of thought. We spoke, for example, of Immanuel Kant and the central position that Euclidean geometry occupies in his philosophical work. After Lobachevsky, new approaches to logic and to philosophy were necessary, and it is here that the real significance of Lobachevsky's discoveries becomes apparent. This has been pointed out by many writers, including Bertrand Russell, but never so clearly as by Richard Von Mises, an Austrian born mathematician who came to this country after World War I. The discussion that follows is based on his writings, which show how the development of non-Euclidean geometry has had effects that go far beyond conventional mathematics.

The concept that there can be different geometries based on different sets of axioms compelled mathematicians to focus their attention more closely on the axiomatic system of thought itself. Those of us who learned Euclidean geometry in school probably regard axiomatics as a straightforward system. You are given certain basic assumptions, or axioms, and you deduce theorems from them by the use of logical thought processes. However, as Lobachevsky showed, the process of selecting axioms can cause two thousand years of confusion, when the axioms are not precisely selected to begin with. Lobachevsky spoke of selecting new axioms that could be tested by physical experiment. From the standpoint of logic, his more important contribution was that he showed a clear example of the concept of independence, that two (or more) geometries can be conceived when you realize that the parallel postulate is independent of the other basic postulates (or axioms) of Euclid.

Subsequent mathematicians and logicians have shown that there are like difficulties with the other "self-evident" axioms that we learned in school, e.g., the axiom that the whole is greater than any of its parts, or that every quantity is equal to itself. These difficulties arise from our failure to make truly precise analyses of the meaning of the words we use. Does it matter from a practical standpoint? Well, no, most of the time it does not, but we shall see that it can matter even to those of us who are not nuclear physicists or mathematicians.

Parenthetically, it should be pointed out that the physical sciences were also turned in new directions in the 19th century. Ernst Mach, in 1883, showed that Newton's mechanics, as expressed by Newton, were based on a somewhat confused combination of axioms and definitions, so that the use of conventional deductive logic may lead to erroneous conclusions.

To illustrate, Newton defines force as something that changes the velocity of a body; he then enunciates the law that velocities are changed by forces. Mach showed us that Newton, had he the benefit of New Math, would have said something like "the circumstances in which a body is at a given time determine its instantaneous change of velocity, (i. e. acceleration) and for different bodies under the same circumstances, their observed accelerations differ by a numerical factor which is a constant for each such body (i. e. mass)."

Mach did not, in his reformed version of Newton, make new discoveries about the physical sciences. Rather, he expressed Newton's ideas far more clearly, with one result being that we appreciate Newton's genius even more.

In 1889, another German mathematician, David Hilbert, re-created geometry based on a logically sound structure of axioms. This approach to geometry, a refinement of the discoveries of Lobachevsky, dealt a death blow to Immanuel Kant's belief that Euclid's vision of space represented a priori truth.

Hilbert and the axiomaticists have concerned themselves with two general principles, namely consistency and independence. We say that a group of axioms are consistent if it is not possible to deduce from them both one statement and its opposite. How then do we tell if a group of axioms are independent? Assume one consistent group of axioms, a second group and a third group. The first and second are assumed to be consistent with each other. If the third group may be substituted for the second, with the result that the first and third are consistent, then the second and third are independent of the first. You will recall that Lobachevsky's discovery may be expressed as the discovery that the parallel postulate is independent of Euclid's other postulates, or axioms.

Let my audience not think that we have now arrived, through the ingenuity of New Math, at a triumphant victory over the forces of darkness and superstition. On the contrary, the axiomatic approach to mathematics and logic has led to new difficulties. It developed, after Hilbert's work, that the consistency of geometry can only be demonstrated by an appeal to the consistency of arithmetic.

You answer, "Well, of course, arithmetic is consistent." I point out that it has never been proven to be consistent, and that one of our prominent modern mathematicians, Kurt Gödel, has proven to the satisfaction of his colleagues that arithmetic cannot be proven consistent. We need not despair, however, because arithmetic and geometry do correspond with most of our real world as we are able to observe and measure it, so that for most purposes we may presume that our more perfectly axiomatized geometries and arithmetics are consistent.

"For most purposes" is of course not good enough for anyone who falls into the grips of New Math. What about the aspects of arithmetic and geometry that are not consistent?

This question leads us to an even more fundamental one--namely the question of the rigor of our logic. Modern studies of mathematics, especially the studies triggered by Lobachevsky's discoveries, have led to far-reaching developments in the study of logic. We can no longer speak with certainty about eternally valid truths of logical method. Logic itself is a developing science; the axiomatization of arithmetic and geometry is valid with respect to today's logic. Tomorrow may require different logical techniques, which could result in new approaches to the axiomatic formulation of arithmetic and geometry and of physics, chemistry and other sciences as well.

The new approaches to logic come to us from a group of mathematicians and philosophers known as the "Vienna Circle", whose most prominent member was Ludwig Wittgenstein. Wittgenstein tells us that meaningful statements may be divided into two groups: The first group consists of statements that may be tested by experience. The second group consists of statements that are either true or false because of the way they are worded. It is the second group that interests logicians. If a statement of this second kind is true, it is referred to as "tautological"; if it is false, it is "contradictory". At this point, please try to forget your ordinary understanding of the word "tautology". Logicians use it differently than normal people.

Let us look at examples of tautologies, as the word is now used. In arithmetic, we may say that the sum of two natural numbers is a natural number. By the logical rules established for arithmetic, this is true, and tautological, independent of experience. The statement, "water freezes at 32° Fahrenheit" is of course dependent on experience and is neither tautological nor contradictory, as we are using those words.

Because of our use of the concept of tautological statements, it may be argued that we have merely returned to Immanuel Kant's conception of *a priori* synthetic statements. This is not so. Tautologies are not *a priori* because they do not come from super-empirical sources, and they are not synthetic, because in and of themselves, they say nothing about reality. They represent only reformulations of arbitrary rules that have been fixed without any necessary connection to experience. Our previous example of a tautology referred to natural numbers, but this is a concept that can be developed independent of experience. Natural numbers are useful because they are related to experience, but for this purpose, they may be considered as a convenient example of arbitrary abstractions.

The objection may then be raised that we are recreating Kant's analytic judgments, that is, those concepts that exist independently and may be discovered through pure thought. However, by definition, our tautologies are composed by arbitrary rules acting upon arbitrary assumptions. This is far removed from Kant's philosophical structure.

There is obviously a limit to how far we wish to go in emphasizing the arbitrariness of a logical system. It is the thesis of most New Mathematicians that mathematics is essentially a chain of tautologies. However, if we lose all connection with experienced reality, we may have a beautiful logical system, but one that is of no use to anyone. Fortunately, the Vienna Circle and other modern logicians concentrate their efforts on logical systems that do have some relationship to reality. The use of the plural, "systems", is deliberate here. There is no general agreement today that any one system of logic is solely and completely valid for human thought.

What is most significant about the new approaches to logic is the mathematical treatment that is given, first to elementary relationships such as "and", "or", "if" and "not" and then to far more complex structures. This has led to the discovery and expression of formal laws, which you may hear referred to as "truth-function theory" or as "propositional calculus". We have thus moved from the logic of mathematics, which we have been speaking of previously, to the mathematics of logic, a new and separate mathematical discipline.

The mathematics of logic has moved us far beyond the logic which is familiar to most of us. Immanuel Kant recognized that Aristotle's logic had never been improved on for more than two thousand years. It has taken great strides since Kant's time, however, and modern approaches to logic have added a great deal to our store of knowledge about mathematics and science.

Progress in the area of mathematical logic or "logistic" has involved not only those areas of higher mathematics that are accessible to the specialist, but also those that become the concern of the parent of students of New Math. In fact, many of the great advances in logistic have involved such mundane areas as arithmetic. Bertrand Russell and Alfred North Whitehead created their Principia Mathematica in 1910 as an attempt to "reduce all mathematical concepts to the simplest logical operations". In doing so, they have shown that arithmetic is not so clear and simple as we once imagined it. They have also shown us a glimpse of what some refer to as a "world free of metaphysics." You may have, as I do, a fondness for metaphysics, but no matter how much you care for the subject, you cannot help being offended by the activities of those who abuse metaphysics and want to extend it to areas where it clearly does not belong. To this extent, a "world free of metaphysics" simply means a world where science takes its proper place while metaphysics is concerned with values that are not of this world.

There are some modern logicians who carry the concept of anti-metaphysics perhaps too far. There is an offshoot of the Vienna Circle known as the "Polish School", whose members include Lukasiewicz, Ajdukiewicz, Kotarbinski, Tarski and others. They say that all truth, all knowledge is possible through an extension of the new techniques of logistic - that is, mathematical logic. They may be right, but they have not convinced most of their colleagues. I mentioned Lukasiewicz as a member of the Polish School. He has attained a certain amount of contemporary fame as the creator of a new form of symbolic logic which has been adapted by the creators of a parlor game, available at any book store, known as WFF N' PROOF. If you have never heard of it, ask any teenager who has had a background in New Math. The introductory exercise in this so-called game is as follows:

"Given the following three statements as premises:

(1) If Bill takes the bus, then Bill misses his appointment, if the bus is late.

(2) Bill shouldn't go home, if (a) Bill misses his appointment, and (b) Bill feels downcast.

(3) If Bill doesn't get the job, then (a) Bill feels downcast and (b) Bill should go home.

is it valid to conclude that if Bill takes the bus, then Bill does get the job, if the bus is late?"

The problem posed has a clear and simple solution, but in order to solve it, it helps to become an expert in manipulating the symbolic logic of Lukasiewicz. You may again say, as I am sure you have said many times this evening, "Well, all this may be very interesting, but so what?" If you're a lawyer, I'll tell you so what. All the problems in this innocent parlor game of WFF N' PROOF are derived from the more abstruse sections of the United States Internal Revenue Code of 1954, as amended. All lawyers today are tax lawyers to some extent. It may be the ultimate Polish joke to point out that all lawyers are therefore devotees of Polish logic.

Whether you are a logician, a mathematician, a lawyer, or simply the parent of a student of New Math, you cannot help wondering where all this will lead us. We have seen mathematics nearly re-created from new fundamentals; we have seen logic stirred from two thousand years of lethargy. Are there any fundamental concepts or foundations of mathematics where we can finally rest? It is this unanswerable question that helps make New Math so interesting.

We have seen, that, thanks to Nicholai Ivanovich Lobachevsky and his successors, mathematics is now a tautological structure, independent of experience. It has foundations and basic assumptions which are debatable like those of any other science.

When we say that the foundations of mathematics are debatable, you may be inclined to say that such self-evident, and fundamental, propositions as two times two equals four are not debatable. This may be true, but bear in mind Lobachevsky's discovery that Euclidean geometry, which at one time was considered to be as self-evident as simple arithmetic, is actually a special case of a much more general geometry, which has itself been shown to be of great utility in describing reality. We may conclude then that self-evidence is not one of the foundations of mathematics.

Another suggested foundation is intuition, which is a more subtle and complex concept than self-evidence. The intuitionist school recognizes that customary patterns of thought, based on experience, dictate the simple ideas of mathematics. The intuitionist believes that the mathematician looks at and operates on these simple ideas, and through the mental process called intuition, recognizes which concepts have the properties of being clear and indubitable. The mathematics resulting from this foundation can never be considered a closed system, since it is always subject to being revised or added to as our experience of the world grows. This is, of course, a far cry from Kantian and scholastic views of mathematics, which hold that its basic concepts are "once and for all impressed upon the human race by the properties of its reasoning power".

Intuitionism is not just idle speculation about the foundations of mathematics. It has had a considerable effect on logistic, and in particular on the "rule of the excluded middle". Either "it is Monday night" or "it is not Monday night". These two statements admit of no third possibilities, or so I was always taught. The intuitionists, however, true to the tenets of New Math, have overturned conventional wisdom. In mathematical problems dealing with infinite sets of numbers, our old friend, the rule of the excluded middle is no longer applicable. This has led to the creation of still another new form of logistic, known as "problem calculus". We previously spoke of "propositional calculus" in which statements were reduced to symbols and operated upon by mathematical techniques. In problem calculus, problems are reduced to symbols, and the question is no longer whether a statement is true or not but rather whether a problem is solvable or not. In the logistic of propositional calculus, we could always assume that the truth or falsity of a statement about experience could be demonstrated. In our new logistic, problem calculus, it may be impossible to determine whether a problem can be solved. This logistic exists, therefore, without a rule of the excluded middle. This discovery is as far-reaching as the discovery that Euclid's parallel axiom is logically independent of the other axioms of geometry. For example, it put into serious doubt all those proofs of mathematics, some of them relied upon for hundreds of years, known as "indirect proofs".

The intuitionists have not taken over, despite the achievements sketched above. The school known as "formalism" objects, not unnaturally, to reliance upon a concept so ill-defined as "intuition". Hilbert, the leader of the formalistic school, has proposed that we restrict our use of the name "mathematics" to those mechanical processes in which symbolically represented axioms are operated on by arbitrary rules. The meaning and interpretation of the symbols in question are then reserved to "metamathematics", which must be used to prove the consistency of the system known as mathematics. Unfortunately for Hilbert, the development of a consistency proof requires the use of the rule of the excluded middle. What Hilbert and his followers were attempting was the construction of a proof, by finite means, that would support a structure that includes conceptions of the infinite. At present, these attempts have failed. Gödel has shown that one cannot prove the consistency of a formal system without going beyond the very same formal elements of the system. The formalists have therefore been unable to establish what they consider most essential to their approach, its consistency. The intuitionists are not bothered by this; the formalists still are.

Meanwhile, there is also a school of logicism, represented by Frege, Peano, Russell and Whitehead, that is attempting to reduce all of the tautological relationships of mathematics to basic logical concepts, with no necessary relation to the world of reality. The logicists have achieved considerable success and have made great strides in pointing out the logical mistakes of other mathematicians. However, they have not succeeded in showing us how an infallible logical system can be related to the world of experience.

We keep returning to this question of the relation of mathematics to experience. The practitioner of New Math may construct any number of tautological systems, but our interest in them rests ultimately on the extent to which they correspond with some reality that we know through experience.

The reality that we "know" today is a world far different from the world as it was known fifty, one hundred, or one hundred fifty years ago. This is the reason why we have New Math and why we parents cannot answer the simple questions of a seventh grader.

It is really no comfort to us at all, but we may close by contemplating the words of Albert Einstein: "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality."